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Perturbation and Stability of the Signature Operator on a Riemannian Manifold

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Abstract

Let M be a Riemannian compact oriented smooth manifold of dimension n multiple of 4. In this paper, we study the signature behavior of M after a perturbation of the signature operator by a multiplication operator of degree q on differential forms. We establish a stability result when the degree q exceeds $n/2$.

Keywords: *Signature operator, index, signature, multiplication operator, stability.*

1 Introduction

The topological invariants in a manifold are traditionally an important objects in differential geometry (see e.g. [1], [11]). For example, the Atiyah-Singer index computes the index of an elliptic differential operator on a compact manifold in terms of topological invariants of the manifold and the differential operator. The Atiyah-Singer index Theorem gives that for every elliptic differential operator P on a compact manifold one has the equality between the Fredholm index of P and the topological index of P. See [10] for review of this result, including an extension to non-compact manifolds. The Atiyah-

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Singer index theorem is a far reaching generalization of classical results like the Gauss-Bonnet theorem, the Riemann-Roch-Hirzebruch theorem, the signature theorem of Hirzebruch and the Lefschetz fixed point theorem. It has many applications in topology, geometry, mathematical physics, etc.

In a smooth compact oriented n-dimensional manifold M, the Atiyah-Singer index theorem of the Laplacian on differential forms and of its square root, which is the Dirac differential operator, does not affect a significant result in geometry and analysis. But the restriction of the Dirac operator on an eigenspace associated to the eigenvalue of an isomorphism denoted τ, creates a differential operator called the Hodge operator or the signature operator of the manifold M. Its index according to Atiyah-Singer theorem is the signature of M. We can calculate the signature through the characteristic classes (see e.g. [14]), or as difference of dimensions of the kernel (the nullity) of signature operator and the kernel of its adjoint (the deficiency). The signature of the manifold is different from zero if n is multiple of 4. If n is not multiple of 4, we decree that the signature of M is zero.

Therefore, the signature is a topological invariant. Then the signature of M is the index of the signature operator as a Fedholm operator on the pre-Hilbert space of all differential forms on M.

It is well known in the context of analysis that the property of being Fredholm operator on Banach spaces is stable under small perturbations, see Kato [9], Cordes-Labrousse [4] and Benharrat-Messirdi [2]. Sz. Nagy [12] extended these results to unbounded operators and relatively bounded perturbations.

The stability theorems are important in many respects. In particular it is one of the powerful methods for proving the existence of solutions of functional equations in Banach spaces (see [9]). It is a natural and important question then to study the stability of the index of elliptic differential operators on a compact manifold. Indeed, let T and A be linear operators in a finite-dimensional space to itself, then the index of $(T + A)$ is always zero if the space is complete. It is in general not easy to see when the index of T is conserved under a perturbation if T is the signature operator (see e.g. [3], [6]). But an elegant result is obtained here if the perturbation A is a multiplication operator by a differential form. By means of the spectral theorem, this result remains true in particular for all selfadjoint perturbations of the signature operator.

More precisely, we establish in this paper that this stability is not true on the space of all differential forms on a compact oriented and smooth manifold M of dimension $n = 2l$ equipped with a natural pre-Hilbert non complete structure. When T is the signature operator and A is a multiplication operator by a differential form of degree q, the index is not still conserved. Indeed, we show that the index of $(T + A)$ is equal to index of T if q exceeds $\frac{n}{2}$.

Our paper is organized as follows:

In section 2, we give some preliminary results upon which our investigation will be based : Hodge theory, Hodge theorem, signature operator.

Section 3 is devoted to the signature of a riemannian manifold M and the study of multiplication operators on the set of real and smooth differential forms on M.

Finally, in section 4 we apply the results of sections 2 and 3 to study the stability of the index of signature operator perturbed by a multiplication operator on a riemannian compact orientable and $4l$-dimensional manifold, $l \in \mathbb{N} \setminus \{0\}$.

2 Preliminaries

In this section we will present the basics of Hodge theory. We start by reviewing the De Rham complex, p-cohomology and the Hodge star operator. We then introduce the Laplacian and establish the connection between harmonic forms and cohomology.

If A is a linear operator defined on a linear space then the null space and the range space of A are denoted respectively by $\ker A$ and $\Im A$, $A^*$is the adjoint operator of A. Let (M, g) be riemannian manifold, compact, n-dimensional and oriented, $n = 2\ell$, (cf. [13],[5],[8]). Let $\Omega^p = C^\infty(\wedge^p T^*M)$, $p \in \mathbb{N}$, be the real and smooth differential p-forms vector space. The exterior derivative d define a sequence of operator linear partial derivatives

$$0 \to \Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \cdots \to \Omega^n \xrightarrow{d} 0 \tag{1}$$

We call (d, Ω^p) the De Rham complex. The quotient

$$H^p = \frac{Z^p(M)}{B^p(M)} = \frac{\{\text{closed } p\text{-forms on } M\}}{\{\text{exacts } p\text{-forms on } M\}}$$

is a vector space called the p-cohomology space, then its homology is the cohomology H^* in Ω^* the algebra of the forms on M.

Let be $x \in M$, $(e_1, .., e_n)$ an orthonormal basis of the tangent space T_xM and $(e_1^*, .., e_n^*)$ its dual basis of the cotangent space T_x^*M. On the space $\wedge^p T_x^* M$ of exteriors p-forms on T_xM, we consider the inner product associated to g which makes the basis $\{e_{i_1}^* \wedge .. \wedge e_{i_p}^*\}_{1 \le i_1 < .. < i_p \le n}$ orthonormal. We define the star operator by

$$\begin{aligned} \star \quad &: \quad \wedge^p T^*M \to \wedge^{n-p} T^*M \\ \star(e_{i_1}^* \wedge .. \wedge e_{i_p}^*) \quad &= \quad \varepsilon(e_{j_1}^* \wedge .. \wedge e_{j_{n-p}}^*) \\ \star e_I^* \quad &= \quad \varepsilon e_J^* \end{aligned} \tag{2}$$

where $J = \{j_1, ..., j_{n-p}\}$ is complementary sequence of $I = \{i_1, ..., i_p\}$ in $N = \{1, ..., n\}$ and ε the signature of $\{N\} \to \{I, J\}$. This transformation can be extended in a same way to

$$\star : \wedge^p T^*M \otimes \mathbb{C} \to \wedge^{n-p} T^*M \otimes \mathbb{C}$$

it induces an $C^\infty(M)$-linear application $\star : \Omega^p \to \Omega^{n-p}$ called the Hodge star operator. A straightforward calculation shows that $\star$ satisfies the identity $\star^2 = (-1)^{p(n-p)} id$, where id is the identity operator on Ω^p.

We define on Ω^p a natural hermitian product by

$$< \alpha, \beta > = \int_M \alpha \wedge \star\beta \quad \text{for all } \alpha, \beta \in \Omega^p \tag{3}$$

Thus, Ω^p is a pre-Hilbert space.

The exterior derivative $d : \Omega^p \to \Omega^{p+1}$ has a unique adjoint δ defined by $< \alpha, d\beta > = < \delta\alpha, \beta >$ if α, β are smooth forms of degrees $p+1$ and p on M.

$$\delta : \Omega^{p+1} \to \Omega^p \ ; \ \delta = (-1)^{n(p+1)+1} \star d \star \tag{4}$$

Clearly, $\delta^2 = 0$, since $d^2 = 0$. This will allow us to define a self adjoint second order partial differential operator Δ by

$$\Delta : \Omega^p \to \Omega^p; \quad \Delta = D^2 = (\delta + d)^2 = \delta d + d\delta \tag{5}$$

called the Laplace operator on forms space and D is the Dirac operator. The solutions of $\Delta\omega = 0$ are called the harmonic forms and we denote by $\mathcal{H}^p$ the set of harmonic p-forms. Then

$$\ker \Delta = \mathcal{H}^p \tag{6}$$

In a system of local coordinates $(x_1.., x_n)$, Δ is expressed by

$$\Delta = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} (\sqrt{|g|}\ g^{ij} \frac{\partial}{\partial x_j})$$

where g is the local matrix metric, $|g|$ its determinant and (g^{ij}) its inverse. Let us recall the following decomposition theorem :

Theorem 1 *For each $p \in \mathbb{N}$, $\mathcal{H}^p$ is finite dimensional, $Im\Delta$ is finite codimensional and there are orthogonal direct sum decomposition*

$$\Omega^p = \mathcal{H}^p \oplus \Delta(\Omega^p) = \mathcal{H}^p \oplus Imd \oplus Im\delta$$

As a consequence, we have the isomorphism identification

$$H^p \approx \mathcal{H}^p \tag{7}$$

Then the cohomology H^p is a finite dimensional. Moreover, it is easy to see that the operator $\star$ satisfies $\star^2\alpha = (-1)^p\alpha$ for all p-form α. Denote by Ω^* the direct sum of all the spaces Ω^p, then the isomorphism

$$\tau : \Omega^* \to \Omega^* \ ; \ \tau = (-1)^{p(p-1)+l}\star \tag{8}$$

is an involution i.e. $\tau^2 = id$; so it admits 1 and -1 as eigenvalues. Let Ω^+ and Ω^- be the eigenspaces associated respectively to these eigenvalues. We have thus the decomposition

$$\Omega^p = \Omega^+ \oplus \Omega^- \tag{9}$$

We can easily show that $D\tau = \tau D$. The restriction of D respectively to the spaces Ω^+ and Ω^- give two operators

$$D^+ : \Omega^+ \to \Omega^- \text{ and } D^- : \Omega^- \to \Omega^+ \tag{10}$$

transversely elliptic and are adjoint to each other. The operators D^+ and D^- are Fredholm then they admits an index. D^+ is called The Hodge operator or the signature operator.

3 Signature and Multiplication Operators

3.1 Signature of a Manifold

We are going to use the Hodge theorem to exhibit the signature of M as the Fredholm index of a differential operator on M. The operator Δ is an elliptic selfadjoint operator, from the point view of index theorem, Δ is not interesting because $index(\Delta) = 0$. A more basic operator is the first order differential operator D

$$D : \Omega^* \to \Omega^*; \quad D = d + \delta \tag{11}$$

D is elliptic and self adjoint, hence $index(D) = 0$. Consequently, D does not produce any useful relationships. D^+ is elliptic, his index have an interesting topological meaning. Indeed, observe that

$$\tau : \ker D \to \ker D \tag{12}$$

if $D^2\omega = 0$, thus $D\omega = 0$, then $\ker D = \ker \Delta$. It then follows from the Hodge theorem that τ is an involution on the cohomology H^*.

Let be the decomposition

$$H^* = H^*_+ \oplus H^*_- \tag{13}$$

in eigenspaces associated respectively to the eigenvalues 1 and -1 of τ.
We have

$$\ker D^+ = H^*_+ \; ; \; \ker D^- = H^*_- \tag{14}$$

and

$$\begin{aligned} index(D^+) &= \dim\ \ker D^+ - \dim\ co\ker D^+ \\ &= \dim\ \ker D^+ - \dim\ \ker D^- \\ &= \dim H^*_+ - \dim H^*_- \end{aligned}$$

Let us calculate $\dim H^*_+ - \dim H^*_-$, for it we consider the restrictions of $\star$ and τ to H^ℓ.

• If ℓ is odd, we have $\tau = \pm i\star$ and $\star^2 = -id$ on H^ℓ. Furthermore, $\star$ is a real isomorphism on H^ℓ and the eigenvalues of $\star$ and τ are conjugates. In this case $\dim H^*_+ = \dim H^*_-$ and consequently $index(D^+) = 0$.

• If ℓ is even, $\tau = \star$ on H^ℓ. From the relation (13) we get

$$\mathcal{H}^* = \mathcal{H}^*_+ \oplus \mathcal{H}^*_- \text{ and } H^* = H^*_+ \oplus H^*_- \tag{15}$$

Thus,

$$index(D^+) = \dim H^*_+ - \dim H^*_- \tag{16}$$

Remark 2 *Let $V_p = H^p \oplus H^{\ell-p}$ for $0 \leqslant p \leqslant \ell - 1$. The action of τ on H^* exchanges H^p and $H^{\ell-p}$; so V_p is invariant by τ.*

We have precisely

$$index(D^+) = \dim H^\ell_+ - \dim H^\ell_- \tag{17}$$

Definition 3 *The signature $Sign(M)$ of M is the signature of the non degenerate quadratic form Q defined by formula* (8) *and equal to the Fredholm index of the signature operator on M.*

$$Sign(M) = index(D^+)$$

As a consequence, the bilinear map :

$$\Phi : H^p \times\ H^{n-p} \to \Phi(\alpha, \beta) = \int_M \alpha \wedge \beta \in \mathbb{R}$$

is non degenerated (this is Poincaré duality). Moreover, Φ is symmetric for $p = 2\ell$; its signature is an element of $\mathbb{Z}$ and is exactly the signature of M. Thus, the signature is defined for any oriented compact manifold and it is a topological invariant. When the dimension of M is not a multiple of 4 we decree that $Sign(M) = 0$.

Example 4 *$Sign(\mathbb{C}P^2) = 1$, where $\mathbb{C}P^2$ is the complex projective space (see [14]).*

3.2 Multiplication Operators on Ω^*

Definition 5 *Let $q \in \mathbb{N}$. The operator A_q defined by*

$$A_q : \Omega^p \to \Omega^{p+q}; \quad \alpha \to A_q\alpha = a_q \wedge \alpha \tag{18}$$

is called the multiplication operator by a q-form a_q on Ω^*.

Taking account of the pre-Hilbert space structure defined on Ω^* by the inner product in formula (3), for all p-form α and $p+q$-form β we have:

$$\begin{aligned} &< \; A_q\alpha, \beta >= (-1)^{p+q} \int_M \alpha \wedge a_q \wedge \star\beta \\ &= \; (-1)^{pq}(-1)^{p(n-p)} \int_M \alpha \wedge (-1)^{p(n-p)} \star (\star(a_q \wedge \star\beta)) \\ &= \; (-1)^{pq+p(n-p)} < \alpha, \star(a_q \wedge \star\beta) > \end{aligned}$$

We get the following result :

Proposition 6 *The adjoint operator A_q^* of A_q with respect to the pre-Hilbert space structure on Ω^* is*

$$\begin{aligned} A_q^* \quad &: \quad \Omega^p \to \Omega^{p-q} \\ \beta \quad &\to \quad A_q^*\beta = (-1)^{pq+p(n-p)} \star (a_q \wedge \star\beta) \end{aligned} \tag{19}$$

We remark that $A_q^* = (-1)^{pq+p(n-p)} \star A_q \star$ is not a multiplication operator on Ω^*, thus A_q is not selfadjoint.

$$\begin{aligned} \ker A_q \quad &= \quad \bigoplus_p \mathcal{A}_q^p \text{ where } \mathcal{A}_q^p = \{\alpha \in \Omega^p \;\; ; \;\; a_q \wedge \alpha = 0\} \\ \ker A_q^* \quad &= \quad \bigoplus_p \mathcal{A}_q^{*p} \text{ where } \mathcal{A}_q^{*p} = \{\beta \in \Omega^p \;\; ; \;\; a_q \wedge \star\beta = 0\} \end{aligned}$$

We can in particular give the two following natural cases :

1) If $q = 0$, then $\mathcal{A}_0^p = \mathcal{A}_0^{*p} = \{0\}$ for all $p \in \mathbb{N}$.

2) If $q = n$, then $\mathcal{A}_n^0 = \mathcal{A}_n^{*0} = \mathcal{A}_n^{*p} = \{0\}$ and $\mathcal{A}_n^p = \Omega^p$, $\mathcal{A}_n^{*p} = \Omega^{n-p}$ for all $0 < p < n$.

It follows that :

Proposition 7 *The Hodge star operator $\star$ exchanges the spaces $\mathcal{A}_q^p$ and $\mathcal{A}_q^{n-p}$ for all $0 \leqslant p \leqslant n$ and the Fredholm index of A_q is given by :*

$$index(A_q) = \dim \bigoplus_p \mathcal{A}_q^p - \dim \bigoplus_p \mathcal{A}_q^{*p} = 0$$

3.3 Stability of Signature Operator

In this section, the manifold is assumed to be riemannian, compact, orientable and n-dimensional where n is multiple of 4 ($n = 2\ell$, ℓ even). Let $D_q^+ = D^+ + A_q^+$ a perturbation of D^+ by a multiplication operator A_q^+ of degree q on Ω_+ :

$$\begin{aligned} D_q^+ : \Omega^+ &\to \Omega^- \\ \alpha_+ &\longmapsto D_q^+ \alpha_+ = D^+ \alpha_+ + a_+ \wedge \alpha_+ \end{aligned} \tag{20}$$

where a_+ is a q-form in Ω^+.

The adjoint D_q^{+*} of D_q^+ is given by

$$\begin{aligned} D_q^{+*} : \Omega^- &\to \Omega^+ \\ \beta_- &\longmapsto D_q^{+*} \beta_- = (D^- + A_q^{+*}) \beta_- \end{aligned}$$

Thus, D_q^+ is not selfadjoint and we have

$$\ker D_q^+ = \bigoplus_p (H_+^p \cap \mathcal{A}_q^{+p}) \text{ and } \ker D_q^{+*} = \bigoplus_p (H_-^p \cap \mathcal{A}_q^{+*p})$$

In particular,

1) if $q = 0$, then $H_+^p \cap \mathcal{A}_0^{+p} = H_-^p \cap \mathcal{A}_0^{+*p} = \{0\}$ for all $p \in \mathbb{N}$.

2) If $q = 1$, then $H_+^p \cap \mathcal{A}_1^{+p} = \{\alpha \in \Omega^+ \; ; \; d\alpha + a_+ \wedge \alpha = 0 \text{ and } \delta\alpha = 0\}$ and $H_-^p \cap \mathcal{A}_1^{+*p} = \{\beta \in \Omega^- \; ; \; d\alpha = 0 \text{ and } \delta\beta + a_+ \wedge \beta = 0\}$.

3) If $q = n$, then $H_+^0 \cap \mathcal{A}_n^{+0} = \{0\}$, $H_+^p \cap \mathcal{A}_n^{+p} = H_+^p$ and $H_-^0 \cap \mathcal{A}_n^{+*0} = H_-^n \cap \mathcal{A}_n^{+*n} = \{0\}$, $H_-^p \cap \mathcal{A}_n^{+*p} = H_-^{n-p}$.

By virtue of remark (3.1) and proposition (3.6), the star operator "$\star$" exchanges $H_+^p \cap \mathcal{A}_q^{+p}$ and $H_-^{n-p} \cap \mathcal{A}_q^{+*(n-p)}$ for $1 \leqslant p < n$, thus we obtain :

$$\sum_{p=1}^{n-1} (dim(H_+^p \cap \mathcal{A}_q^{+p}) - dim(H_-^p \cap \mathcal{A}_q^{+*p})) = dim(H_+^{n/2} \cap \mathcal{A}_q^{+n/2}) - dim(H_-^{n/2} \cap \mathcal{A}_q^{+*n/2}) \tag{21}$$

So the Fredholm index of the operator D_q^+ is given by :

$$index(D_q^+) = \begin{cases} 0 & \text{if } q = 0 \\ \dim(H_+^\ell \cap \mathcal{A}_q^{+\ell}) - \dim(H_-^\ell \cap \mathcal{A}_q^{+*\ell}) & \text{if } 1 \leq q \leqslant \frac{n}{2} = \ell \\ index(D^+) & \text{if } \frac{n}{2} < q \leqslant n \end{cases} \tag{22}$$

Finally, we obtain our main stability result :

Theorem 8 *The Fredholm index of signature operator on a riemannian, compact, orientable and n-dimensional manifold, where n is multiple of 4, is stable under perturbations by a multiplication operator of degree q if $\frac{n}{2} < q \leqslant n$.*

In conclusion, by virtue of Atiyah-Singer index theorem we get the following corollary :

Corollary 9 *The class of operators D_q^+ has a stable index on M, when q exceeds $\frac{n}{2}$.*

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