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On Fixed-point Theorems in Intuitionistic Fuzzy Metric Space I

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Abstract

In this paper, first we have established two sets of sufficient conditions for a TS-IF contractive mapping to have unique fixed point in a intuitionistic fuzzy metric space. Then we have defined $(\,\varepsilon\,,\,\lambda\,)$ *IF-uniformly locally contractive mapping and* $\eta-$ *chainable space, where it has been proved that the* $(\,\varepsilon\,,\,\lambda\,)$ *IF-uniformly locally contractive mapping possesses a fixed point.*

Keywords: *Intuitionistic fuzzy set, intuitionistic fuzzy metric spaces, Fuzzy Banach Space, contraction mapping, Fixed piont*

1 Introduction

Fuzzy set theory was first introduce by Zadeh[13] in 1965 to describe the situation in which data are imprecise or vague or uncertain. Thereafter the concept of fuzzy set was generalized as intuitionistic fuzzy set by K. Atanassov[6, 7] in 1984. It has a wide range of application in the field of population dynamics , chaos control , computer programming , medicine , etc.
The concept of fuzzy metric was first introduced by Kramosil and Michalek[9] but using the idea of intuitionistic fuzzy set, Park[5] introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms, which is a generalization of fuzzy metric space due to George and Veeramani[1].

Introducing the contraction mapping with the help of the membership function for fuzzy metric, several authors[2, 8, 10] established the Banach fixed point theorem in fuzzy metric space. In the paper[2], to prove the Banach Fixed Point theorem in intuitionistic fuzzy metric space, Mohamad[2] also introduced one concept of contractive mapping, which is not so natural. There he proved that every iterative sequence is a contractive sequence and then assumed that every contractive sequences are Cauchy. But all these contraction mappings, which they have considered to establish different type fixed point theorem, do not bear the intension of the contraction mapping with respect to a fuzzy metric, where a fuzzy metric gives the degree of nearness of two points with respect to a parameter t. Considering this meaning of fuzzy metric, in our paper[11], we have redefined the notion of contraction mapping in an intuitionistic fuzzy metric space and then directly, it has been proved that the every iterative sequence is a Cauchy sequence, that is, we don't need to assume that every contractive sequences are Cauchy sequences. Thereafter we have established the Banach Fixed Point theorem there. In this paper, first we have established two sets of sufficient conditions for a TS-IF contractive mapping to have unique fixed point in a intuitionistic fuzzy metric space. Then we have defined (ε, λ) IF-uniformly locally contractive mapping and $\eta-$chainable space, where it has been proved that the (ε, λ) IF-uniformly locally contractive mapping possesses a fixed point.

2 Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1 *[3]. A binary operation* $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ *is continuous* t *- norm if* $*$ *satisfies the following conditions* :
(i) $*$ *is commutative and associative ,*
(ii) $*$ *is continuous ,*
(iii) $a * 1 = a \quad \forall\, a \,\varepsilon\, [0, 1]$,
(iv) $a * b \leq c * d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \,\varepsilon\, [0, 1]$.

Definition 2.2 *[3]. A binary operation* $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ *is continuous t-conorm if* $\diamond$ *satisfies the following conditions* :
(i) $\diamond$ *is commutative and associative ,*
(ii) $\diamond$ *is continuous ,*
(iii) $a \diamond 0 = a \quad \forall\, a \in [0, 1]$,
(iv) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \in [0, 1]$.

Result 2.3 *[4].* (a) *For any* $r_1, r_2 \in (0, 1)$ *with* $r_1 > r_2$*, there exist* $r_3, r_4 \in (0, 1)$ *such that* $r_1 * r_3 > r_2$ *and* $r_1 > r_4 \diamond r_2$.

(b) For any $r_5 \in (0,1)$ *, there exist* $r_6, r_7 \in (0,1)$ *such that* $r_6 * r_6 \geq r_5$ *and* $r_7 \diamond r_7 \leq r_5$.

Definition 2.4 *[5] Let* $*$ *be a continuous t-norm ,* $\diamond$ *be a continuous t-conorm and* X *be any non-empty set. An* ***intuitionistic fuzzy metric*** *or in short* ***IFM*** *on* X *is an object of the form*
$A = \{((x,y,t), \mu(x,y,t), \nu(x,y,t)) : (x,y,t) \in X^2 \times (0,\infty)\}$ *where* μ, ν *are fuzzy sets on* $X^2 \times (0,\infty)$ *,* μ *denotes the degree of nearness and* ν *denotes the degree of non−nearness of* x *and* y *relative to* t *satisfying the following conditions : for all* $x, y, z \in X$, $s, t > 0$
(i) $\mu(x,y,t) + \nu(x,y,t) \leq 1 \quad \forall\ (x,y,t) \in X^2 \times (0,\infty)$;
(ii) $\mu(x,y,t) > 0$;
(iii) $\mu(x,y,t) = 1$ *if and only if* $x = y$
(iv) $\mu(x,y,t) = \mu(y,x,t)$;
(v) $\mu(x,y,s) * \mu(y,z,t) \leq \mu(x,z,s+t)$;
(vi) $\mu(x,y,\cdot) : (0,\infty) \to (0,1]$ *is continuous;*
(vii) $\nu(x,y,t) > 0$;
$(viii)$ $\nu(x,y,t) = 0$ *if and only if* $x = y$;
(ix) $\nu(x,y,t) = \nu(y,x,t)$;
(x) $\nu(x,y,s) \diamond \mu(y,z,t) \geq \nu(x,z,s+t)$;
(xi) $\nu(x,y,\cdot) : (0,\infty) \to (0,1]$ *is continuous.*

If A *is a* ***IFM*** *on* X *, the pair* (X, A) *will be called a* ***intuitionistic fuzzy metric space*** *or in short* ***IFMS***.

We further assume that (X, A) *is a* ***IFMS*** *with the property:*
(xii) *For all* $a \in (0,1)$, $a * a = a$ *and* $a \diamond a = a$

Remark 2.5 *[5] In intuitionistic fuzzy metric space* X, $\mu(x,y,\cdot)$ *is non-decreasing and* $\nu(x,y,\cdot)$ *is non-increasing for all* $x, y \in X$.

Definition 2.6 *[1] A sequence* $\{x_n\}_n$ *in a intuitionistic fuzzy metric space is said to be a* ***Cauchy sequence*** *if and only if for each* $r \in (0,1)$ *and* $t > 0$ *there exists* $n_0 \in N$ *such that* $\mu(x_n, x_m, t) > 1 - r$ *and* $\nu(x_n, x_m, t) < r$ *for all* $n, m \geq n_0$.
A sequence $\{x_n\}$ *in a intuitionistic fuzzy metric space is said to converge to* $x \in X$ *if and only if for each* $r \in (0,1)$ *and* $t > 0$ *there exists* $n_0 \in N$ *such that* $\mu(x_n, x, t) > 1 - r$ *and* $\nu(x_n, x, t) < r$ *for all* $n, m \geq n_0$.

Note 2.7 *[12] A sequence* $\{x_n\}_n$ *in an intuitionistic fuzzy metric space is a Cauchy sequence if and only if*

$$\lim_{n \to \infty} \mu(x_n, x_{n+p}, t) = 1 \text{ and } \lim_{n \to \infty} \nu(x_n, x_{n+p}, t) = 0$$

A sequence $\{x_n\}_n$ *in an intuitionistic fuzzy metric space converges to* $x \in X$ *if and only if*

$$\lim_{n \to \infty} \mu(x_n, x, t) = 1 \text{ and } \lim_{n \to \infty} \nu(x_n, x, t) = 0$$

Definition 2.8 *[2] Let* (X, A) *be a intuitionistic fuzzy metric space. We will say the mapping* $f : X \to X$ *is* ***t-uniformly continuous*** *if for each* ε*, with* $0 < \varepsilon < 1$*, there exists* $0 < r < 1$*, such that* $\mu(x, y, t) \geq 1 - r$ *and* $\nu(x, y, t) \leq r$ *implies* $\mu(f(x), f(y), t) \geq 1 - \varepsilon$ *and* $\nu(f(x), f(y), t) \leq \varepsilon$ *for each* $x, y \in X$ *and* $t > 0$.

Definition 2.9 *[11] Let* (X, A) *be* ***IFMS*** *and* $T : X \to X$. T *is said to be TS-IF contractive mapping if there exists* $k \in (0, 1)$ *such that*

$$k\, \mu(T(x), T(y), t) \geq \mu(x, y, t)$$

$$\text{and} \qquad \frac{1}{k}\, \nu(T(x), T(y), t) \leq \nu(x, y, t) \;\; \forall\; t > 0.$$

Proposition 2.10 *Let* (X, A) *be a intuitionistic fuzzy metric space. If* $f : X \to X$ *is TS-IF contractive then f is t-uniformly continuous.*

Proof. Obvious

Theorem 2.11 *[11] Let* (X, A) *be a complete* ***IFMS*** *and* $T : X \to X$ *be TS-IF contractive mapping with k its contraction constant. Then* T *has a unique fixed point.*

3 Fixed-Point Theorems

Definition 3.1 *Let* (X, A) *be an* ***IFMS*** , $x \in X$, $r \in (0, 1)$, $t > 0$,

$$B(x, r, t) = \{y \in X / \mu(x, y, t) > 1 - r, \nu(x, y, t) < r\}.$$

Then $B(x, r, t)$ *is called an* ***open ball*** *centered at x of radius r w.r.t. t.*

Definition 3.2 *Let* (X, A) *be an* ***IFMS*** *and* $P \subseteq X$. P *is said to be a* ***closed set*** *in* (X, A) *if and only if any sequence* $\{x_n\}$ *in* P *converges to* $x \in P$ *i.e, iff.* $\lim\limits_{n \to \infty} \mu(x_n, x, t) = 1$ *and* $\lim\limits_{n \to \infty} \nu(x_n, x, t) = 0 \Rightarrow x \in P$.

Definition 3.3 *Let* (X, A) *be an* ***IFMS,*** $x \in X$, $r \in (0, 1)$, $t > 0$,

$$S(x, r, t) = \{y \in X / \mu(x, y, t) > 1 - r, \nu(x, y, t) < r\}.$$

Hence $S(x, r, t)$ *is said to be a* ***closed ball*** *centered at x of radius r w.r.t. t iff. any sequence* $\{x_n\}$ *in* $S(x, r, t)$ *converges to y then* $y \in S(x, r, t)$.

Theorem 3.4 (*Contraction on a closed ball*) *:- Suppose* (X, A) *is a complete IFMS. Let* $T : X \to X$ *be TS-IF contractive mapping on* $S(x, r, t)$ *with contraction constant k. Moreover, assume that*

$$k\,\mu\,(x, T(x), t) > (1 - r) \;\; \textit{and} \;\; \frac{1}{k}\,\nu\,(x, T(x), t) < r$$

Then T has unique fixed point in $S(x, r, t)$.

Proof. Let $x_1 = T(x)$, $x_2 = T(x_1) = T^2(x)$, $\cdots$, $x_n = T(x_{n-1})$ i.e, $x_n = T^n(x)$ for all $n \in N$. Now

$$k\,\mu\,(x, T(x), t) > (1 - r)$$

$$\Rightarrow \mu\,(x, T(x), t) > \frac{(1 - r)}{k} > (1 - r)$$

$$\Rightarrow \mu\,(x, x_1, t) > (1 - r) \qquad \cdots \qquad (i)$$

Again,

$$\frac{1}{k}\,\nu\,(x, T(x), t) < r$$

$$\Rightarrow \nu\,(x, T(x), t) < r\,k < r$$

$$\Rightarrow \nu\,(x, x_1, t) < r \qquad \cdots \qquad (ii)$$

(i) and (ii) $\Rightarrow x_1 \in S(x, r, t)$.

Assume that $x_1, x_2, \cdots, x_{n-1} \in S(x, r, t)$. We show that $x_n \in S(x, r, t)$.

$$k\,\mu\,(x_1, x_2, t) = k\,\mu\,(T(x), T(x_1), t)$$

$$\geq \mu\,(x, x_1, t)$$

$$\Rightarrow \mu\,(x_1, x_2, t) > \frac{(1-r)}{k} > (1-r)$$

$$k\,\mu\,(x_2, x_3, t) = k\,\mu\,(T(x_1), T(x_2), t)$$

$$\geq \mu\,(x_1, x_2, t)$$

$$\Rightarrow \mu\,(x_2, x_3, t) \geq \frac{1}{k}\,\mu\,(x_1, x_2, t)$$

$$> \frac{1-r}{k} > (1-r)$$

Again,

$$\frac{1}{k}\,\nu\,(x_1, x_2, t) = \frac{1}{k}\,\nu\,(T(x), T(x_1), t)$$

$$\leq \nu(x, x_1, t)$$

$$\Rightarrow \nu(x_1, x_2, t) \leq k\,\nu(x, x_1, t) < k\,r < r$$

$$\frac{1}{k}\,\nu(x_2, x_3, t) = \frac{1}{k}\,\nu(T(x_1), T(x_2), t)$$

$$\leq \nu(x_1, x_2, t)$$

$$\Rightarrow \nu(x_2, x_3, t) \leq k\,\nu(x_1, x_2, t) < k\,r < r$$

Similarly it can be shown that ,
$\mu(x_3, x_4, t) > 1-r$, $\nu(x_3, x_4, t) < r$, $\cdots$, $\mu(x_{n-1}, x_n, t) > 1-r$ and $\nu(x_{n-1}, x_n, t) < r$.
Thus, we see that ,

$$\mu(x, x_n, t) \geq \mu\left(x, x_1, \frac{t}{n}\right) * \mu\left(x_1, x_2, \frac{t}{n}\right) * \cdots * \mu\left(x_{n-1}, x_n, \frac{t}{n}\right)$$

$$> (1-r) * (1-r) * \cdots * (1-r) = 1-r$$

i.e., $\mu(x, x_n, t) > 1-r$

$$\nu(x, x_n, t) \leq \nu\left(x, x_1, \frac{t}{n}\right) \diamond \nu\left(x_1, x_2, \frac{t}{n}\right) \diamond \cdots \diamond \nu\left(x_{n-1}, x_n, \frac{t}{n}\right)$$

$$< r \diamond r \diamond \cdots \diamond r = r$$

Thus , $\mu(x, x_n, t) > 1 - r$ *and* $\nu(x, x_n, t) < r$
$\Rightarrow x_n \in S(x, r, t)$
Hence, by the theorem 3.10[11] and the definition3.3, T has unique fixed point in $S(x, r, t)$.

Note 3.5 *It follows from the proof of Theorem 3.10[11] that for any* $x \in X$ *the sequence of iterates* $\{T^n(x)\}$ *converges to the fixed point of T.*

Lemma 3.6 *Let* (X, A) *be* ***IFMS*** *and* $T : X \rightarrow X$ *be t-uniformly continuous on X. If* $x_n \rightarrow x$ *as* $n \rightarrow \infty$ *in* (X, A) *then* $T(x_n) \rightarrow T(x)$ *as* $n \rightarrow \infty$ *in* (X, A) .

Proof. Proof directly follows from the definitions of t-uniformly continuity and convergence of a sequence in a ***IFMS***.

Lemma 3.7 *Let* (X, A) *be* ***IFMS***. *If* $x_n \rightarrow x$ *and* $y_n \rightarrow y$ *in* (X, A) *then* $\mu(x_n, y_n, t) \rightarrow \mu(x, y, t)$ *and* $\nu(x_n, y_n, t) \rightarrow \nu(x, y, t)$ *as* $n \rightarrow \infty$ *for all* $t > 0$ *in* R .

Proof. We have,

$$\lim_{n \to \infty} \mu(x_n, x, t) = 1, \lim_{n \to \infty} \nu(x_n, x, t) = 0$$

$$\textit{and} \lim_{n \to \infty} \mu(y_n, y, t) = 1, \lim_{n \to \infty} \nu(y_n, y, t) = 0$$

$$\begin{aligned}\mu(x_n, y_n, t) &\geq \mu\left(x_n, x, \frac{t}{2}\right) * \mu\left(x, y_n, \frac{t}{2}\right) \\ &\geq \mu\left(x_n, x, \frac{t}{2}\right) * \mu\left(x, y, \frac{t}{4}\right) * \mu\left(y, y_n, \frac{t}{4}\right)\end{aligned}$$

$$\Rightarrow \lim_{n \to \infty} \mu(x_n, y_n, t) \geq \mu(x, y, t)$$

$$\begin{aligned}\mu(x, y, t) &\geq \mu\left(x, x_n, \frac{t}{2}\right) * \mu\left(x_n, y, \frac{t}{2}\right) \\ &\geq \mu\left(x, x_n, \frac{t}{2}\right) * \mu\left(x_n, y_n, \frac{t}{4}\right) * \mu\left(y_n, y, \frac{t}{4}\right)\end{aligned}$$

$$\Rightarrow \mu(x, y, t) \geq \lim_{n \to \infty} \mu(x_n, y_n, t) \quad \forall\, t > 0.$$

Then,

$$\lim_{n \to \infty} \mu(x_n, y_n, t) = \mu(x, y, t) \text{ for all } t > 0,$$

Similarly, $\lim_{n \to \infty} \nu(x_n, y_n, t) = \nu(x, y, t)$ for all $t > 0$.

Theorem 3.8 *Let (X, A) be a complete* ***IFMS*** *and $T : X \to X$ be a t-uniformly continuous on X. If for same positive integer m, T^m is a TS-IF contractive mapping with k its contractive constant then T has a unique fixed point in X.*

Proof. Let $B = T^m$, n be an arbitrary but fixed positive integer and $x \in X$.
we now show that $B^n T(x) \to B^n(x)$ in (X, A).
Now,

$$\begin{aligned}k\,\mu(B^n T(x), B^n(x), t) &= k\,\mu\left(B(B^{n-1}T(x)), B(B^{n-1}(x)), t\right) \\ &\geq \mu(B^{n-1}T(x), B^{n-1}(x), t)\end{aligned}$$

$$\begin{aligned}\textit{i.e, } \mu(B^n T(x), B^n(x), t) &\geq \frac{1}{k}\,\mu(B^{n-1}T(x), B^{n-1}(x), t) \\ &= \frac{1}{k}\,\mu\left(B(B^{n-2}T(x)), B(B^{n-2}(x)), t\right) \\ &\geq \frac{1}{k^2}\,\mu(B^{n-2}T(x), B^{n-2}(x), t)\end{aligned}$$

$$\geq \quad \cdots$$
$$\geq \frac{1}{k^n}\,\mu\,(\,T(x)\,,\,x\,,\,t\,)$$
$$\Rightarrow \lim_{n\to\infty}\mu\,(\,B^nT(x)\,,\,B^n(x)\,,\,t\,) \geq \lim_{n\to\infty}\frac{1}{k^n}\,\mu\,(\,T(x)\,,\,x\,,\,t\,)$$
$$\Rightarrow \lim_{n\to\infty}\mu\,(\,B^nT(x)\,,\,B^n(x)\,,\,t\,) = 1$$

Similarly, $\lim\limits_{n\to\infty}\nu\,(\,B^n\,T(x)\,,\,B^n(x)\,,\,t\,) = 0,$ for all $t > 0.$
Thus, $B^n\,T(x) \to B^n\,(x)$ in $(\,X\,,\,A\,)$.
Again, by the theorem 3.10[11], we see that B has a unique fixed point y(say), and from the note [3.5], it follows that $B^n(x) \to y$ as $n \to \infty$ in $(\,X\,,\,A\,)$.

Since T is t-uniformly continuous on X, it follows from the above lemma[3.6] that $B^n\,T(x) = T\,B^n(x) \to T(y)$ as $n \to \infty$ in $(\,X\,,\,A\,)$.

Again since $\lim\limits_{n\to\infty}\mu\,(\,B^n\,T(x)\,,\,B^n(x)\,,\,t\,) = 1$ and $\lim\limits_{n\to\infty}\nu\,(\,B^n\,T(x)\,,\,B^n(x)\,,\,t\,) = 0,$ we have by the lemma [3.7]
$\lim\limits_{n\to\infty}\mu\,(\,T(y)\,,\,y\,,\,t\,) = 1$ and $\lim\limits_{n\to\infty}\nu\,(\,T(y)\,,\,y\,,\,t\,) = 0,$ for all $t > 0,$
i.e., $\mu\,(\,T(y)\,,\,y\,,\,t\,) = 1$ and $\nu\,(\,T(y)\,,\,y\,,\,t\,) = 0,$ for all $t > 0.$
$\Rightarrow T(y) = y \Rightarrow y$ is a fixed point of T.
If y' is a fixed point of T, *i.e.*, $T(y') = y'$, then $T^m(y') = T^{m-1}(T(y')) = T^{m-1}(y') = \cdots = y' \Rightarrow B(y') = y' \Rightarrow y'$ is a fixed point of B.
But y is the unique fixed point of B, therefore $y = y'$ which implies that y is the unique fixed point of T. This completes the proof.

Definition 3.9 *Let $(\,X\,,\,A\,)$ be a* ***IFMS*** *and $T : X \to X$. For $\varepsilon > 0$ and $0 < \lambda < 1$, T is said to be $(\,\varepsilon\,,\,\lambda\,)$* ***IF-uniformly locally contractive*** *if*

$$\mu\,(\,x\,,\,y\,,\,t\,) > \varepsilon \Longrightarrow \lambda\,\mu\,(\,Tx\,,\,Ty\,,\,t\,) > \mu\,(\,x\,,\,y\,,\,t\,)$$
$$\nu\,(\,x\,,\,y\,,\,t\,) < 1 - \varepsilon \Longrightarrow \frac{1}{\lambda}\,\nu\,(\,Tx\,,\,Ty\,,\,t\,) < \nu\,(\,x\,,\,y\,,\,t\,)$$

Definition 3.10 *Let $0 < \eta < 1$ and $(\,X\,,\,A\,)$ be a* ***IFMS***. *Then $(\,X\,,\,A\,)$ is said to be* ***IF*** *$\eta -$* ***chainable*** *space if for every $a\,,\,b \in X$ there exist a finite set of points $a = x_0\,,\,x_1\,,\,\cdots\,,\,x_n = b$ such that*

$$\mu\,(\,x_{i-1}\,,\,x_i\,,\,t\,) > \eta \text{ and } \nu\,(\,x_{i-1}\,,\,x_i\,,\,t\,) < 1-\eta\;,\quad i = 1, 2, \cdots, n$$

Theorem 3.11 *Let $(\,X\,,\,A\,)$ be a complete* ***IFMS*** *and IF$\varepsilon -$chainable space. If $T : X \to X$ is $(\,\varepsilon\,,\,\lambda\,)$ IF-uniformly locally contractive then T has a fixed point in X.*

Proof. Let x be an arbitrary but fixed point of X. If $Tx = x$ then a fixed point is assured. We assume therefore that $Tx \neq x$. Since X is IF $\varepsilon -$ chainable space, there exists a finite set of points $x = x_0 , x_1 , \cdots , x_n = Tx$ such that

$$\mu (x_{i-1} , x_i , t) > \varepsilon \ \textit{and} \ \nu (x_{i-1} , x_i , t) < 1 - \varepsilon , \quad i = 1, 2, \cdots , n$$

Again, since T is (ε , λ) IF-uniformly locally contractive, we have

$$\mu (x_{i-1} , x_i , t) > \varepsilon \Longrightarrow \lambda \, \mu (Tx_{i-1} , Tx_i , t) > \mu (x_{i-1} , x_i , t) > \varepsilon$$

$$\textit{i.e.,} \quad \mu (Tx_{i-1} , Tx_i , t) > \frac{\varepsilon}{\lambda} > \varepsilon ;$$

$$\nu (x_{i-1} , x_i , t) < 1 - \varepsilon \Longrightarrow \frac{1}{\lambda} \nu (Tx_{i-1} , Tx_i , t) < \nu (x_{i-1} , x_i , t) < 1 - \varepsilon$$

$$\textit{i.e.,} \quad \nu (Tx_{i-1} , Tx_i , t) < \lambda (1 - \varepsilon) < 1 - \varepsilon$$

and therefore,

$$\begin{aligned} \lambda^2 \mu (T^2 x_{i-1} , T^2 x_i , t) &= \lambda \ (\lambda \, \mu (T (Tx_{i-1}) , T (Tx_i) , t)) \\ &> \lambda \, \mu (Tx_{i-1} , Tx_i , t) > \lambda \, \varepsilon \end{aligned}$$

$$\Longrightarrow \ \mu (T^2 x_{i-1} , T^2 x_i , t) > \varepsilon ;$$

$$\begin{aligned} \frac{1}{\lambda^2} \nu (T^2 x_{i-1} , T^2 x_i , t) &= \frac{1}{\lambda} \left(\frac{1}{\lambda} \nu (T (Tx_{i-1}) , T (Tx_i) , t) \right) \\ &< \frac{1}{\lambda} \nu (Tx_{i-1} , Tx_i , t) < \frac{1}{\lambda} (1 - \varepsilon) \end{aligned}$$

$$\Longrightarrow \ \nu (T^2 x_{i-1} , T^2 x_i , t) < 1 - \varepsilon$$

In the similar way we have,

$$\begin{aligned} \lambda^3 \mu (T^3 x_{i-1} , T^3 x_i , t) &= \lambda^2 \left(\lambda \, \mu (T (T^2 x_{i-1}) , T (T^2 x_i) , t) \right) \\ &> \lambda^2 \mu (T^2 x_{i-1} , T^2 x_i , t) > \lambda^2 \varepsilon \end{aligned}$$

$$\Longrightarrow \ \mu (T^3 x_{i-1} , T^3 x_i , t) > \varepsilon$$

$$\cdots$$

$$\begin{aligned} \lambda^m \mu (T^m x_{i-1} , T^m x_i , t) &= \lambda^{m-1} \left(\lambda \, \mu (T (T^{m-1} x_{i-1}) , T (T^{m-1} x_i) , t) \right) \\ &> \lambda^{m-1} \mu (T^{m-1} x_{i-1} , T^{m-1} x_i , t) > \lambda^{m-1} \varepsilon \end{aligned}$$

$$\Longrightarrow \ \mu (T^m x_{i-1} , T^m x_i , t) > \varepsilon ;$$

$$\frac{1}{\lambda^3}\,\nu\,(\,T^3 x_{i-1}\,,\,T^3 x_i\,,\,t\,)\;=\;\frac{1}{\lambda^2}\,\left(\frac{1}{\lambda}\,\nu\,(\,T\,(\,T^2 x_{i-1}\,)\,,\,T\,(\,T^2 x_i\,)\,,\,t\,)\right)$$

$$<\;\frac{1}{\lambda^2}\,\nu\,(\,T^2 x_{i-1}\,,\,T^2 x_i\,,\,t\,)\;<\;\frac{1}{\lambda^2}\,(\,1\,-\,\varepsilon\,)$$

$$\Longrightarrow\;\;\nu\,(\,T^3 x_{i-1}\,,\,T^3 x_i\,,\,t\,)\;<\;(\,1-\varepsilon\,)$$

$$\cdots$$

$$\frac{1}{\lambda^m}\,\nu\,(\,T^m x_{i-1}\,,\,T^m x_i\,,\,t\,)\;=\;\frac{1}{\lambda^{m-1}}\,\left(\frac{1}{\lambda}\,\nu\,(\,T\,(\,T^{m-1} x_{i-1}\,)\,,\,T\,(\,T^{m-1} x_i\,)\,,\,t\,)\right)$$

$$<\;\frac{1}{\lambda^{m-1}}\,\nu\,(\,T^{m-1} x_{i-1}\,,\,T^{m-1} x_i\,,\,t\,)\;<\;\frac{1}{\lambda^{m-1}}\,(\,1-\varepsilon\,)$$

$$\Longrightarrow\;\;\nu\,(\,T^m x_{i-1}\,,\,T^m x_i\,,\,t\,)\;<\;1-\varepsilon$$

Now,

$$\begin{aligned}\mu\,(\,T^m x\,,\,T^{m+1} x\,,\,t\,)\;&=\;\mu\,(\,T^m x_0\,,\,T^m x_n\,,\,t\,)\\ &\geq\;\left(\mu\left(T^m x_0\,,\,T^m x_1\,,\,\frac{t}{n}\right)\,*\,\mu\left(T^m x_1\,,\,T^m x_2\,,\,\frac{t}{n}\right)\right.\\ &\qquad\left.*\;\cdots\;*\,\mu\left(T^m x_{n-1}\,,\,T^m x_n\,,\,\frac{t}{n}\right)\right)\\ &>\;\varepsilon\end{aligned}$$

i.e. $\mu\,(\,T^m x\,,\,T^{m+1} x\,,\,t\,)\;>\;\varepsilon$ *for all* $t\,>\,0$ *and for all* $m\,\in\,\mathbf{N}$;

and,

$$\begin{aligned}\nu\,(\,T^m x\,,\,T^{m+1} x\,,\,t\,)\;&=\;\nu\,(\,T^m x_0\,,\,T^m x_n\,,\,t\,)\\ &\leq\;\left(\nu\left(T^m x_0\,,\,T^m x_1\,,\,\frac{t}{n}\right)\,\diamond\,\nu\left(T^m x_1\,,\,T^m x_2\,,\,\frac{t}{n}\right)\right.\\ &\qquad\left.\diamond\;\cdots\;\diamond\,\nu\left(T^m x_{n-1}\,,\,T^m x_n\,,\,\frac{t}{n}\right)\right)\\ &<\;1\,-\,\varepsilon\end{aligned}$$

i.e. $\nu\,(\,T^m x\,,\,T^{m+1} x\,,\,t\,)\;<\;1-\varepsilon$ *for all* $t\,>\,0$ *and for all* $m\,\in\,\mathbf{N}$.

Now, for all $t\,>\,0$ and $j\,<\,k$ we have,

$$\begin{aligned}\mu\,(\,T^j x\,,\,T^k x\,,\,t\,)\;\geq\;&\mu\left(T^j x\,,\,T^{j+1} x\,,\,\frac{t}{k\,-\,j}\right)\,*\,\mu\left(T^{j+1} x\,,\,T^{j+2} x\,,\,\frac{t}{k\,-\,j}\right)\\ &*\;\cdots\;*\;\mu\left(T^{k-1} x\,,\,T^k x\,,\,\frac{t}{k\,-\,j}\right)\;>\;\varepsilon\,;\end{aligned}$$

$$\nu\,(T^{j}x\,,\,T^{k}x\,,\,t\,)\;\leq\;\nu\left(T^{j}x\,,\,T^{j+1}x\,,\,\frac{t}{k\,-\,j}\right)\diamond\;\nu\left(T^{j+1}x\,,\,T^{j+2}x\,,\,\frac{t}{k\,-\,j}\right)$$

$$\diamond\quad\cdots\quad\diamond\;\nu\left(T^{k-1}x\,,\,T^{k}x\,,\,\frac{t}{k\,-\,j}\right)\;<\;1-\varepsilon.$$

$\Longrightarrow$ $\{T^{j}x\}$ is a Cauchy sequence in $(X\,,\,A)$. Since $(X\,,\,A)$ is complete, there exists $\xi\,\in\,X$ such that $T^{i}x\,\longrightarrow\,\xi$ as $i\,\longrightarrow\,\infty$ in $(X\,,\,A)$. Again, since T is IF-uniformly locally contractive, it follows that T is t-uniformly continuous on X and hence by the lemma(3.6), we get

$$T\xi\;=\;\lim_{i\,\to\,\infty}T\,T^{i}x\;=\;\lim_{i\,\to\,\infty}T^{i+1}x\;=\;\xi$$

which shows that ξ is a fixed point of T.

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