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Solution of General Fractional Oscillation Relaxation Equation by Adomian's Method

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Abstract

We show the efficiency of Adomian decomposition method to deal with the General Fractional Oscillation Relaxation Equation, a generalization of oscillation and relaxation equations, under nonhomogeneous initial value conditions. The analytical solution is obtained in compact and elegant forms in terms of generalized Mittag-Leffler functions.

Keywords: *Bagley-Torvik equation, Basset problem, Caputo fractional derivative, Fractional decomposition method, Mittag-Leffler function.*

1 Introduction

Fractional calculus has been the subject of intensive research since its first international conference in 1974. Nowadays, it founds numerous applications in different areas of applied sciences and engineering especially by the introduction of fractional differential equations which allow a better description of nonhomogenous natural phenomena.

Whereas solving these kinds of equations is difficult by classical methods like Laplace transform method. In recent times, several new techniques including analytical decomposition [15] have been proposed to obtain analytical or approximate analytical solution of fractional differential equations.

In this paper, we adopt the Adomian's method to solve a more general 2-order fractional differential equation, the so called General Fractional Oscillation Relaxation Equation. We get its solution in terms of series of generalized Mittag-Leffler functions.

The outline of this work is as follow. We begin in Section 2 by giving some useful notions related to Fractional calculus and Adomian decomposition method (ADM). In Section 3, the fractional oscillation relaxation equation is solved by using ADM. Section 4 is devoted to numerical illustrations of the second order case such the resolution of Basset problem and Bagley-Torvik equation. Concluding remarks are given in Section 5.

2 Preliminaries

In this section, we recall some necessary results relative to Fractional calculus and to Adomian's Method.

2.1 Fractional Calculus

Let $\alpha > 0$ be an arbitrary real and $f(t)$ a sufficiently well-behaved function. Following [10], the Riemann-Liouville fractional integral of order α is defined by

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau)\, d\tau, \quad t > 0, \tag{1}$$

its left inverse, the Riemann-Liouville fractional derivative of same order in the form of

$$D^{\alpha} f(t) = D^m J^{m-\alpha} f(t) \tag{2}$$

$$= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau, & m-1 < \alpha < m \\[2ex] \dfrac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \tag{3}$$

and the Caputo fractional derivative of order α by

$$D_*^{\alpha} f(t) = J^{m-\alpha} D^m f(t) \tag{4}$$

$$= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau, & m-1 < \alpha < m \\ \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m \in \mathbb{N}. \end{cases} \tag{5}$$

For the Caputo derivative, we have the following composition rule with the Riemann-Liouville fractional integral for $\beta > \alpha > 0$ and $m-1 < \alpha \leq m$

$$J^{\beta}\left[D_*^{\alpha} f(t)\right] = J^{\beta-\alpha} f(t) - \sum_{p=0}^{m-1} f^{(p)}(0) \frac{t^{\beta-\alpha+p}}{\Gamma(\beta-\alpha+p+1)}. \tag{6}$$

These results are helpful to deal with fractional differential equations by Adomian decomposition method. The solution of such equations often involves special functions like Mittag-Leffler type functions. We would recall the definition of the Mittag-Leffler function, for $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, $\gamma \in \mathbb{C}$, $\Re(\gamma) > 0$,

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{7}$$

Wiman [16] introduced a generalization of the Mittag-Leffler function in the general form,

$$E_{\alpha,\,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{8}$$

Another generalization of (7) was proposed by Prabhakar [14] in the form,

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k\, z^k}{k!\,\Gamma(\alpha k + \beta)}, \tag{9}$$

where $\Gamma(z)$, $z \in \mathbb{C}$, is the Gamma Euler function and $(\gamma)_k$ the Pochhammer symbol defined by

$$(\gamma)_0 = 1,\ (\gamma)_k = \gamma(\gamma+1)...(\gamma+k-1) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)},\ \gamma \neq 0.$$

The generalized Mittag-Leffler function (9) is an entire function of order $\rho = [\Re(\alpha)]^{-1}$ and type $\sigma = \frac{1}{\rho}\left[\{\Re(\alpha)\}^{\Re(\alpha)}\right]^{-\rho}$. For some particular values of parameters, we have

$$E_{\alpha}(z) = E_{\alpha,1}^{1}(z),\ E_{\alpha,\beta}(z) = E_{\alpha,\beta}^{1}(z), \tag{10}$$

$$\phi(\alpha,\beta;z) = {}_1F_1(\alpha;\beta;z) = \Gamma(\beta) E_{1,\beta}^{\alpha}(z) \tag{11}$$

where $\phi(\alpha,\beta;z)$ is the Kummer's confluent hypergeometric function.

2.2 Adomian Decomposition Method

The technique presented by Adomian [4, 5, 6] consists of splitting the given equation into linear and nonlinear parts. Then the solution is decomposed in a series of functions where the nonlinear contribution is obtained in the form of "Adomian's polynomials" from its expansion into power series. It was proven that the series solution converge accurately [7, 9, 1, 2, 11].

To illustrate the method, we consider the following general nonlinear system

$$\left| \begin{array}{ccc} Lu(t)+Ru(t)+Nu(t) & = & g(t) \\ u(0) & = & u_0 \end{array} \right. \tag{12}$$

where L is the highest order derivative which assumed to be invertible, R the remaining linear part, N represents a nonlinear operator and g a well-behaved function.

Applying the inverse operator L^{-1} to both side of (12), we have

$$u(t) = f(t) - L^{-1}Ru(t) - L^{-1}Nu(t) \tag{13}$$

where $f(t) = u_0 + L^{-1}g(t)$

The next step is to introduce the series form of the general solution and of the nonlinear operator into eq.(13),

$$u = \sum_{n=0}^{\infty} u_n \quad \text{and} \quad Nu = \sum_{n=0}^{\infty} A_n \tag{14}$$

The polynomials (A_n) in $(u_1, \ldots, u_n)$ are the Adomian's polynomials generated by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} \tag{15}$$

Therefore, by identification, we obtain the successive terms of the series solution by the following recurrent relation

$$\left| \begin{array}{ccc} u_0 & = & f(t) \\ u_{n+1} & = & -L^{-1}(Ru_n) - L^{-1}(A_n) \end{array} \right. \tag{16}$$

3 General Solution

Fractional differential equations are main application of the Fractional calculus. These types of equations appear frequently in many physical and technical areas [13].

In this section, we deal with 2-order fractional differential equation by the Adomian decomposition method. These kind of equations are called general fractional oscillation relaxation equation as a fractional generalization of oscillation and relaxation equations. We apply some basic transformation and integration to obtain the solution in elegant form.

Let

$$\begin{array}{cc} \beta_i \in \mathbb{R}, & \beta_0 = 0 < \beta_1 < \beta_2 \leq 2 \\ {\scriptstyle m_i - 1 < \beta_i \leq m_i}, & m_i \in \mathbb{N},\ i = 0,\, 1,\, 2. \end{array}$$

Consider the following form of general fractional oscillation relaxation equation

$$\left| \begin{array}{rcl} \lambda_2 D_*^{\beta_2} u(t) + \lambda_1 D_*^{\beta_1} u(t) + \lambda_0 u(t) & = & f(t) \\ u^{(p)}(0^+)_{p=0,\,1,\,m_2-1} & = & k_p, \end{array} \right. \tag{17}$$

with $D_*^{\beta_i} u(t)$ denotes the Caputo fractional derivative of order β_i of the field variable $u(t)$, λ_i and k_p are real constants, $\lambda_2 \neq 0$.

For the fractional Adomian's method, we choose for the linear operator the Caputo fractional derivative of order β_2: $L = D_*^{\beta_2}$, and the inverse the Riemann-Liouville fractional integral of same order: $L^{-1} = J^{\beta_2}$.

In virtue of the composition rule (6), applying L^{-1} to both sides of the Eq.(17) leads

$$\begin{aligned} &\lambda_2 \left[u(t) - \sum_{p=0}^{m_2-1} k_p \frac{t^p}{\Gamma(p+1)} \right] + \lambda_1 \left[J^{\beta_2-\beta_1} u(t) - \right. \\ &\left. \sum_{p=0}^{m_1-1} k_p \frac{t^{\beta_2-\beta_1+p}}{\Gamma(\beta_2-\beta_i+p+1)} \right] + \lambda_0 J^{\beta_2} u(t) = J^{\beta_2} f(t). \end{aligned} \tag{18}$$

Then,

$$u(t) = \sum_{i=1}^{2} \frac{\lambda_i}{\lambda_2} \sum_{p=0}^{m_i-1} k_p \frac{t^{\beta_2-\beta_i+p}}{\Gamma(\beta_2-\beta_i+p+1)} + \frac{1}{\lambda_2} J^{\beta_2} f(t) - \sum_{i=0}^{1} \frac{\lambda_i}{\lambda_2} J^{\beta_2-\beta_i} u(t) \tag{19}$$

Setting $u(t) = \sum_{n=0}^{\infty} u_n$ and by identification, we get the iteration process

$$\left| \begin{array}{rcl} u_0 & = & \displaystyle\sum_{i=1}^{2} \frac{\lambda_i}{\lambda_2} \sum_{p=0}^{m_i-1} k_p \frac{t^{\beta_2-\beta_i+p}}{\Gamma(\beta_2-\beta_i+p+1)} + \frac{1}{\lambda_2} J^{\beta_2} f(t) \\ u_{n+1} & = & \displaystyle -\left(\frac{\lambda_1}{\lambda_2} J^{\beta_2-\beta_1} + \frac{\lambda_0}{\lambda_2} J^{\beta_2} \right) (u_n). \end{array} \right. \tag{20}$$

Thus,

$$\begin{array}{rcl} u_1 & = & \displaystyle -\left(\frac{\lambda_1}{\lambda_2} J^{\beta_2-\beta_1} + \frac{\lambda_0}{\lambda_2} J^{\beta_2} \right) (u_0) \\ u_2 & = & \displaystyle \left(\frac{\lambda_1}{\lambda_2} J^{\beta_2-\beta_1} + \frac{\lambda_0}{\lambda_2} J^{\beta_2} \right)^2 (u_0) \\ & \vdots & \\ u_q & = & \displaystyle (-1)^q \left(\frac{\lambda_1}{\lambda_2} J^{\beta_2-\beta_1} + \frac{\lambda_0}{\lambda_2} J^{\beta_2} \right)^q (u_0) \end{array} \tag{21}$$

Applying the multinomial theorem [3] yields

$$u_q = (-1)^q \sum_{\substack{r+s=q \\ r\geq 0,\, s\geq 0}} \frac{q!}{r!s!} \left(\frac{\lambda_0}{\lambda_2}\right)^r \left(\frac{\lambda_1}{\lambda_2}\right)^s J^{(\beta_2-\beta_1)s+\beta_2 r}(u_0) \tag{22}$$

Reconstituting the decomposition series, the exact solution of Eq.(17) reads

$$\begin{aligned} u(t) &= \sum_{q=0}^{\infty} u_q \\ &= \sum_{q=0}^{\infty} (-1)^q \sum_{\substack{r+s=q \\ r\geq 0,\, s\geq 0}} \frac{q!}{r!s!} \left(\frac{\lambda_0}{\lambda_2}\right)^r \left(\frac{\lambda_1}{\lambda_2}\right)^s J^{(\beta_2-\beta_1)s+\beta_2 r}(u_0) \end{aligned} \tag{23}$$

Next, we substitute the value of u_0 (20) and have

$$\begin{aligned} u(t) &= \sum_{i=1}^{2} \frac{\lambda_i}{\lambda_2} \sum_{p=0}^{m_i-1} k_p \sum_{q=0}^{\infty} (-1)^q \sum_{\substack{r+s=q \\ r\geq 0,\, s\geq 0}} \frac{q!}{r!s!} \left(\frac{\lambda_0}{\lambda_2}\right)^r \left(\frac{\lambda_1}{\lambda_2}\right)^s \frac{t^{\zeta_{i,p}}}{\Gamma(\xi_{i,p})} \\ &+ \frac{1}{\lambda_2} \sum_{q=0}^{\infty} (-1)^q \sum_{\substack{r+s=q \\ r\geq 0,\, s\geq 0}} \frac{q!}{r!s!} \left(\frac{\lambda_0}{\lambda_2}\right)^r \left(\frac{\lambda_1}{\lambda_2}\right)^s J^{\gamma+\beta_2 r} f(t) \end{aligned} \tag{24}$$

where $\gamma = (\beta_2 - \beta_1)s + \beta_2$, $\zeta_{i,p} = \gamma + \beta_2 r - \beta_i + p$ and $\xi_{i,p} = \zeta_{i,p} + 1$.

According to the definition of the Riemann-Liouville fractional integral, we can write

$$\begin{aligned} u(t) &= \sum_{i=1}^{2} \frac{\lambda_i}{\lambda_2} \Bigg\{ \sum_{p=0}^{m_i-1} k_p \sum_{s=0}^{\infty} (-1)^s \sum_{r=0}^{\infty} \frac{(r+s)!}{r!s!} \left(\frac{\lambda_1}{\lambda_2}\right)^s t^{\gamma-\beta_i+p} \times \\ &\left(-\frac{\lambda_0}{\lambda_2}\right)^r \frac{t^{\beta_2 r}}{\Gamma(\gamma+\beta_2 r-\beta_i+p+1)} \Bigg\} + \frac{1}{\lambda_2} \sum_{s=0}^{\infty} (-1)^s \sum_{r=0}^{\infty} \frac{(r+s)!}{r!s!} \\ &\left(\frac{\lambda_1}{\lambda_2}\right)^s \left(-\frac{\lambda_0}{\lambda_2}\right)^r \left\{ \frac{1}{\Gamma(\gamma+\beta_2 r)} \int_0^t (t-\tau)^{\gamma+\beta_2 r-1} f(\tau) d\tau \right\}. \end{aligned} \tag{25}$$

Therefore

$$\begin{aligned} u(t) &= \sum_{i=1}^{2} \frac{\lambda_i}{\lambda_2} \Bigg\{ \sum_{p=0}^{m_i-1} k_p \sum_{s=0}^{\infty} (-1)^s \left(\frac{\lambda_1}{\lambda_2}\right)^s t^{\gamma-\beta_i+p} \Bigg(\sum_{r=0}^{\infty} \frac{(r+s)!}{r!\, s!} \times \\ &\left[-\frac{\lambda_0}{\lambda_2}\right]^r \frac{t^{\beta_2 r}}{\Gamma(\gamma+\beta_2 r-\beta_i+p+1)} \Bigg) \Bigg\} + \frac{1}{\lambda_2} \int_0^t \sum_{s=0}^{\infty} (-1)^s \left(\frac{\lambda_1}{\lambda_2}\right)^s \\ &(t-\tau)^{\gamma-1} \left\{ \sum_{r=0}^{\infty} \frac{(r+s)!}{r!\, s!} \left[-\frac{\lambda_0}{\lambda_2}\right]^r \frac{(t-\tau)^{\beta_2 r}}{\Gamma(\gamma+\beta_2 r)} \right\} f(\tau) d\tau. \end{aligned} \tag{26}$$

So

$$
\begin{aligned}
u(t) \quad &= \sum_{i=1}^{2} \frac{\lambda_i}{\lambda_2} \left\{ \sum_{p=0}^{m_i-1} k_p \sum_{s=0}^{\infty} (-1)^s \left(\frac{\lambda_1}{\lambda_2}\right)^s t^{\gamma-\beta_i+p} E^{s+1}_{\beta_2,\,\gamma-\beta_i+p+1}\left(-\frac{\lambda_0}{\lambda_2} t^{\beta_2}\right)\right\} \\
&+ \frac{1}{\lambda_2} \int_0^t \sum_{s=0}^{\infty} (-1)^s \left(\frac{\lambda_1}{\lambda_2}\right)^s (t-\tau)^{\gamma-1} E^{s+1}_{\beta_2,\,\gamma}\left(-\frac{\lambda_0}{\lambda_2}\{t-\tau\}^{\beta_2}\right) f(\tau) d\tau \\
&= \sum_{p=0}^{m_2-1} k_p \sum_{s=0}^{\infty} \left(-\frac{\lambda_1}{\lambda_2}\right)^s t^{(\beta_2-\beta_1)s+p} E^{s+1}_{\beta_2,\,(\beta_2-\beta_1)s+p+1}\left(-\frac{\lambda_0}{\lambda_2} t^{\beta_2}\right) \\
&+ \frac{\lambda_1}{\lambda_2} \sum_{p=0}^{m_1-1} k_p \sum_{s=0}^{\infty} \left(-\frac{\lambda_1}{\lambda_2}\right)^s t^{(\beta_2-\beta_1)(s+1)+p} E^{s+1}_{\beta_2,\,(\beta_2-\beta_1)(s+1)+p+1}\left(-\frac{\lambda_0}{\lambda_2} t^{\beta_2}\right) \\
&+ \frac{1}{\lambda_2} \int_0^t \sum_{s=0}^{\infty} \left(-\frac{\lambda_1}{\lambda_2}\right)^s \tau^{(\beta_2-\beta_1)s+\beta_2-1} E^{s+1}_{\beta_2,\,(\beta_2-\beta_1)s+\beta_2}\left(-\frac{\lambda_0}{\lambda_2} \tau^{\beta_2}\right) f(t-\tau) d\tau
\end{aligned}
\tag{27}
$$

where $E^{\rho}_{\alpha,\,\beta}(z)$ is the generalized Mittag-Leffler function (9).

Finally, introducing the fractional Green's function $\mathcal{G}_2(t)$

$$
\mathcal{G}_2(t) = \frac{1}{\lambda_2} \sum_{s=0}^{\infty} \left(-\frac{\lambda_1}{\lambda_2}\right)^s t^{(\beta_2-\beta_1)s+\beta_2-1} E^{s+1}_{\beta_2,\,(\beta_2-\beta_1)s+\beta_2}\left(-\frac{\lambda_0}{\lambda_2} t^{\beta_2}\right). \tag{28}
$$

we obtain an elegant form of the analytical solution of eq.(17)

$$
u(t) = \sum_{i=1}^{2} \lambda_i \sum_{p=0}^{m_i-1} k_p J^{1+p} D^{\beta_i} \{\mathcal{G}_2(t)\} + \mathcal{G}_2 * f(t) \tag{29}
$$

with J^{α}, D^{α} and $h{*}g(t)$ denote respectively the Riemann-Liouvile fractional integral of order α, the Riemann-Liouvile fractional derivative of order α and the Laplace convolution defined by

$$
h * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau \tag{30}
$$

We summarize above results by the following theorem,

Theorem 3.1 *Let* $0 < \beta_1 < \beta_2 \leq 2$, $m_i - 1 < \beta_i \leq m_i$, $m_i \in I\!N$, λ_i *and* k_p *be real constants,* $\lambda_2 \neq 0$, $i = 0,\, 1,\, 2$.
The solution of the initial value problem for the general fractional oscillation relaxation equation

$$
\left|\begin{array}{rcl}
\lambda_2 D_*^{\beta_2} + \lambda_1 D_*^{\beta_1} u(t) + \lambda_0 u(t) & = & f(t) \\
u^{(p)}(0^+)_{p=0,\,1,\,m_2-1} & = & k_p
\end{array}\right. \tag{31}
$$

writes

$$
\begin{aligned}
u(t) &= \sum_{p=0}^{m_2-1} k_p \sum_{s=0}^{\infty} \left(-\frac{\lambda_1}{\lambda_2}\right)^s t^{(\beta_2-\beta_1)s+p} E^{s+1}_{\beta_2,\,(\beta_2-\beta_1)s+p+1}\left(-\frac{\lambda_0}{\lambda_2}t^{\beta_2}\right) \\
&+\frac{\lambda_1}{\lambda_2}\sum_{p=0}^{m_1-1} k_p \sum_{s=0}^{\infty} \left(-\frac{\lambda_1}{\lambda_2}\right)^s t^{(\beta_2-\beta_1)(s+1)+p} E^{s+1}_{\beta_2,\,(\beta_2-\beta_1)(s+1)+p+1}\left(-\frac{\lambda_0}{\lambda_2}t^{\beta_2}\right) \\
&+\frac{1}{\lambda_2}\int_0^t \sum_{s=0}^{\infty} \left(-\frac{\lambda_1}{\lambda_2}\right)^s \tau^{(\beta_2-\beta_1)s+\beta_2-1} E^{s+1}_{\beta_2,\,(\beta_2-\beta_1)s+\beta_2}\left(-\frac{\lambda_0}{\lambda_2}\tau^{\beta_2}\right) f(t-\tau)d\tau \\
&= \sum_{i=1}^{2}\lambda_i \sum_{p=0}^{m_i-1} k_p J^{1+p} D^{\beta_i}\left\{\mathcal{G}_2(t)\right\} + \mathcal{G}_2 * f(t)
\end{aligned}
\tag{32}
$$

where

$$
\mathcal{G}_2(t) = \frac{1}{\lambda_2}\sum_{s=0}^{\infty}\left(-\frac{\lambda_1}{\lambda_2}\right)^s t^{(\beta_2-\beta_1)s+\beta_2-1} E^{s+1}_{\beta_2,\,(\beta_2-\beta_1)s+\beta_2}\left(-\frac{\lambda_0}{\lambda_2}t^{\beta_2}\right). \tag{33}
$$

4 Illustrations

For the applications, we study below the Basset problem [12] and the Bagley-Torvik equation [8], two special cases of the general fractional relaxation oscillation equation.

4.1 Basset Problem

This classical problem of Fluid dynamics concerns the unsteady motion of a spherical particle accelerating in a viscous fluid under the action of the gravity. The motion is governed by the composite fractional relaxation equation [12, 10]

$$
\frac{du}{dt} + aD_*^{\alpha}u(t) + u(t) = 1 \tag{34}
$$

with $0 < \alpha < 1$, $a = \beta^{\alpha} > 0$, $\beta = \frac{9}{1+2\chi}$ and $\chi = \frac{\rho_p}{\rho_f}$. β and χ are related to the densities ρ_f, ρ_p of the fluid and particle.

For the application of Adomian's method, we consider the generalized Basset problem where $\alpha = \frac{3}{4}$, $\chi = \frac{3}{4}$, $u(0) = 1$ namely

$$
\frac{du}{dt} + \left[\frac{9}{2}\right]^{\frac{3}{4}} D_*^{\frac{3}{4}}u(t) + u(t) = 1 \tag{35}
$$

By the theorem 3.1, we have

$$\begin{aligned} u(t) &= \left[\frac{9}{2}\right]^{\frac{3}{4}} \sum_{s=0}^{\infty} \left[-\frac{9}{2}\right]^{\frac{3}{4}s} t^{\frac{1}{4}(s+1)} E^{s+1}_{1,\, \frac{1}{4}(s+1)+1}(-t) + \\ &\sum_{s=0}^{\infty} \left[-\frac{9}{2}\right]^{\frac{3}{4}s} t^{\frac{1}{4}s} E^{s+1}_{1,\, \frac{1}{4}s+1}(-t) + \int_0^t \sum_{s=0}^{\infty} \left[-\frac{9}{2}\right]^{\frac{3}{4}s} \tau^{\frac{1}{4}s} E^{s+1}_{1,\, \frac{1}{4}s+1}(-\tau) d\tau. \end{aligned}$$

After simple calculations, we get

$$\begin{aligned} u(t) &= \sum_{s=0}^{\infty} \left[-\frac{9}{2}\right]^{\frac{3}{4}s} t^{\frac{1}{4}s} E^{s+1}_{1,\, \frac{1}{4}s+1}(-t) + \left[\frac{9}{2}\right]^{\frac{3}{4}} t^{\frac{1}{4}} \sum_{s=0}^{\infty} \left[-\frac{9}{2}\right]^{\frac{3}{4}s} t^{\frac{1}{4}s} E^{s+1}_{1,\, \frac{1}{4}s+\frac{5}{4}}(-t) \\ &+ t \sum_{s=0}^{\infty} \left[-\frac{9}{2}\right]^{\frac{3}{4}s} t^{\frac{1}{4}s} E^{s+1}_{1,\, \frac{1}{4}s+2}(-t). \end{aligned} \tag{36}$$

4.2 Bagley-Torvik Equation

The Bagley-Torvik equation arises in the modelling of the motion of a rigid plate immersed in a Newtonian fluid [8]. It is a composite fractional oscillation equation [10]

$$\lambda_2 \frac{d^2u}{dt^2} + \lambda_1 D_*^{\frac{3}{2}} u(t) + \lambda_0 u(t) = f(t). \tag{37}$$

For numerical application, we set $\lambda_2 = \lambda_1 = \lambda_0 = 1$, $u(0) = u'(0) = 1$ and $f(t) = 1 + t$ namely

$$\frac{d^2u}{dt^2} + D_*^{\frac{3}{2}} u(t) + u(t) = 1 + t. \tag{38}$$

Applying the theorem 3.1 holds

$$\begin{aligned} u(t) &= \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}(s+1)} E^{s+1}_{2,\, \frac{1}{2}(s+1)+1}(-t^2) + t^{\frac{3}{2}} \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}(s+1)+2}(-t^2) \\ &+ \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}s+1}(-t^2) + t \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}s+2}(-t^2) \\ &+ \int_0^t \sum_{s=0}^{\infty} (-1)^s \tau^{\frac{1}{2}s+1} E^{s+1}_{2,\, \frac{1}{2}s+2}(-\tau^2)(1+t-\tau) d\tau, \end{aligned}$$

hence,

$$\begin{aligned} u(t) &= \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}s+1}(-t^2) + t^{\frac{1}{2}} \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}s+\frac{3}{2}}(-t^2) \\ &+ t \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}s+2}(-t^2) + t^{\frac{3}{2}} \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}s+\frac{5}{2}}(-t^2) \\ &+ t^2 \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}s+3}(-t^2) + t^3 \sum_{s=0}^{\infty} (-1)^s t^{\frac{1}{2}s} E^{s+1}_{2,\, \frac{1}{2}s+4}(-t^2). \end{aligned} \tag{39}$$

5 Conclusion

In this paper, we have applied the so-called Adomian's method for solving the general fractional oscillation relaxation equations. We get the same general exact solution as the Laplace transform technique in terms of generalized Mittag-Leffler functions. All results prove the effectiveness of the Adomian decomposition method to deal with fractional differential equations.

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