Gen. Math. Notes, Vol. 4, No. 1, May 2011, pp. 1-15
ISSN 2219-7184; Copyright © ICSRS Publication, 2011
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Some Results on Intuitionistic Fuzzy Ideals in BCK-Algebras

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(Received: 20-11-10 /Accepted: 7-4-11)

Abstract

In this paper, we give some results on the intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals, intuitionistic fuzzy commutative ideals.

Keywords: *BCK-algebra, Fuzzy (implicative, positive implicative and commutative) ideal.*

1 Introduction

After the introduction of the concept of fuzzy sets by Zadeh [12] several researches were conducted on the generalizations of the notion of fuzzy sets. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1, 2] as a generalization of the notion of fuzzy set. The first author (together with Hong, Kim, Meng, Roh and Song) [3, 5, 6, 7] considered the fuzzification of ideals and sub- algebras in BCK-algebras (cf. [3, 4, 5, 6). In this paper we give some results on the intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals, intuitionistic fuzzy commutative ideals.

2 Preliminaries

First we present the fundamental definitions. By a BCK-algebra (see [7, 8, 9]) we mean a nonempty set X with a binary operation * and a constant 0 satisfying the axioms:

(BCK-1) $((x * y) * (x * z)) \leq (z * y)$,
(BCK-2) $(x * (x * y)) \leq y$,
(BCK-3) $x \leq x$,
(BCK-4) $x \leq y$ and $y \leq x$ imply that $x = y$,
(BCK-5) $0 \leq x$

for all $x, y, z \in X$.
A partial ordering "$\leq$" on X can be defined by $x \leq y$ if and only if $x * y = 0$. In any BCK-algebra X the following holds:

(P1) $x * 0 = x$
(P2) $x * y \leq x$
(P3) $(x * y) * z = (x * z) * y$
(P4) $(x * z) * (y * z) \leq x * y$
(P5) $x * (x * (x * y)) = x * y$
(P6) $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$, for all $x, y, z \in X$.

A BCK-algebra X is said to be implicative if $x = x * (y * x)$, for all $x,y \in X$.
A BCK-algebra X is said to be positive implicative if $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.
A BCK-algebra X is said to be commutative if $x * (x * y) = y * (y * x)$ for all $x, y, z \in X$.
A non-empty subset I of a BCK-algebra X is called an ideal of X,
(I_1) $0 \in I$
(I_2) $x * y$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.
A non-empty subset I of a BCK-algebra X is said to be sub-algebra of X if $x * y \in X$ whenever $x, y \in X$
A non-empty subset I of a BCK-algebra X is called an implicative ideal of X if it satisfies (I_1) and (I_3) $(x * (y * x)) * z \in I$ and $z \in I$ imply $x \in I$ for all $x, y, z \in X$.
A non-empty subset I of a BCK-algebra X is called a commutative ideal of X if it satisfies (I_1) and (I_4) $(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$ for $x, y, z \in X$.
A non-empty subset I of a BCK-algebra X is said to be positive implicative ideal of X if it satisfies (I_1) and (I_5) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$ for all $x, y, z \in X$.
Let μ and λ be the fuzzy sets in a set X. For s, t ε [0, 1], the set U (μ, s) = $\{x \in X /$ μ(x) ≥ s} is called a upper level of μ and the set L (λ, t) = { $x \in X$ / λ(x) ≤ t} is called a lower level of λ.

An intuitionistic fuzzy set A in a non-empty set X is an object having the form $A = \{x, \mu_A(x), \lambda_A(x) / x \in X\}$, where the function $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denoted the degree of membership (namely $\mu(x)$) and the degree of non membership (namely $\lambda(x)$) of each element $x \in X$ to the set A respectively, and $0 \le \mu_A(x) + \lambda_A(x) \le 1$ for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (X, \mu_A, \lambda_A)$ or $A = (\mu_A, \lambda_A)$.

Definition 2.1. *Let* $A = (\mu_A, \lambda_A)$ *and* $B = (\mu_B, \lambda_B)$ *be intuitionistic fuzzy sets in X. Then*

(i) $A = \{(x, \mu_A(x), \bar{\mu}_A(x)) / x \in X\}$

(ii) $\Diamond A = \{(x, \bar{\lambda}_A(x), \lambda_A(x)) / x \in X\}$.

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 2.2. *An IFS* $A = (X, \mu_A, \lambda_A)$ *in X is an intuitionistic fuzzy sub-algebra of X if it satisfies*

(IFS 1) $\mu_A(x * y) \ge \min\{\mu_A(x), \mu_A(y)\}$
(IFS 2) $\lambda_A(x * y) \le \max\{\lambda_A(x), \lambda_A(y)\}$ *for all* $x, y \in X$.

Example 2.3. *Consider a BCK-algebra X = {0, a, b, c} with the following Cayley table:*

*	0	*a*	*b*	*c*
0	0	0	0	0
a	*a*	0	0	*a*
b	*b*	*a*	0	*b*
c	*c*	*c*	*c*	0

Let $A = (X, \mu_A, \lambda_A)$ *be an IFS in X defined by*

$$\mu_A(0) = \mu_A(a) = \mu_A(c) = 0.7 > 0.3 = \mu_A(b)$$

and

$$\lambda_A(0) = \lambda_A(a) = \lambda_A(c) = 0.2 < 0.5 = \lambda_A(b).$$

Then $A = (X, \mu_A, \lambda_A)$ *is an IF subalgebra of X.*

Proposition 2.4. *Let* $A = (X, \mu_A, \lambda_A)$ *be an intuitionistic fuzzy sub-algebra of X, then*

$\mu_A(0) \ge \mu_A(x)$ *and* $\lambda_A(0) \le \lambda_A(x)$ *for all* $x \in X$.

Definition 2.5. *An IF* $A = (X, \mu_A, \lambda_A)$ *in X is an intuitionistic fuzzy ideal (IF-ideal) of X if it satisfies*

(IF1) $\mu_A(0) \geq \mu_A(x)$ *and* $\lambda_A(0) \leq \lambda_A(x)$
(IF2) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(x)\}$
(IF3) $\lambda_A(x) \leq \min\{\lambda_A(x * y), \lambda_A(y)\}$, *for all* $x, y \in X$.

Theorem 2.6. *[4]Let* $A = (X, \mu_A, \lambda_A)$ *be an intuitionistic fuzzy ideal of X. If* $x \leq y$ *in X, then*

$$\mu_A(x) \geq \mu_A(y), \ \ \lambda_A(x) \leq \lambda_A(y),$$

that is μ_A *is order-reversing and* λ_A *is order-preserving.*

Theorem 2.7. *[4]Every intuitionistic fuzzy ideal of X is an intuitionistic fuzzy sub-algebra of X.*

Theorem 2.8. *[4]* $A = (X, \mu_A, \lambda_A)$ *is an intuitionistic fuzzy ideal of X if and only if for*

$$x, y, z \in X, x * y \leq z \Rightarrow \mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\} \text{ and } \lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(z)\}.$$

Proposition 2.9. *[4]* $A = (X, \mu_A, \lambda_A)$ *is an intuitionistic fuzzy ideal of X if and only if the non-empty upper s-level cut* $U(\mu_A; s)$ *and the non-empty lower t-level cut* $L(\lambda_A; t)$ *are ideals of X, for any* $s, t \in [0,1]$.

Corollary 2.10. $A = (X, \mu_A, \lambda_A)$ *is an intuitionistic fuzzy subalgebra of X if and only if the non-empty upper s-level cut* $U(\mu_A; s)$ *and the non-empty lower t-level cut* $L(\lambda_A; t)$ *are sub-algebras of X, for any* $s, t \in [0,1]$.

Proposition 2.11. *[11]In a BCK-algebra X, the following holds, for all* $x, y, z \in X$,

(i) $((x * z) * z) * (y * z) \leq (x * y) * z$.
(ii) $(x * z) * (x * (x * z)) = (x * z) * z$
(iii) $(x * (y * (y * x))) * (y * (x * (y * (y * x)))) \leq x * y$.

3 Main Results

In this section we present the results on the intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals and intuitionistic fuzzy commutative ideals.

Definition 3.1. *[11]An IFS* $A=(X,\mu_A,\lambda_A)$ *in a BCK-algebra X is an intuitionistic fuzzy implicative ideal (IFI-ideal) of X if it satisfies*

(IFI 1) $\mu_A(0)\geq\mu_A(x)$ *and* $\lambda_A(0)\leq\lambda_A(x)$
(IFI 2) $\mu_A(x)\geq\min\{\mu_A((x*(y*x))*z),\mu_A(z)\}$
(IFI 3) $\lambda_A(x)\leq\max\{\lambda_A((x*(y*x))*z),\lambda_A(z)\}$, *for all* $x,y,z\in X$.

Definition 3.2. *[11]An IFS* $A=(X,\mu_A,\lambda_A)$ *in X is an intuitionistic fuzzy commutative ideal (IFCI-ideal) of X if it satisfies*

(IFCI 1) $\mu_A(0)\geq\mu_A(x)$ *and* $\lambda_A(0)\leq\lambda_A(x)$
(IFCI 2) $\mu_A(x*(y*(y*x))\geq\min\{\mu_A((x*y)*z),\mu_A(z)\}$
(IFCI 3) $\lambda_A(x*(y*(y*x))\leq\max\{\lambda_A((x*y)*z),\lambda_A(z)\}$ *for all* $x,y,z\in X$.

Definition 3.3. *[11]An IFS* $A=(X,\mu_A,\lambda_A)$ *in a BCK-algebra X is an intuitionistic fuzzy positive implicative ideal (IFPI-ideal) of X if it satisfies*

(IFPI 1) $\mu_A(0)\geq\mu_A(x)$ *and* $\lambda_A(0)\leq\lambda_A(x)$
(IFPI 2) $\mu_A(x*z)\geq\min\{\mu_A((x*y)*z),\mu_A(y*z)\}$
(IFPI 3) $\lambda_A(x*z)\leq\max\{\lambda_A((x*y)*z),\lambda_A(y*z)\}$ *for all* $x,y,z\in X$.

Theorem 3.4. *An intuitionistic fuzzy ideal* $A=(X,\mu_A,\lambda_A)$ *of X is an intuitionistic fuzzy implicative if and only if A is both intuitionistic commutative and intuitionistic fuzzy positive implicative.*

Proof: Assume that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy implicative ideal of X. By (2.11(i) and 2.8), we have

$$\begin{aligned}\min\{\mu_A((x*y)*z),\mu_A(y*z)\}&\leq\mu_A((x*z)*z)\\&=\mu_A((x*z)*(x*(x*z)))\quad\text{(by 2.11(ii))}\\&=\mu_A(x*z)\quad\text{(by [11, 3.7(iii)])}\end{aligned}$$

and
$$\begin{aligned}\max\{\lambda_A((x*y)*z),\lambda_A(y*z)\}&\geq\lambda_A((x*z)*z)\\&=\lambda_A((x*z)*(x*(x*z)))\\&=\lambda_A(x*z),\text{for all }x,y,z\in X.\end{aligned}$$

Then $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of X. And by theorem 2.6, 2.11(iii) and 3.7(iii),

$$\mu_A(x*y)\leq\mu_A((x*(y*(y*x)))*(y*(x*(y*(y*x)))))=\mu_A(x*(y*(y*x)))$$

and

$$\lambda_A(x*y) \geq \lambda_A((x*(y*(y*x)))*(y*(x*(y*(y*x))))) = \lambda_A(x*(y*(y*x))).$$

It follows from [11, 4.6] that $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy commutative. Conversely, suppose that $A=(X,\mu_A,\lambda_A)$ is both intuitionistic fuzzy positive implicative and intuitionistic fuzzy commutative.
Since, $(y*(y*x))*(y*x) \leq x*(y*x)$, it follows from theorem 2.6.

$$\mu_A(y*(y*x))*(y*x)) \geq \mu_A(x*(y*x)) \text{ and } \lambda_A(y*(y*x))*(y*x)) \leq \lambda_A(x*(y*x)).$$

Using [11, 5.8], we have

$$\mu_A(y*(y*x))*(y*x)) = \mu_A(y*(y*x))$$

and

$$\lambda_A(y*(y*x))*(y*x)) = \lambda_A(y*(y*x)).$$

Therefore

$$\mu_A(x*(y*x)) \leq \mu_A(y*(y*x)) \text{ and } \lambda_A(x*(y*x)) \geq \lambda_A(y*(y*x)) \ldots \quad (1)$$

On the other hand since $x*y \leq x*(y*x)$, we have, by theorem 2.6

$$\mu_A(x*y) \geq \mu_A(x*(y*x)) \text{ and } \lambda_A(x*y) \leq \lambda_A(x*(y*x)).$$

Since $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy commutative ideal of X, by [11, 4.7] we have

$$\mu_A(x*y) = \mu_A(x*(y*(y*x))) \text{ and } \lambda_A(x*y) = \lambda_A(x*(y*(y*x))).$$

Hence

$$\mu_A(x*(y*x)) \leq \mu_A(x*(y*(y*x))) \text{ and } \lambda_A(x*(y*x)) \geq \lambda_A(x*(y*(y*x))) \ldots (2)$$

Combining (1) and (2), we obtain

$$\mu_A(x*(y*x) \leq \min\{\mu_A(x*(y*(y*x))), \mu_A(y*(y*x))\} \leq \mu_A(x)$$

and

$$\lambda_A(x*(y*x) \geq \max\{\lambda_A(x*(y*(y*x))), \lambda_A(y*(y*x))\} \geq \lambda_A(x).$$

So $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy implicative ideal of X. The proof is complete.

Theorem 3.5. *If* $A=(X,\mu_A,\lambda_A)$ *is an intuitionistic fuzzy ideal of X with the following conditions holds*

(i) $\mu_A(x*y) \geq \min\{\mu_A(((x*y)*y)*z), \mu_A(z)\}$
(ii) $\lambda_A(x*y) \leq \max\{\lambda_A(((x*y)*y)*z), \lambda_A(z)\}$, *for all* $x,y,z \in X$. *Then A is intuitionistic fuzzy positive implicative ideal of X.*

Proof: Suppose $A = (X, \mu_A, \lambda_A)$ is intuitionistic fuzzy ideal of X. with condition (i) and (ii). Using (P3) and (P4), we have

$$((x * z) * z) * (y * z)) \leq (x * z) * y = (x * y) * z, \text{ for all } x, y, z \in X,$$

therefore by theorem 2.6

$$\mu_A(((x * z) * z) * (y * z))) \geq \mu_A((x * y) * z)$$

And

$$\lambda_A(((x * z) * z) * (y * z))) \leq \lambda_A((x * y) * z).$$

Now

$$\mu_A(x * z) \geq \min\{\mu_A(((x * z) * z) * (y * z)), \mu_A(y * z)\}$$
$$\geq \min\{\mu_A((x * y) * z), \mu_A(y * z)\}, \text{ for all } x, y, z \in X$$

and

$$\lambda_A(x * z) \leq \max\{\lambda_A(((x * z) * z) * (y * z)), \lambda_A(y * z)\}$$
$$\leq \max\{\lambda_A((x * y) * z), \lambda_A(y * z)\}, \text{ for all } x, y, z \in X.$$

Hence $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of X.

Lemma 3.6. *Let* $A = (X, \mu_A, \lambda_A)$ *be a fuzzy ideal of X, then A is an intuitionistic fuzzy positive implicative ideal of X if and only if*

$$\mu_A((x * z) * (y * z)) \geq \mu_A((x * y) * z)) \textit{ and } \lambda_A((x * z) * (y * z)) \leq \lambda_A((x * y) * z)),$$

for all $x, y, z \in X$.

Proof: Suppose that $A = (X, \mu_A, \lambda_A)$ is a fuzzy ideal of X and

$$\mu_A((x * z) * (y * z)) \geq \mu_A((x * y) * z)) \text{ and } \lambda_A((x * z) * (y * z)) \leq \lambda_A((x * y) * z)),$$

for all $x, y, z \in X$ Therefore

$$\mu_A(x * z) \geq \min\{\mu_A((x * z) * (y * z)), \mu_A(y * z)\} \quad \geq \min\{\mu_A((x * y) * z), \mu_A(y * z)\}$$

$$\lambda_A(x * z) \leq \max\{\lambda_A((x * z) * (y * z)), \lambda_A(y * z)\} \leq \max\{\lambda_A((x * y) * z), \lambda_A(y * z)\},$$

for all $x, y, z \in X$. Thus A is an intuitionistic fuzzy positive implicative ideal of X. Conversely, assume that $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of X implies that $A = (X, \mu_A, \lambda_A)$ is an IF-ideal of X.

Let $a = x * (y * z)$ and $b = x * y$,

Since $((x*(y*z))*(x*y)) \leq y*(y*z)$,

we have that

$$\mu_A((a*b)*z) = \mu_A(((x*(y*z))*(x*y)*z) \geq \mu_A((y*(y*z))*z) = \mu_A(0)$$

and so,

$$\mu_A((x*z)*(y*z)) = \mu_A((x*(y*z)*z) = \mu_A(a*z)$$
$$\geq \min\{\mu_A((a*b)*z), \mu_A(b*z)\} \geq \min\{\mu_A(0), \mu_A(b*z)\}$$
$$= \mu_A(b*z) = \mu_A((x*y)*z).$$

Therefore

$$\mu_A((x*z)*(y*z)) \geq \mu_A((x*y)*z), \text{ for all } x, y, z \in X.$$

And

$$\lambda_A((a*b)*z) = \lambda_A(((x*(y*z))*(x*y)*z) \leq \lambda_A((y*(y*z))*z) = \lambda_A(0)$$

And so,

$$\lambda_A((x*z)*(y*z)) = \lambda_A((x*(y*z)*z) = \lambda_A(a*z)$$
$$\leq \max\{\lambda_A((a*b)*z), \lambda_A(b*z)\} \leq \max\{\lambda_A(0), \lambda_A(b*z)\}$$
$$= \lambda_A(b*z) = \lambda_A((x*y)*z).$$

Therefore

$$\lambda_A((x*z)*(y*z)) \leq \lambda_A((x*y)*z), \text{ for all } x, y, z \in X.$$

Thus

$$\mu_A((x*z)*(y*z)) \geq \mu_A((x*y)*z)), \ \lambda_A((x*z)*(y*z)) \leq \lambda_A((x*y)*z)),$$

for all $x, y, z \in X$.

Theorem 3.7. *If* $A = (X, \mu_A, \lambda_A)$ *is intuitionistic fuzzy positive implicative ideal of X then (PI 1) for any*

$$x,y,a,b \in X, ((x*y)*y)*a \leq b \Rightarrow \mu_A(x*y) \geq \min\{\mu_A(a), \mu_A(b)\}$$

and

$$\lambda_A(x*y) \leq \max\{\lambda_A(a), \lambda_A(b)\}.$$

(PI 2) For any

$$x,y,z,a,b \in X, ((x*y)*z)*a \leq b \Rightarrow \mu_A((x*z)*(y*z)) \geq \min\{\mu_A(a), \mu_A(b)\}$$

and

$$\lambda_A((x*z)*(y*z)) \leq \max\{\lambda_A(a), \lambda_A(b)\}.$$

Proof: Suppose, $A = (X, \mu_A, \lambda_A)$ is intuitionistic fuzzy positive implicative ideal of X.

(PI1). Let $x, y, z \in X$ be such that $((x*y)*y)*a \leq b$. Using 2.6,

we have

$$\mu_A((x*y)*y) \geq \min\{\mu_A(a), \mu_A(b)\} \text{ and } \lambda_A((x*y)*y) \leq \max\{\lambda_A(a), \lambda_A(b)\}.$$

It follows that

$$\mu_A(x*y) \geq \min\{\mu_A((x*y)*y), \mu_A(y*y)\} = \min\{\mu_A((x*y)*y), \mu_A(0)\}$$

$$= \mu_A((x*y)*y) \geq \min\{\mu_A(a), \mu_A(b)\}.$$

And

$$\lambda_A(x*y) \leq \max\{\lambda_A((x*y)*y), \lambda_A(y*y)\}$$

$$= \max\{\lambda_A((x*y)*y), \lambda_A(0)\} = \lambda_A((x*y)*y) \leq \max\{\lambda_A(a), \lambda_A(b)\}.$$

(ii) Now let $x, y, z \in X$ be such that $((x*y)*z)*a \leq b$.
Since $A = (X, \mu_A, \lambda_A)$ intuitionistic fuzzy positive implicative ideal of X, it follows from known lemma 3.6,

$$\mu_A((x*z)*(y*z)) \geq \mu_A((x*y)*z) \geq \min\{\mu_A(a), \mu_A(b)\}$$

and

$$\lambda_A((x*z)*(y*z)) \leq \lambda_A((x*y)*z) \leq \max\{\lambda_A(a), \lambda_A(b)\}$$

This completes the proof.

Theorem.3.8. *Let* $A = (X, \mu_A, \lambda_A)$ *be IFS in X satisfying the condition*

$$((x*y)*y)*a \leq b \Rightarrow \mu_A(x*y) \geq \min\{\mu_A(a), \mu_A(b)\}$$

and

$$\lambda_A(x*y) \leq \max\{\lambda_A(a), \lambda_A(b)\},$$

for any $x, y, a, b \in X$, *Then* $A = (X, \mu_A, \lambda_A)$ *intuitionistic fuzzy positive implicative ideal of X.*

Proof: First we prove that $A = (X, \mu_A, \lambda_A)$ is an IF-ideal of X.
Let $x, y, z \in X$ be such that $x*y \leq z$.

Then $(((x*0)*0)*y)*z = (x*y)*z) = 0$, that is $(((x*0)*0)*y) \leq z$

Since, for $x,y,a,b \in X$,

$$((x*y)*y)*a \leq b \Rightarrow \mu_A(x*y) \geq \min\{\mu_A(a), \mu_A(b)\}$$

and

$$\lambda_A(x*y) \leq \max\{\lambda_A(a), \lambda_A(b)\}$$

Put $y = 0, a = y, b = z$,

we get

$$\mu_A(x) = \mu(x * 0) \geq \min\{\mu_A(y), \mu_A(z)\}$$

and

$$\lambda_A(x) = \lambda_A(x * 0) \leq \max\{\lambda_A(y), \lambda_A(z)\}.$$

It follows that A =(X, μ_A, λ_A) is IF- ideal of X.

Note that

$$(((x * y) * y) * ((x * y) * y)) * 0 = 0$$

implies

$$(((x * y) * y) * ((x * y) * y)) \leq 0, \forall x, y \in X.$$

From hypothesis we have

$$\mu_A(x * y) \geq \min\{\mu_A((x * y) * y), \mu_A(0)\} = \mu_A((x * y) * y)$$

and

$$\lambda_A(x * y) \leq \max\{\lambda_A((x * y) * y), \lambda_A(0)\} = \lambda_A((x * y) * y).$$

And so $A = (X, \mu_A, \lambda_A)$ is intuitionistic fuzzy positive implicative ideal of X.

Theorem 3.9. *Let* $A = (X, \mu_A, \lambda_A)$ *be an IFS in X satisfying* $((x * y) * z) * a \leq b$ *imply* $\mu_A((x * y) * (y * z)) \geq \min\{\mu_A(a), \mu_A(b)\}$ *and* $\lambda_A((x * y) * (y * z)) \leq \max\{\lambda_A(a), \lambda_A(b)\}$ *for any* $x, y, z, a, b \in X$.

Then $A = (X, \mu_A, \lambda_A)$ *is an intuitionistic fuzzy positive implicative ideal of X.*

Proof: Let $x, y, a, b \in X$ be such that $((x * y) * y) * a \leq b$,

that is

$$(((x * y) * y) * a) * b = 0$$

therefore

$$\mu_A(x * y) = \mu_A((x * y) * 0) = \mu_A((x * y) * (y * y)) \geq \min\{\mu_A(a), \mu_A(b)\}$$

And

$$\lambda_A(x * y) = \lambda_A((x * y) * 0) = \lambda_A((x * y) * (y * y)) \geq \min\{\lambda_A(a), \lambda_A(b)\}.$$

It follows from 3.8, $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of X.

Theorem 3.10. *Let* $A = (X, \mu_A, \lambda_A)$ *be an intuitionistic fuzzy positive implicative ideal of BCK-algebra X, then so is* $A = (X, \mu_A, \bar{\mu}_A)$.

Proof: We have $\mu_A(0) \geq \mu_A(x) \Rightarrow 1 - \bar{\mu}_A(0) \geq 1 - \bar{\mu}_A(x) \Rightarrow \bar{\mu}_A(0) \leq \bar{\mu}_A(x), \forall x \in X$.

Consider for any $x, y, z \in X$,

$$\mu_A(x*z) \geq \min\{\mu_A((x*y)*z), \mu_A(y*z)\}$$
$$\Rightarrow 1-\overline{\mu}_A(x*z) \geq \min\{1-\overline{\mu}_A((x*y)*z), 1-\overline{\mu}_A(y*z)\}$$
$$\Rightarrow \overline{\mu}_A(x*z) \leq 1-\min\{1-\overline{\mu}_A((x*y)*z), 1-\overline{\mu}(y*z)\}$$
$$\Rightarrow \overline{\mu}_A(x*z) \leq \max\{\overline{\mu}_A((x*y)*z), \overline{\mu}_A(y*z)\}$$

Hence $A=(X,\mu_A,\overline{\mu}_A)$ is an intuitionistic fuzzy positive implicative ideal of BCK-algebra X.

Theorem 3.11. *Let* $A=(X,\mu_A,\lambda_A)$ *be an intuitionistic fuzzy positive implicative ideal of BCK-algebra X then so is* $\Diamond A=(X,\overline{\lambda}_A,\lambda_A)$.

Proof: We have $\lambda_A(0) \leq \lambda_A(x) \Rightarrow 1-\overline{\lambda}_A(0) \leq 1-\overline{\lambda}_A(x) \Rightarrow \lambda_A(0) \geq \lambda_A(x), \forall x \in X$.
Consider for any $x, y, z \in X$

$$\lambda_A(x*z) \leq \max\{\lambda_A((x*y)*z), \lambda_A(y*z)\}$$
$$\Rightarrow 1-\overline{\lambda}_A(x*z) \leq \max\{1-\overline{\lambda}_A((x*y)*z), 1-\overline{\lambda}_A(y*z)\}$$
$$\Rightarrow \overline{\lambda}_A(x*z) \leq 1-\max\{1-\overline{\lambda}_A((x*y)*z), 1-\lambda_A(y*z)\}$$
$$\Rightarrow \overline{\lambda}_A(x*z) \geq \min\{\overline{\lambda}_A((x*y)*z), \overline{\lambda}_A(y*z)\}.$$

Hence $\Diamond A=(X,\overline{\lambda}_A,\lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of BCK-algebra X.

Theorem 3.12. $A=(X,\mu_A,\lambda_A)$ *is an intuitionistic fuzzy positive implicative ideal of BCK-algebra X if and only if* $A=(X,\mu_A,\overline{\mu}_A)$ *and* $\Diamond A=(X,\overline{\lambda}_A,\lambda_A)$ *are intuitionistic fuzzy positive implicative ideal of BCK-algebra.*

Theorem 3.13. $A=(X,\mu_A,\lambda_A)$ *is an intuitionistic fuzzy positive implicative ideal of BCK-algebra X if and only if the non-empty upper s-level cut* $U(\mu_A;s)$ *and the non-empty lower t-level cut* $L(\lambda_A;t)$ *are PI-ideals of X, for any* $s,t \in [0,1]$.

Proof: Suppose $A=(X,\mu_A,\lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of X and $U(\mu_A;s) \neq \phi$ for any $s \in [0,1]$. It is clear that for any $x \in X$,

$$\mu_A(0) \geq \mu_A(x) \Rightarrow \mu_A(0) \geq \mu_A(x) \geq s \Rightarrow \mu_A(0) \geq s \text{ implies } 0 \in U(\mu_A;s).$$

Furthermore if $(x*y)*z \in U(\mu_A;s), y*z \in U(\mu_A;s)$
implies

$$\mu_A((x*y)*z)) \geq s \text{ and } \mu_A(y*z) \geq s.$$

Therefore

$$\mu_A(x*z) \geq \min\{\mu_A((x*y)*z), \mu_A(y*z)\} \geq \min\{s,s\} = s$$

implies $x*z \in U(\mu_A;s)$.

This shows that $U(\mu_A;s)$ is positive implicative ideal of X.
Similarly, we can prove $L(\lambda_A, t)$ is positive implicative ideal of X, $\forall s, t \in [0,1]$
Conversely, assume that for any $s,t \in [0,1]$, $U(\mu_A;s)$ and $L(\lambda_A, t)$ are either empty or positive implicative ideals of X.

Put $\mu_A(x) = s$, $\lambda_A(x) = t$ for any $x \in X$.

Since $0 \in U(\mu_A;s) \Rightarrow \mu_A(0) \geq s = \mu_A(x)$ and $0 \in L(\lambda_A, t) \Rightarrow \lambda_A(0) \leq t = \lambda_A(x)$

thus

$$\mu_A(0) \geq \mu_A(x) \text{ and } \lambda_A(0) \leq \lambda_A(x) \text{ for all } x \in X.$$

Now we only need to show that (IFPI 3),

then take $s_1 = \min\{\mu_A((x*y)*z), \mu_A(y*z)\} \Rightarrow (x*y)*z, y*z \in U(\mu_A;s_1)$.

Since $U(\mu_A;s_1)$ is implicative ideal of X

we have

$$y*z \in U(\mu_A;s_1) \Rightarrow \mu_A(x*z) \geq s_1 = \min\{\mu_A((x*y)*z), \mu_A(y*z)\}.$$

Therefore

$$\mu_A(x*z) \geq \min\{\mu_A((x*y)*z), \mu_A(y*z)\} \text{ for all } x, y, z \in X$$

Similarly we can prove $\lambda_A(x*z) \leq \min\{\lambda_A((x*y)*z), \lambda_A(y*z)\}$ for all $x, y, z \in X$.

Hence $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy positive implicative ideal of BCK-algebra X.

Theorem 3.14. $A = (X, \mu_A, \lambda_A)$ *is an intuitionistic fuzzy implicative or commutative ideals of BCK-algebra X if and only if the non-empty upper s-level cut* $U(\mu_A;s)$ *and the non-empty lower t-level cut* $L(\lambda_A;t)$ *are implicative or commutative ideals of X, for any* $s, t \in [0,1]$.

Corollary 3.15. $A = (X, \mu_A, \lambda_A)$ *is an intuitionistic fizzy implicative ideal of BCK-algebra X if and only if the non-empty upper s-level cut* $U(\mu_A;s)$ *and the non-empty* *lower t-level cut* $L(\lambda_A;t)$ *are both commutative and positive ideals of X, for any* $s,t \in [0,1]$.

Corollary 3.16. $A=(X,\mu_A,\lambda_A)$ *is an intuitionistic fuzzy commutative and intuitionistic fuzzy positive implicative ideals of BCK-algebra X if and only if the non-empty upper s-level cut* $U(\mu_A;s)$ *and the non-empty lower t-level cut* $L(\lambda_A;t)$ *are implicative ideals of X, for any* $s,t \in [0,1]$.

Theorem 3.17. *Let* $A=(X,\mu_A,\lambda_A)$ *be an IFS of a BCK-algebra. If A is an intuitionistic fuzzy positive implicative ideal of X then the set* $J=\{x\in X/\mu_A(x)=\mu_A(0)\}$ *and* $K=\{x\in X/\lambda_A(x)=\lambda_A(0)\}$ *are an PI-ideal of X.*

Proof: Assume that $A=(X,\mu_A,\lambda_A)$ intuitionistic fuzzy positive implicative ideal of X. Since, $\mu_A(0)=\mu_A(0)\Rightarrow 0\in J$.

If $(x*y)*z, y*z\in J \Rightarrow \mu_A((x*y)*z)=\mu_A(0)$ and $\mu_A(y*z)=\mu_A(0)$.

Since

$$\mu_A(x*z)\geq \min\{\mu_A((x*y)*z),\mu_A(y*z)\}=\min\{\mu_A(0),\mu_A(0)\}=\mu_A(0),$$

but,

$$\mu_A(x*z)\leq \mu_A(0). \text{ Therefore, } \mu_A(x*z)=\mu_A(0)\Rightarrow x*z\in J.$$

Thus, J is an implicative ideal of X and $\lambda_A(0)=\lambda_A(0)\Rightarrow 0\in K$

If $(x*y)*z, y*z\in K$

Then

$$\lambda_A((x*y)*z)=\lambda_A(0)$$

And

$$\lambda_A(y*z)=\lambda_A(0).$$

Since,

$$\lambda_A(x*z)\leq \max\{\lambda_A((x*y)*z),\lambda_A(y*z)\}=\max\{\lambda_A(0),\lambda_A(0)\}=\lambda_A(0)$$

but,

$$\lambda_A(x*z)\geq \lambda_A(0).$$

Therefore, $\lambda_A(x*z)=\lambda_A(0)\Rightarrow x*z\in K$.
Thus, K is an implicative ideal of X

Theorem 3.18. *(Extension property for intuitionistic fuzzy positive implicative ideals) Let* $A=(X,\mu_A,\lambda_A)$ *and* $B=(X,\mu_B,\lambda_B)$ *are two fuzzy ideals of X such that* $A(0)=B(0)$ *and* $A\subseteq B$ *(that is* $\mu_A(0)=\mu_B(0),\lambda_A(0)=\lambda_B(0)$ *and* $\mu_A(x)\leq\mu_B(x)$,

$\lambda_A(x) \geq \lambda_B(x), \forall x \in X$*). If* $A = (X, \mu_A, \lambda_A)$ *is an intuitionistic fuzzy positive implicative ideal of X,, then so is B.*

Proof: Suppose that $A = (X, \mu_A, \lambda_A)$ is intuitionistic fuzzy positive implicative ideal of X

$$\begin{aligned}
\mu_B(((x*z)*(y*z))*((x*y)*z)) &= \mu_B(((x*z)*((x*y)*z))*(y*z)) && \text{(by P2)}\\
&= \mu_B(((x*((x*y)*z))*z)*(y*z)) && \text{(by P2)}\\
&\geq \mu_A(((x*((x*y)*z))*z)*(y*z)) && (\text{Since } \mu_A \subseteq \mu_B)\\
&\geq \mu_A(((x*((x*y)*z))*y)*z) && \text{(by lemma 3.6)}\\
&= \mu_A(((x*y)*((x*y)*z)*z) && \text{(by P2)}\\
&= \mu_A(((x*y)*z)*((x*y)*z)) && \text{(by P2)}\\
&= \mu_A(0) = \mu_B(0) && \text{(by BCK-3).}
\end{aligned}$$

It follows from (F1) and (F2) that

$$\mu_B((x*z)*(y*z)) \geq \min\{\mu_B(((x*z)*(y*z)*((x*y)*z)), \mu_B((x*y)*z)\}$$

$$\geq \min\{\mu_B(0), \mu_B((x*y)*z))\} = \mu_B((x*y)*z) \text{ for all } x, y, z \in X.$$

Therefore, for any $x, y, z \in X$, $\mu_B((x*z)*(y*z)) \geq \mu_B((x*y)*z)$ and

$$\begin{aligned}
\lambda_B(((x*z)*(y*z))*((x*y)*z)) &= \lambda_B(((x*z)*((x*y)*z))*(y*z)) && \text{(by P2)}\\
&= \lambda_B(((x*((x*y)*z))*z)*(y*z)) && \text{(by P2)}\\
&\leq \lambda_A(((x*((x*y)*z))*z)*(y*z)) && (\text{Since } \lambda_B \subseteq \lambda_A)\\
&\leq \lambda_A(((x*((x*y)*z))*y)*z) && \text{(by 3.6)}\\
&= \lambda_A(((x*y)*((x*y)*z)*z) &&\\
&= \lambda_A(((x*y)*z)*((x*y)*z)) &&\\
&= \lambda_A(0) = \lambda_B(0) && \text{(by BCK-3)}
\end{aligned}$$

It follows from (F1) and (F2) that

$$\lambda_B((x*z)*(y*z)) \leq \max\{\lambda_B(((x*z)*(y*z)*((x*y)*z)), \lambda_B((x*y)*z)\}$$

$$\leq \max\{\lambda_B(0), \lambda_B((x*y)*z))\} = \lambda_B((x*y)*z) \text{ for all } x, y, z \in X.$$

Therefore $\lambda_B((x*z)*(y*z)) \leq \lambda_B((x*y)*z)$, for all $x, y, z \in X$.
Hence $B = (X, \mu_B, \lambda_B)$ is an intuitionistic fuzzy positive implicative ideal of X.

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