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# A Non-Uniform Bound on the Poisson-Negative Binomial Relative Error

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**Abstract**

*The Stein-Chen method is used to determine a non-uniform bound for the relative error of the negative binomial cumulative distribution function with parameters $n$ and $p$ and the Poisson cumulative distribution functions with mean $nq = n(1-p)$. In view of this bound, it is indicated that the Poisson cumulative distribution function with this mean can be used as an estimate of the negative binomial cumulative distribution function when $q$ is sufficiently small.*

**Keywords:** *Cumulative distribution function, negative binomial distribution, Poisson distribution, non-uniform bound, relative error, Stein-Chen method.*

## 1 Introduction

It is well-known that the negative binomial distribution with parameters $n > 0$ and $p \in (0, 1)$ is a discrete distribution with a long history and is widely used in many areas of probability and statistics as same as other important discrete distributions. Its applications appear in fields such as automobile insurance, inventory analysis, telecommunications networks analysis and population genetics. For $n \in \mathbb{N}$, it is called the Pascal distribution, which can be thought of as the distribution of the number of failures before the number of successes reaches a fixed integer $n$ in a sequence of independent Bernoulli trials, where success occurs on each trial with a probability of $p$ and failure occurs on each trial with a probability of $q = 1 - p$. A special case, $n = 1$, it is referred to as the geometric distribution with parameter $p$, which models the number of failures before the first success. Let $X$ be the negative binomial random variable with parameters $n$ and $p$, then its probability distribution function is of the form

$$p_X(x) = \frac{\Gamma(n+x)}{\Gamma(n)x!}q^x p^n, \; x = 0, 1, ..., \tag{1}$$

and the mean and variance of $X$ are $E(X) = \frac{nq}{p}$ and variance $Var(X) = \frac{nq}{p^2}$, respectively. Under parametrization, $\lambda = \frac{nq}{p}$ and $p = \frac{n}{n+\lambda}$, (1) becomes

$$p_X(x) = \frac{\lambda^x}{x!}\frac{\Gamma(n+x)}{\Gamma(n)(n+\lambda)^x}\left(\frac{1}{1+\frac{\lambda}{n}}\right)^n, \; x = 0, 1, .... \tag{2}$$

It is a well-known result that if $n \to \infty$ and $q \to 0$ while $\lambda = \frac{nq}{p}$ remains fixed, then $p_X(x) \to \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x \in \mathbb{N} \cup \{0\}$, that is, the negative binomial distribution with parameters $n$ and $p$ converges to the Poisson distribution with mean $\lambda$. With this Poisson mean, some authors have tried to approximate the negative binomial distribution by the Poisson distribution in terms of the total variation distance together with its uniform bound, which can be found in Vervaat [12], Romanowska [6], Gerber [3], Pfeifer [5], Majsnerowska [4], and Roos [7]. Let us consider the probability distribution function (1), by setting $\lambda = nq$ and $p = \frac{n-\lambda}{n}$, it can be expressed as

$$p_X(x) = \frac{\lambda^x}{x!}\frac{\Gamma(n+x)}{\Gamma(n)n^x}\left(1 - \frac{\lambda}{n}\right)^n, \; x = 0, 1, .... \tag{3}$$

Observe that if $n \to \infty$ and $q \to 0$ while $\lambda = nq$ remains fixed, then $p_X(x) \to \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x \in \mathbb{N} \cup \{0\}$. Therefore, the negative binomial distribution with parameters $n$ and $p$ also converges to the Poisson distribution with mean $\lambda = nq$ when $n$ is large and $q$ is small. Therefore, the Poisson distribution with parameter $\lambda = nq$ can be used as an estimate of the negative binomial distribution with parameters $n$ and $p$ if $n$ is sufficiently large and $q$ is sufficiently small. In this case, Teerapabolarn [10] determined a non uniform bound for the difference of the negative binomial and Poisson cumulative distribution functions in the form of

$$-(e^{\lambda} - 1)\min\left\{1, \frac{1}{p(x_0+1)}q\right\} \leq \mathbb{NB}_{n,p}(x_0) - \mathbb{P}_{\lambda}(x_0) \leq 0, \tag{4}$$

where $\mathbb{NB}_{n,p}(x_0) = \sum_{j=0}^{x_0} \frac{\Gamma(n+j)}{\Gamma(n)j!} q^j p^n$ and $\mathbb{P}_\lambda(x_0) = \sum_{j=0}^{x_0} \frac{e^{-\lambda}\lambda^j}{j!}$ are the negative binomial and Poisson cumulative distribution functions at $x_0 \in \mathbb{N} \cup \{0\}$. Subsequently, Teerapabolarn [11] determined the least upper bound

$$\sup_{x_0 \geq 0} \left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{NB}_{n,p}(x_0)} - 1 \right| = e^{-\lambda} p^{-n} - 1 \tag{5}$$

for the relative error between the negative binomial and Poisson cumulative distribution functions. This bound is a good criteria for measuring the accuracy of the approximation; however, it is a uniform bound. So, our interesting is to determine a good bound for each $x_0 \in \mathbb{N} \cup \{0\}$. In this study, we focus on determining a non-uniform bound for $\left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{NB}_{n,p}(x_0)} - 1 \right|$.

The Stein-Chen method is utilized to provide all results in the present study as mentioned in Section 2. In Section 3, we use the Stein-Chen method to yield a non-uniform bound for the approximation. In Section 4, we give some examples to illustrate the results obtained. Concluding remarks are presented in the last section.

# 2 Method

The classical Stein's method was first introduced by Stein [8]. It is a tool for approximating the distribution of random elements. His original work was applied to the central limit theorem for sums of dependent random variables. The Stein-Chen method as applied to the Poisson case was first developed by Chen [2] and is also used to determine the main result presented in this study.

Consider Stein's equation for the Poisson distribution with parameter $\lambda > 0$, for given $h$, of the form

$$h(x) - \mathcal{P}_\lambda(h) = \lambda f(x+1) - x f(x), \tag{6}$$

where $\mathcal{P}_\lambda(h) = e^{-\lambda} \sum_{l=0}^{\infty} h(l) \frac{\lambda^l}{l!}$ and $f$ and $h$ are bounded real-valued functions defined on $\mathbb{N} \cup \{0\}$.

For $x_0 \in \mathbb{N} \cup \{0\}$, let function $h_{x_0} : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$h_{x_0}(x) = \begin{cases} 1 & \text{if } \ x \leq x_0, \\ 0 & \text{if } \ x > x_0. \end{cases}$$

Putting $h = h_{x_0}$ for $x_0 \in \mathbb{N} \cup \{0\}$, (6) becomes

$$h_{x_0}(x) - \mathbb{P}(x_0; \lambda) = \lambda f_{x_0}(x+1) - x f_{x_0}(x), \tag{7}$$

where $x \in \mathbb{N} \cup \{0\}$.
Following Barbour et al. [1], the solution $f$ of (7) can be expressed in the form

$$f_{x_0}(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[\mathbb{P}_\lambda(x-1)[1-\mathbb{P}_\lambda(x_0)]] & \text{if } x \le x_0, \\ (x-1)!\lambda^{-x}e^{\lambda}[\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(x-1)]] & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \tag{8}$$

It should be noted that $f_{x_0}(x) \ge 0$ for every $x \in \mathbb{N} \cup \{0\}$. The following lemma yields a non-uniform bound of (8) that is used to determine the main result.

**Lemma 2.1.** *For* $x_0 \in \mathbb{N} \cup \{0\}$, let $p_\lambda(x_0) = \frac{e^{-\lambda}\lambda^{x_0}}{x_0!}$. *Then the following inequality holds:*

$$\sup_{x \ge 2} f_{x_0}(x) \le \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))}{p_\lambda(x_0+1)(x_0+1)}. \tag{9}$$

**Proof.** It follows from Teerapabolarn [9] that $f_{x_0}$ is an increasing function for $x \le x_0$ and a decreasing function for $x > x_0$. Therefore, we have $f_{x_0}(x) \le f_{x_0}(x_0)$ for $x \le x_0$ and $f_{x_0}(x) \le f_{x_0}(x_0+1)$ for $x > x_0$. Because

$$\begin{aligned} f_{x_0}(x_0+1) - f_{x_0}(x_0) &= (x_0-1)!\lambda^{-(x_0+1)}e^{\lambda}(1-\mathbb{P}_\lambda(x_0))\sum_{k=0}^{x_0}(x_0-k)\frac{\lambda^k}{k!} \\ &> 0, \end{aligned}$$

we obtain $f_{x_0}(x) \le f_{x_0}(x_0+1)$ for every $x \in \mathbb{N}$. Therefore, by (8), we have

$$\begin{aligned} f_{x_0}(x_0+1) &= x_0!\lambda^{-(x_0+1)}e^{\lambda}\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0)) \\ &= \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))}{p_\lambda(x_0+1)(x_0+1)}. \end{aligned}$$

Hence, the inequality (9) holds. □

**Lemma 2.2.** *Let* $x_0 \in \mathbb{N} \cup \{0\}$ *and* $\lambda = nq$, *then we have the following:*

$$\mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) \le \left(e^{-\lambda}p^{-n} - 1\right)\mathbb{NB}_{n,p}(x_0) \tag{10}$$

*and*

$$\mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) \le \left(1 - e^{\lambda}p^{n}\right)\mathbb{P}_\lambda(x_0). \tag{11}$$

**Proof.** It is clear that (10) and (11) hold for $x_0 = 0$. Next, we shall show that the two inequalities hold for $x_0 \in \mathbb{N}$.

$$\mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) = e^{-\lambda}\sum_{k=0}^{x_0}\frac{\lambda^k}{k!} - p^n\sum_{k=0}^{x_0}\frac{\lambda^k}{k!}\left(\frac{n+k-1}{n}\cdots\frac{n}{n}\right)$$

$$
\begin{aligned}
&\leq e^{-\lambda}\sum_{k=0}^{x_0}\frac{\lambda^k}{k!}\left(\frac{n+k-1}{n}\cdots\frac{n}{n}\right)-p^n\sum_{k=0}^{x_0}\frac{\lambda^k}{k!}\left(\frac{n+k-1}{n}\cdots\frac{n}{n}\right)\\
&=(e^{-\lambda}-p^n)\sum_{k=0}^{x_0}\frac{\lambda^k}{k!}\left(\frac{n+k-1}{n}\cdots\frac{n}{n}\right)\\
&=\frac{(e^{-\lambda}-p^n)}{p^n}p^n\sum_{k=0}^{x_0}\frac{(nq)^k}{k!}\left(\frac{n+k-1}{n}\cdots\frac{n}{n}\right)\\
&=\left(e^{-\lambda}p^{-n}-1\right)\mathbb{NB}_{n,p}(x_0)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{P}_\lambda(x_0)-\mathbb{NB}_{n,p}(x_0) &\leq e^{-\lambda}\sum_{k=0}^{x_0}\frac{\lambda^k}{k!}-p^n\sum_{k=0}^{x_0}\frac{\lambda^k}{k!}\\
&=(e^{-\lambda}-p^n)\sum_{k=0}^{x_0}\frac{\lambda^k}{k!}\\
&=\left(1-e^{\lambda}p^n\right)\mathbb{P}_\lambda(x_0).
\end{aligned}
$$

Therefore, (10) and (11) hold. □

# 3 Main result

The following theorem shows a result of the Poisson approximation to the negative binomial cumulative distribution in terms of the relative error between these cumulative distribution functions and its non uniform bound, which can be derived by the Stein-Chen method and the properties in the two Lemmas.

**Theorem 3.1.** *For $x_0 \in \mathbb{N}\cup\{0\}$, if $\lambda = nq$ then we have the following:*

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{NB}_{n,p}(x_0)}-1\right| \leq \min\left\{e^{-\lambda}p^{-n}-1, \frac{(1-\mathbb{P}_\lambda(x_0))q}{e^{\lambda}p^{n+1}p_\lambda(x_0)}\right\}. \tag{12}$$

**Proof.** Substituting $x$ by $X$ and taking expectation in (7), yields

$$
\begin{aligned}
\mathbb{NB}_{n,p}(x_0)-\mathbb{P}_\lambda(x_0) &= E[\lambda f_{x_0}(X+1)-Xf_{x_0}(X)]\\
&=\sum_{k=0}^{\infty}[nqf_{x_0}(x+1)-xf_{x_0}(x))]p_X(x)
\end{aligned}
$$

$$\begin{aligned}
&= [nqf_{x_0}(1)]p_X(0) + [nqf_{x_0}(2) - f_{x_0}(1)]p_X(1) \\
&\quad + [nqf_{x_0}(3) - 2f_{x_0}(2)]p_X(2) + [nqf_{x_0}(4) - 3f_{x_0}(3)]p_X(3) \\
&\quad + [nqf_{x_0}(5) - 4f_{x_0}(4)]p_X(4) + \sum_{x=5}^{\infty}[nqf_{x_0}(x+1) - xf_{x_0}(x)]p_X(x) \\
&= nqp^n f_{x_0}(1) + n^2q^2p^n f_{x_0}(2) - nqp^n f_{x_0}(1) \\
&\quad + \frac{n^2(n+1)q^3p^n f_{x_0}(3)}{2} - \frac{2n(n+1)q^2p^n f_{x_0}(2)}{2} \\
&\quad + \frac{n^2(n+1)(n+2)q^4p^n f_{x_0}(4)}{3!} - \frac{n(n+1)(n+2)q^3p^n f_{x_0}(3)}{2} \\
&\quad + \frac{n^2(n+1)(n+2)(n+3)q^5p^n f_{x_0}(5)}{4!} \\
&\quad - \frac{n(n+1)(n+2)(n+3)q^4p^n f_{x_0}(4)}{3!} + \sum_{x=5}^{\infty}[nqf_{x_0}(x+1) - xf_{x_0}(x)]p_X(x) \\
&= -nq^2p^n f_{x_0}(2) - n(n+1)q^3p^n f_{x_0}(3) \\
&\quad - \frac{n(n+1)(n+2)q^4p^n f_{x_0}(4)}{2} - \frac{n(n+1)(n+2)(n+3)q^5p^n f_{x_0}(5)}{3} \\
&\quad - \cdots - \frac{n(n+1)(n+2)(n+3)\cdots(n+k)q^{x+2}p^n f_{x_0}(x+2)}{x} - \cdots \\
&= -\sum_{x=1}^{\infty} xqp_X(x)f_{x_0}(x+1) \\
&< 0. \qquad (13)
\end{aligned}$$

From which it follows that

$$\begin{aligned}
0 &\le \mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) \\
&\le \sum_{x=1}^{\infty} xqp_X(x)f_{x_0}(x+1) \\
&\le \sup_{x\ge 2} f_{x_0}(x)\sum_{x=1}^{\infty} xqp_X(x) \\
&\le \frac{\mathbb{P}_\lambda(x_0)(1-\mathbb{P}_\lambda(x_0))q}{p_\lambda(x_0)p}, \qquad (14)
\end{aligned}$$

and follows from (10) in Lemma 2.2, we obtain

$$0 \le \mathbb{P}_\lambda(x_0) - \mathbb{NB}_{n,p}(x_0) \le \left(e^{-\lambda}p^{-n} - 1\right)\mathbb{NB}_{n,p}(x_0). \qquad (15)$$

Dividing (14) and (15) by $\mathbb{NB}_{n,p}(x_0)$, we have

$$\begin{aligned} 0 \leq \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{NB}_{n,p}(x_0)} - 1 &\leq \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{NB}_{n,p}(x_0)} \frac{(1-\mathbb{P}_\lambda(x_0))q}{p_\lambda(x_0)p} \\ &= \frac{(1-\mathbb{P}_\lambda(x_0))q}{e^\lambda p^{n+1} p_\lambda(x_0)} \end{aligned}$$

and

$$0 \leq \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{NB}_{n,p}(x_0)} - 1 \leq e^{-\lambda} p^{-n} - 1,$$

which implies the inequality (12). Hence, (12) is obtained. $\square$

Similarly, dividing (14) and (15) by $\mathbb{P}_\lambda(x_0)$ and using Lemma 2.2, we also obtain

$$0 \leq 1 - \frac{\mathbb{NB}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} \leq \frac{(1-\mathbb{P}_\lambda(x_0))q}{p_\lambda(x_0)p}$$

and

$$0 \leq 1 - \frac{\mathbb{NB}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} \leq 1 - e^\lambda p^n,$$

which gives the following corollary.

**Corollary 3.1.** *For $x_0 \in \mathbb{N} \cup \{0\}$, if $\lambda = nq$ then we have the following:*

$$\left| \frac{\mathbb{NB}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1 \right| \leq \min\left\{ 1 - e^\lambda p^n, \frac{(1-\mathbb{P}_\lambda(x_0))q}{p_\lambda(x_0)p} \right\}. \tag{16}$$

Note that, the non-uniform bounds in Theorem 3.1 and Corollary 3.1 can be adapted to uniform bounds, which are the same results reported in Teerapabolarn (2012).

**Corollary 3.2.** *For $\lambda = nq$, we have the following:*

$$\sup_{x_0 \geq 0} \left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{NB}_{n,p}(x_0)} - 1 \right| = e^{-\lambda} p^{-n} - 1$$

*and*

$$\sup_{x_0 \geq 0} \left| \frac{\mathbb{NB}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1 \right| = 1 - e^\lambda p^n.$$

# 4 Conclusion

In this study, a non-uniform bound on the relative error in Theorem 3.1, determined by the Stein-Chen method, is an approximation of the relative error between the negative binomial cumulative distribution function with parameters $n$ and $p$ and the Poisson cumulative distribution with mean $\lambda = nq = n(1-p)$. With this bound, it is found that the result gives a good Poisson approximation when $q$ is sufficiently small.

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