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Arens Regularity of Module Extensions of Banach Algebras

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Abstract

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}''$ be its second dual equipped with the first or second Arens product, X be a Banach $\mathcal{A}$-bimodule and let $\mathcal{U}=\mathcal{A}\oplus X$ as a module extensions of Banach algebra. In this paper we study the topological centres of $\mathcal{U}''$ and show that under certain conditions Arens regularity of $\mathcal{A}$ implies that of $\mathcal{U}$.

Keywords: *Arens regular, Module action, Topological centres, Bounded bilinear map.*

1 Introduction

In 1951, Arens showed that each bounded bilinear map m on normed spaces has two natural but different extensions [1]. When these extensions coincide, m is said to be Arens regular. If the product of a Banach algebra $\mathcal{A}$ enjoys this property, then $\mathcal{A}$ is called Arens regular.

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. Then $\mathcal{U}=\mathcal{A}\oplus X$, with norm $\|(a,x)\|=\|a\|+\|x\|$, and product

$$(a,x)(b,y)=(ab, a\cdot y+x\cdot b) \quad (a,b\in\mathcal{A},\ x,y\in X),$$

is a Banach algebra which is known as a module extension Banach algebra. Some aspects of algebras of this form have been discussed in [6] and [7].

In this paper we study Arens regularity of this class of Banach algebras for the special case $X = \mathcal{A}'$. We give a criterion for certain bounded bilinear map to be Arens regular (Theorem 3.1 below), and then we apply the above criterion to show that Arens regularity of $\mathcal{A}$ implies that of $\mathcal{U}$ with special hypothesis. Moreover, we present some properties of Banach algebra $\mathcal{A}$ that are inherited by module extensions.

Throughout the paper we identify an element of a Banach space X with its canonical image in X''.

The second dual space $\mathcal{A}''$ of a Banach algebra $\mathcal{A}$ admits two Banach algebra products known as the first and second Arens products, each extending the product on $\mathcal{A}$. These products which we denote by $\Box$ and $\Diamond$, respectively, can be defined as follows

$$\Phi\Box\Psi = w^* - \lim_i \lim_j a_i b_j, \quad \Phi\Diamond\Psi = w^* - \lim_j \lim_i a_i b_j,$$

where (a_i) and (b_j) are nets in $\mathcal{A}$ that converge, in w^*-topologies, to Φ and Ψ, respectively. The Banach algebra $\mathcal{A}$ is said to be Arens regular if the product map $\pi : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ is regular in the sense of [1], or $\Phi\Box\Psi = \Phi\Diamond\Psi$ on the whole of $\mathcal{A}''$. For any fixed $\Phi \in \mathcal{A}''$, the maps $\Psi \longmapsto \Psi\Box\Phi$ and $\Psi \longmapsto \Phi\Diamond\Psi$ are w^*-w^* continuous on $\mathcal{A}''$. Thus, with the w^*-topology, $(\mathcal{A}'', \Box)$ is a right topological semigroup and $(\mathcal{A}'', \Diamond)$ is a left topological semigroup. The following sets

$$Z_t^1(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \longmapsto \Phi\Box\Psi \text{ is } w^* - w^* \text{ continuous on } \mathcal{A}''\},$$

$$Z_t^2(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \longmapsto \Psi\Diamond\Phi \text{ is } w^* - w^* \text{ continuous on } \mathcal{A}''\},$$

are called the first and the second topological centres of $\mathcal{A}''$, respectively. It is easy to check that $\mathcal{A}$ is Arens regular if and only if $Z_t^1(\mathcal{A}'') = Z_t^2(\mathcal{A}'') = \mathcal{A}''$. For example, each C^*-algebra is Arens regular and for locally compact group G, the group algebra $L^1(G)$ is Arens regular if and only if G is finite [12]. For more information on Arens product and topological centres, we refer the reader to [3] and [5].

A bounded net $(e_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ is a bounded approximate identity (BAI for short) if, for each $a \in \mathcal{A}$, $ae_\alpha \longrightarrow a$ and $e_\alpha a \longrightarrow a$. An element $\Phi_0 \in \mathcal{A}''$ is called mixed unit if it is a right unit for $(\mathcal{A}'', \Box)$ and a left unit for $(\mathcal{A}'', \Diamond)$. It is well-known that an element $\Phi_0 \in \mathcal{A}''$ is a mixed unit if and only if it is a weak* cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ [2]. We denote by WAP$(\mathcal{A})$ the closed subspace of $\mathcal{A}'$ consisting of all the weakly almost periodic functionals in $\mathcal{A}'$ [5].

2 Topological Centres of Module Extensions

Suppose that X is a Banach $\mathcal{A}$-bimodule with the left and right module actions $\pi_1 : \mathcal{A} \times X \longrightarrow X$ and $\pi_2 : X \times \mathcal{A} \longrightarrow X$, respectively. According to [4], X'' is a Banach $\mathcal{A}''$-bimodule, where $\mathcal{A}''$ equipped with the first Arens product. The module actions are defined by

$$\Phi \cdot \nu = w^* - \lim_i \lim_j \widehat{a_i \cdot x_j}, \quad \nu \cdot \Phi = w^* - \lim_j \lim_i \widehat{x_j \cdot a_i},$$

where (a_i) and (x_j) are nets in $\mathcal{A}$ and X that converge, in w^*-topologies, to Φ and ν, respectively.

The second dual $\mathcal{U}''$ of $\mathcal{U} = \mathcal{A} \oplus X$ is identified with $\mathcal{A}'' \oplus X''$, as a Banach space. Also the first Arens product $\Box$ on $\mathcal{U}''$ is given by

$$(\Phi, \mu)\Box(\Psi, \nu) = (\Phi\Box\Psi, \Phi \cdot \nu + \mu \cdot \Psi),$$

where $\Phi\Box\Psi$ is as usual the first Arens product of Φ and Ψ in $\mathcal{A}''$. An easy argument shows that the first topological centre $Z_t^1(\mathcal{U}'')$ of $\mathcal{U}''$ consists of the elements of the form $(\Phi, \mu) \in \mathcal{U}''$ such that:
a) $\Phi \in Z_t^1(\mathcal{A}'')$;
b) $\nu \longmapsto \Phi \cdot \nu : X'' \longrightarrow X''$ is w^*-w^* continuous;
c) $\Psi \longmapsto \mu \cdot \Psi : \mathcal{A}'' \longrightarrow X''$ is w^*-w^* continuous; (see [6], [7]).

In a similar way X'' can be made into an $(\mathcal{A}'', \diamondsuit)$-bimodule. We denote this module action by the symbol "$\bullet$". Thus, the Banach algebra $\mathcal{U}$ is Arens regular if and only if π_1, π_2 and π are regular, or equivalently, $\mathcal{A}$ is Arens regular and

$$\Phi \cdot \nu = \Phi \bullet \nu, \quad \nu \cdot \Phi = \nu \bullet \Phi \qquad (\Phi \in \mathcal{A}'',\ \nu \in X'').$$

3 Main Results

Let $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}'$. Then clearly Arens regularity of $\mathcal{U}$ implies that of $\mathcal{A}$, but the converse is not true in general, even if $\mathcal{A}$ is commutative. For example, let $\mathcal{A} = c_0$, the sequence space of the sequences that converges to zero. Then $\mathcal{A}$ is commutative and Arens regular under pointwise multiplication. Since $\mathcal{A}$ is not reflexive, we can find two bounded sequences (x_n) in $\mathcal{A}$ and (f_m) in $\mathcal{A}'$ such that

$$\lim_n \lim_m \langle f_m, x_n \rangle = 1, \qquad \lim_m \lim_n \langle f_m, x_n \rangle = 0.$$

Now let $a_n = (x_n, 0)$, $b_m = (0, f_m)$ in $\mathcal{U}$. Then $a_n b_m = (0, x_n \cdot f_m)$. Suppose that $\lambda = (0, 1)$ in $\mathcal{U}'$=$\mathcal{A}' \times \mathcal{A}''$, where $1 = (1, 1, ..., 1, ...)$ is the unit element of $\mathcal{A}''$. Then we have

$$\langle \lambda, a_n b_m \rangle = \langle 0, 0 \rangle + \langle 1 \cdot x_n, f_m \rangle = \langle f_m, x_n \rangle.$$

It follows that

$$\lim_n \lim_m \langle \lambda, a_n b_m \rangle = 1, \quad \lim_m \lim_n \langle \lambda, a_n b_m \rangle = 0.$$

Thus, $\lim_n \lim_m \langle \lambda, a_n b_m\rangle \neq \lim_m \lim_n \langle \lambda, a_n b_m\rangle$ and so $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}'$ dose not Arens regular by the double limit theorem [11].

The next result, which is the main one in the paper, provides a criterion for Arens regularity of π_1 and π_2.

Theorem 3.1 *Let $\mathcal{A}$ be a Banach algebra.*
(i) If $R_\Psi : \mathcal{A}'' \longrightarrow \mathcal{A}''$ $(\Phi \longmapsto \Phi \Box \Psi)$ is w^*-w continuous for all $\Psi \in \mathcal{A}''$, then π_2 is regular. In particular, if $\mathcal{A}''\Box\mathcal{A}'' \subseteq \mathcal{A}$, then π_2 is regular.
(ii) If $L_\Psi : \mathcal{A}'' \longrightarrow \mathcal{A}''$ $(\Phi \longmapsto \Psi \Diamond \Phi)$ is w^*-w continuous for all $\Psi \in \mathcal{A}''$, then π_1 is regular. In particular, if $\mathcal{A}''\Diamond\mathcal{A}'' \subseteq \mathcal{A}$, then π_1 is regular.

Proof: We only prove (i).
Let (a_i) and (f_j) be a nets in $\mathcal{A}$ and $\mathcal{A}'$ such that $a_i \longrightarrow \Phi$ and $f_j \longrightarrow \mu$ in w^*-topology. As R_Ψ is w^*-w continuous, we have that $a_i \Box \Psi \longrightarrow \Phi \Box \Psi$ in the weak topology. Thus,

$$\begin{aligned}
\langle \mu \bullet \Phi, \Psi\rangle &= \lim_i \lim_j \langle \Psi, f_j \cdot a_i\rangle \\
&= \lim_i \lim_j \ \langle \widehat{a_i} \cdot \Psi, f_j\rangle \\
&= \lim_i \ \langle \mu, a_i \cdot \Psi\rangle = \langle \mu, \Phi \Box \Psi\rangle \\
&= \lim_j \langle \Phi \Box \Psi, f_j\rangle \\
&= \lim_j \lim_i \ \langle \widehat{a_i} \cdot \Psi, f_j\rangle \\
&= \lim_j \lim_i \langle \Psi, f_j \cdot a_i\rangle \\
&= \langle \mu \cdot \Phi, \Psi\rangle.
\end{aligned}$$

Therefore $\mu \bullet \Phi = \mu \cdot \Phi$ and so $\pi_2 : \mathcal{A}' \times \mathcal{A} \longrightarrow \mathcal{A}'$ is regular. Now assume that $\mathcal{A}''\Box\mathcal{A}'' \subseteq \mathcal{A}$ and let $\Phi_\alpha \longrightarrow \Phi$ in w^*-topology. Since $\mathcal{A}''' = \mathcal{A}' \oplus \mathcal{A}^\perp$ [3], where $\mathcal{A}^\perp$ is the annihilator of $\mathcal{A}$, so for each $\mu \in \mathcal{A}'''$ there exist $f \in \mathcal{A}'$ and $\rho \in \mathcal{A}^\perp$ such that $\mu = \widehat{f} + \rho$. This shows that $\Phi_\alpha \Box \Psi \longrightarrow \Phi \Box \Psi$ in weak topology of $\mathcal{A}''$. Thus, R_Ψ is w^*-w continuous and so the result follows.

As an consequence of this theorem we have the following result.

Corollary 3.2 *Let $\mathcal{A}$ be an Arens regular Banach algebra and $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}'$. Then the following assertions hold.*
(i) If $\mathcal{A}''\Box\mathcal{A}'' \subseteq \mathcal{A}$, then $\mathcal{U}$ is Arens regular and $\mathcal{U}''\Box\mathcal{U}'' \subseteq \mathcal{U}$.
(ii) If $\mathcal{A}$ is commutative and $R_\Psi : \mathcal{A}'' \longrightarrow \mathcal{A}''$ $(\Phi \longmapsto \Phi \Box \Psi)$ is w^*-w continuous, then $\mathcal{U}$ is Arens regular.

Example 3.3 *Let $\mathcal{A} = \ell^1$ with pointwise product. Then $\mathcal{A}$ is an Arens regular Banach algebra which is not reflexive, but $\mathcal{A}''\Box\mathcal{A}'' \subseteq \mathcal{A}$ by example 4.1 of [5]. Therefore by corollary* 3.2, *$\mathcal{U}$ is Arens regular and $\mathcal{U}''\Box\mathcal{U}'' \subseteq \mathcal{U}$.*

We recall that $\mathcal{A}$ is called weakly sequentially complete, (WSC for short) if every weakly Cauchy sequence in $\mathcal{A}$ is weakly convergent.

Remark 3.4 (i) *Let $\mathcal{A}$ be a nonunital Banach algebra with a BAI. Then $\mathcal{A}$ cannot be both WSC and Arens regular [10]. A similar fact is valid for $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}'$, since $\mathcal{U}$ contains $\mathcal{A}$ as a closed subalgebra.*
(ii) *Suppose that the linear mappings $\varphi : \mathcal{A} \longrightarrow \mathcal{A}''$ $(a \longmapsto \Phi \cdot a)$ and $\psi : \mathcal{A}' \longrightarrow \mathcal{A}'$ $(f \longmapsto \Phi \cdot f)$ are weakly compact for each $\Phi \in \mathcal{A}''$. Then both of π_1 and π_2 are regular by theorem* 2.1 *of [9]. Therefore in this case, Arens regularity of $\mathcal{A}$ implies that of $\mathcal{U}$.*

The converse of theorem 3.1 is not true in general. Indeed, let $\mathcal{A}$ be a non-reflexive Banach space and let φ be a non-zero element of $\mathcal{A}'$ such that $\|\varphi\| \leq 1$. Then the product $a \cdot b = \varphi(a)b$ turns $\mathcal{A}$ into a regular Banach algebra for which $\pi_2 : \mathcal{A}' \times \mathcal{A} \longrightarrow \mathcal{A}'$ is regular but $\pi_1 : \mathcal{A} \times \mathcal{A}' \longrightarrow \mathcal{A}'$ is not regular [6]. Therefore the inclusion $\mathcal{A}''\Box\mathcal{A}''(= \mathcal{A}''\Diamond\mathcal{A}'') \subseteq \mathcal{A}$ is not valid by theorem 3.1.

Proposition 3.5 *Let $\mathcal{A}$ be a Banach algebra which is a right ideal in $\mathcal{A}''$. If the right module action of $\mathcal{A}$ on $\mathcal{A}'$ is regular, then $\mathcal{A}''\Box\mathcal{A}'' \subseteq \mathcal{A}$.*

Proof: Assume that (a_i) be a net in $\mathcal{A}$ such that $a_i \longrightarrow \Phi$ in w^*-topology. Then Arens regularity of the right module action of $\mathcal{A}$ on $\mathcal{A}'$ implies that for all $\Psi \in \mathcal{A}''$, $a_i\Box\Psi \longrightarrow \Phi\Box\Psi$ in the weak topology. Since $\mathcal{A}$ is a right ideal in $\mathcal{A}''$, it follows that $\Phi\Box\Psi \in \mathcal{A}$. Thus, $\mathcal{A}''\Box\mathcal{A}'' \subseteq \mathcal{A}$.

If $\mathcal{A}$ is an ideal in $\mathcal{A}''$ then $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}'$ is not an ideal in $\mathcal{U}''$, in general. For example let $\mathcal{A}$ be the group algebra of a compact group G. Then $\mathcal{A}$ is an ideal in $\mathcal{A}''$, as is well-known. However $\mathcal{U}$ is not an ideal in $\mathcal{U}''$. The next result deals with this question that when $\mathcal{U}$ is an ideal in $\mathcal{U}''$.

Proposition 3.6 *Suppose that $\mathcal{A}$ is an Arens regular Banach algebra which is an ideal in $\mathcal{A}''$. Then $\mathcal{U}$ is an ideal in $\mathcal{U}''$.*

Proof: Assume that $\mathcal{A}$ is an ideal in $\mathcal{A}''$, $(a, f) \in \mathcal{U}$ and $(\Phi, \mu) \in \mathcal{U}''$. Since $\mathcal{A}''' = \mathcal{A}' \oplus \mathcal{A}^{\perp}$, so there exists $g \in \mathcal{A}'$ and $\rho \in \mathcal{A}^{\perp}$ such that $\mu = \widehat{g} + \rho$. Then by hypotheses $\rho \cdot \widehat{a} = 0$, therefore $\mu \cdot \widehat{a} = \widehat{g} \cdot \widehat{a}$. It follows that $\mu \cdot \widehat{a}$ is w^*-continuous linear functional on $\mathcal{A}''$, and so $\mu \cdot \widehat{a} \in \mathcal{A}'$. One can verify that $\Phi \cdot \widehat{f} \in \mathcal{A}'$ because $\mathcal{A}$ is Arens regular. Thus, $\Phi \cdot \widehat{f} + \mu \cdot \widehat{a} \in \mathcal{A}'$. Therefore by definition we have $(\Phi, \mu)\Box(a, f) \in \mathcal{U}$, so $\mathcal{U}$ is a left ideal in $\mathcal{U}''$. Similarly, $\mathcal{U}$ is a right ideal in $\mathcal{U}''$.

Let $\mathcal{A}$ be a Banach algebra with a BAI. We say that $\mathcal{A}'$ factors on the left (right) if $\mathcal{A}' = \mathcal{A}' \cdot \mathcal{A}$ $\big(\mathcal{A}' = \mathcal{A} \cdot \mathcal{A}'\big)$, and factors if both equalities $\mathcal{A} \cdot \mathcal{A}' = \mathcal{A}' = \mathcal{A}' \cdot \mathcal{A}$ hold [8]. It is well-known that if $\mathcal{A}$ is Arens regular, then $\mathcal{A}'$ factors and so $\mathcal{U}$=$\mathcal{A} \oplus \mathcal{A}'$ has a BAI [7].

As an immediate corollary of above proposition and Theorem 3.1 of [11] we have the following.

Corollary 3.7 *Let $\mathcal{A}$ be an Arens regular Banach algebra with a BAI and let $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}'$. If $\mathcal{A}$ is an ideal in $\mathcal{A}''$, then $\mathcal{U}' \cdot \mathcal{U} = WAP(\mathcal{U}) = \mathcal{U} \cdot \mathcal{U}'$.*

Let $\mathcal{A}$ be a Banach algebra with a BAI and $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}'$. If $\mathcal{U}$ is Arens regular, then $\mathcal{U}'$ factors. However, the converse is false in general. Indeed, let $\mathcal{A}$ be the group algebra of the discrete group $\mathbb{Z}$. Then $\mathcal{U}$ being unital and so $\mathcal{U}'$ factors, but $\mathcal{U}$ does not Arens regular.

Theorem 3.8 *Let $\mathcal{A}$ be an Arens regular Banach algebra with a BAI. If $\mathcal{U}$ is a right $(left)$ ideal in $\mathcal{U}''$, and $\mathcal{U}'$ factors on the left $(right)$, then $\mathcal{U}$ is Arens regular.*

Proof: Assume that (Φ, μ), (Ψ, ν), $(\Lambda, \rho) \in \mathcal{U}''$ and let (a_i, f_j) be a net in $\mathcal{U}$ such that $(a_i, f_j) \longrightarrow (\Lambda, \rho)$ in w^*-topology. Since $\mathcal{U}$ is a right ideal in $\mathcal{U}''$, we have

$$\begin{aligned}(a_i, f_j)\Box\big((\Phi, \mu)\Box(\Psi, \nu)\big) &= \big((a_i, f_j)\Box(\Phi, \mu)\big)\Box(\Psi, \nu) \\ &= \big((a_i, f_j)\Diamond(\Phi, \mu)\big)\Diamond(\Psi, \nu) \\ &= (a_i, f_j)\Diamond\big((\Phi, \mu)\Diamond(\Psi, \nu)\big) \\ &= (a_i, f_j)\Box\big((\Phi, \mu)\Diamond(\Psi, \nu)\big).\end{aligned}$$

It follows that $(\Lambda, \rho)\Box\big((\Phi, \mu)\Box(\Psi, \nu)\big) = (\Lambda, \rho)\Box\big((\Phi, \mu)\Diamond(\Psi, \nu)\big)$ for all (Λ, ρ) in $\mathcal{U}''$. Now since $\mathcal{U}'$ factors on the left, $(\mathcal{U}'', \Box)$ has an identity $(\Phi_0, 0)$ where Φ_0 is a unit element of $(\mathcal{A}'', \Box)$. Take $(\Lambda, \rho) = (\Phi_0, 0)$, therefore we have $(\Phi, \mu)\Box(\Psi, \nu) = (\Phi, \mu)\Diamond(\Psi, \nu)$, as well.

Example 3.9 *Let $\mathcal{A} = c_0$, with pointwise product. Then $\mathcal{A}$ is Arens regular Banach algebra with a BAI. Since $\mathcal{A}$ is an ideal in $\mathcal{A}''$, therefore $\mathcal{U}$ is an ideal in $\mathcal{U}''$ by proposition 3.6. Now since $\mathcal{U}$ is not Arens regular, $\mathcal{U}'$ does not factors on the left or right by above result. Note that by corollary 3.7, we have $\mathcal{U}' \cdot \mathcal{U} = WAP(\mathcal{U}) = \mathcal{U} \cdot \mathcal{U}'$.*

Now let $\mathcal{A} = L^1(G)$ for locally compact group G and $\mathcal{U} = \mathcal{A} \oplus \mathcal{A}'$. It is easy to see that if $\mathcal{A}'$ factors, then $\mathcal{A}$ is unital (i.e. G is discrete) and so $\mathcal{U}$ is unital. Hence $\mathcal{U}'$ factors. Thus, we have the next result which its proof is immediate by Theorem 2.6 of [8].

Proposition 3.10 *Let $\mathcal{A}$ be a WSC Banach algebra with a sequential BAI. Then $\mathcal{A}'$ factors if and only if $\mathcal{U}'$ factors.*

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