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On the Fine Spectrum of the Forward Difference Operator on the Hahn Space

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Abstract

The main purpose of this paper is to determine the fine spectrum with respect to Goldberg's classification of the difference operator over the sequence space h. As a new development, we give the approximate point spectrum, defect spectrum and compression spectrum of the difference operator on the sequence space h.

Keywords: *Spectrum of an operator, spectral mapping theorem, the Hahn sequence space, Goldberg's classification, forward difference operator.*

1 Introduction

Spectral Theory is one of the most important mathematics fields that can be applied to branches of science and technology. It has been seen to be a very useful tool because of its convenient and easy applicability of the different fields such as numerical analysis, probability theory, quantum mechanics, structural mechanics, aeronautics, electrical engineering, ecology etc. We can give some examples of the spectral theory of how to use these and other disciplines: The flow over a wing may have determined whether laminar or turbulent by the spectral values in aeronautics. The frequency response of an amplifier or the reliability of a power system may have identified by the Spectral Theory in electrical engineering. The rate of convergence of a Markov process may have computed by the Spectral Theory in the Probability Theory. A food network may have defined whether or not settle into a steady equilibrium by the spectral values in ecology. The atomic energy levels and thus, the frequency of a laser or the spectral signature of a star may have determined by the Spectral Theory in quantum mechanics. Also, the spectral values may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge, in numerical analysis and it may determine whether an automobile is too noisy or whether a building will collapse in an earthquake, in structural mechanics.

There are several studies about the spectrum of the linear operators defined by some triangle matrices over certain sequence spaces. This long-time behavior was intensively studied over many years, starting with the work by Wenger [30], who established the fine spectrum of the integer power of the Cesàro operator in c. The generalization of [30] to the weighted mean methods is due to Rhoades [27]. The study of the fine spectrum of the operator on the sequence space ℓ_p, $(1 < p < \infty)$ was initiated by Gonzàlez [14]. The method of the spectrum of the Cesàro operator prepared by Reade [26], Akhmedov and Başar [1], and Okutoyi [22], respectively, whose established to this idea on the sequence spaces c_0 and bv.In [32], the fine spectrum of the Rhaly operators on the sequence spaces c_0 and c was given. The spectrum and fine spectrum for p-Cesàro operator acting on the space c_0 was studied by Coşkun [9]. The investigation of the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c was made by Malafosse [21] and Altay and Başar [4], respectively, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k\in\mathbb{N}}|x_k|/r^k$, $(r > 0)$. The idea of the fine spectrum applied to the Zweier matrix which is a band matrix as an operator over the sequence spaces ℓ_1 and bv by Altay and Karakus [5]. Let Δ_ν is double sequential band matrix on ℓ_1 such that $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_k)$. The spectra and the fine spectra of matrix Δ_ν were determined by Srivastava and Kumar [28]. Afterwards, these results of the double sequential band matrix Δ_ν generalized to the double sequential band matrix Δ_{uv} such that defined by $\Delta_{uv}x = (u_nx_n + v_{n-1}x_{n-1})_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$ (see [29]). In [6], the fine spectra of the Toeplitz operators represented by an upper and lower triangular n-band infinite matrices, over the sequence spaces c_0 and c was computed. The fine spectra of upper triangular double-band matrices over the sequence spaces c_0 and c was obtained by Karakaya and Altun [16]. Let $\Delta_{a,b}$ is a double band matrix with the convergent sequences $\widetilde{a} = (a_k)$ and $\widetilde{b} = (b_k)$ having certain properties, over the sequence space c. The fine spectrum of the matrix $\Delta_{a,b}$

was examined by Akhmedov and El-Shabrawy [3]. The approach to the fine spectrum with respect to Goldberg's classification studied with of the operator $B(r,s,t)$ defined by a triple band matrix over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$ by Furkan et al. [11]. Quite recently, the fine spectrum with respect to Goldberg's classification of the operator defined by the lambda matrix over the sequence spaces c_0 and c computed by Yeşilkayagil and Başar [31].

Hahn sequence space is defined as

$$h = \left\{ x : \sum_{k=1}^{\infty} k|\Delta x_k| < \infty \quad \text{and} \quad \lim_{k\to\infty} x_k = 0 \right\},$$

where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. Here and after Δ denotes the forward difference operator. The space h is a $BK-$space with the norm

$$\|x\|_h = \sum_k k|\Delta x_k| + \sup_k |x_k|$$

The space h was firstly defined by Hahn [15]. Goes and Goes [12] and Rao [23], [24], [25] constructed and studied some important properties of Hahn space.

Hahn proved that the space h is a Banach space, $h \subset \ell_1 \cap \int c_0$ and $h^\beta = \rho_\infty$, where $\int \lambda = \big\{ x = (x_k) \in \omega : (kx_k) \in \lambda \big\}$ and $\rho_\infty = \big\{ x = (x_k) \in \omega : \sup_n n^{-1} \big| \sum_{k=1}^n x_k \big| < \infty \big\}$. Functional analytic properties of the $BK-$space $bv_0 \cap d\ell_1$ was studied by Goes and Goes [12], where $d\ell_1 = \{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \frac{1}{k}|x_k| < \infty \}$. Also, in [12], the arithmetic means of sequences in bv_0 and $bv_0 \cap d\ell_1$ were considered, and used the fact that the Cesàro transform $(n^{-1} \sum_{k=1}^n x_k)$ of order one $x \in bv_0$ is a quasiconvex null sequence.

Rao [23] studied some geometric properties of Hahn sequence space and gave the characterizations of some classes of matrix transformations. Also, in [24] and [25], Rao examined the different properties of Hahn sequence space.

Balasubramanian and Pandiarani [8] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that $\beta-$ and $\gamma-$duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences.

After these works, it weren't done any study on Hahn space until the Kirişci's works ([17], [18], [19]). In [17], it was compiled all studies about to Hahn sequence space and in [18], new Hahn space was constructed with the Cesaro mean. Also, Kirişci have defined p-Hahn sequence space in [19].

In the present paper, our propose is to investigate the fine spectrum of the difference operator Δ on the sequence space h. And also, we define the approximate point spectrum, defect spectrum and compression spectrum of the difference operator on the sequence space h, as a new approach.

2 Preliminaries and Definition

Let X and Y be Banach spaces, $T : X \to Y$ be a bounded linear operator. We give

$$R(T) = \{y \in Y : y = Tx, \quad x \in X\}.$$

$B(X,Y) = \{T : X \to Y : \ T$ *is continuous and linear* $\}$ *and* $B(X) = B(X,X)$.

The sets $R(T)$, $B(X,Y)$ and $B(X)$ are called *the range of the operator* T, *the set of all bounded linear operators from* X *to* Y and *the set of all bounded linear operators from* X *to* X, respectively.

Definition 2.1. *Let* X *be any Banach space and* $T \in B(X)$. *Then* T^* *of* T *is said to be adjoint, if* T^* *is a bounded linear operator on the dual* X^* *of* X *such that* $(T^*f)(x) = f(Tx)$ *for all* $f \in Y^*$ *and* $x \in X$.

Let $X \neq \{\theta\}$ be a non-trivial complex normed space, $D(T) \subseteq X$ and T is a linear operator such that $T : D(T) \to X$. Let $D(T)$ is not dense in X or T hasn't closed graph $\{(x, Tx) : x \in D(T)\} \subseteq X \times X$. If we say T *is invertible*, then we have to mention that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear. Again if we say *the boundedness of* S then we have to understand that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$.

If $T_\alpha = \alpha I - T$ such that associated with each complex number α, then T_α is called the perturbed operator defined on the same domain $D(T)$ as T.

Definition 2.2. *The* spectrum *is*

$$\sigma(T,X) = \{\alpha \in \mathbb{C} : T_\alpha \textit{ is not invertible } \}.$$

The resolvent set *of* T *is* $\rho(T,X) = \mathbb{C}\backslash\sigma(T,X)$, *the complement of spectrum. The resolvent of* T *is the function defined on* $\rho(T,X)$ *by* $\alpha \mapsto T_\alpha^{-1}$.

Definition 2.3. *The set of all sequences denotes with* $\omega := \mathbb{C}^{\mathbb{N}} := \{x = (x_k) : x : \mathbb{N} \to \mathbb{C}, k \to x_k := x(k)\}$ *where* $\mathbb{C}$ *denotes the complex field and* $\mathbb{N}$ *is the set of positive integers. Each linear subspace of* ω *(with the induced addition and scalar multiplication) is called a* sequence space.

The following subsets of ω are obviously sequence spaces:

$$\ell_\infty = \{x = (x_k) \in \omega : \sup_k |x_k| < \infty\} \qquad c = \{x = (x_k) \in \omega : \lim_k x_k \ \text{ exists } \}$$

$$c_0 = \{x = (x_k) \in \omega : \lim_k x_k = 0\} \qquad bs = \left\{ x = (x_k) \in \omega : \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}$$

$$cs = \left\{ x = (x_k) \in \omega : \left(\sum_{k=1}^{n} x_k \right) \in c \right\}$$

$$\ell_p = \left\{ x = (x_k) \in \omega : \sum_k |x_k|^p < \infty, \quad 1 \leq p < \infty \right\}.$$

These sequence spaces are Banach space with the norms; $\|x\|_{\ell_\infty} = \sup_k |x_k|$, $\|x\|_{bs} = \|x\|_{cs} = \sup_n |\sum_{k=1}^{n} x_k|$ and $\|x\|_{\ell_p} = \left(\sum_k |x_k|^p\right)^{1/p}$ as usual, respectively. Let X is one of the above mentioned sequence spaces.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} and $x = (x_k) \in \omega$, where $k, n \in \mathbb{N}$. Then the sequence Ax is called as the $A-$transform of x defined by the usual matrix product. Hence, we transform the sequence x into the sequence $Ax = \{(Ax)_n\}$ where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1}$$

for each $n \in \mathbb{N}$, provided the series on the right hand side of (1) converges for each $n \in \mathbb{N}$.

3 Subdivision of the Spectrum

In this section, we define the parts called point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

3.1 The Point Spectrum, Continuous Spectrum and Residual Spectrum

Let $\alpha \in \mathbb{C}$. Because of $(\alpha I - T)^{-1}$ provides convenience to the solution of the equation $T_\alpha x = y$, there is no problem with use the word resolvent. Therefore, the existence of T_α^{-1} is ensured by $x = T_\alpha^{-1} y$. Furthermore, if we will investigate of properties of T_α^{-1}, then it will help us obtained information about the operator T itself. Let's also note that many properties of T_α and T_α^{-1} which will be investigated, depends on α. Indeed, we know that the Spectral Theory is relevant to the investigation of such properties. In this case, if T_α^{-1} exists, then we will study the set of all α's in the complex plane. Also, we will deal another important property which name is boundedness of T_α^{-1}. We will also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects.

Now, we will give some definitions for use in the study of some properties of T, T_α and T_α^{-1} (see Kreyszig [20, pp. 370-371]).

Definition 3.1. *A regular value α of T is a complex number such that T_α^{-1} exists and is bounded whose domain is dense in X.*

The resolvent set $\rho(T,X)$ of T is the set of all regular values α of T.

The point (discrete) spectrum $\sigma_p(T,X)$ is the set such that T_α^{-1} does not exist. An $\alpha \in \sigma_p(T,X)$ is called an eigenvalue of T.

The continuous spectrum $\sigma_c(T,X)$ is the set such that T_α^{-1} exists and is unbounded, and the domain of T_α^{-1} is dense in X.

The residual spectrum $\sigma_r(T,X)$ is the set such that T_α^{-1} exists (and may be bounded or not) but the domain of T_α^{-1} is not dense in X.

Therefore, these three subspectra form a disjoint subdivision such that

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X). \tag{2}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite-dimensional case.

3.2 The Approximate Point Spectrum, Defect Spectrum and Compression Spectrum

In this subsection, three more subdivision of the spectrum called the *approximate point spectrum*, *defect spectrum* and *compression spectrum* have been defined as in Appell et al. [7].

Definition 3.2. *Let X is a Banach space and T is a bounded linear operator. A $(x_k) \in X$ Weyl sequence for T defined by $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$.*

The approximate point spectrum of T is the set

$$\sigma_{ap}(T,X) := \{\alpha \in \mathbb{C} : \text{there exists a Weyl sequence for } \alpha I - T\} \tag{3}$$

The subspectrum

$$\sigma_{\delta}(T,X) := \{\alpha \in \mathbb{C} : \alpha I - T \text{ is not surjective}\} \tag{4}$$

is called defect spectrum of T.

The two subspectra given by (3) and (4) form a (not necessarily disjoint) subdivision

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{\delta}(T,X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T,X) = \{\alpha \in \mathbb{C} : \overline{R(\alpha I - T)} \neq X\}$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$$

of the spectrum. Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_{\delta}(T,X)$. Moreover, comparing these subspectra with those in (2) we note that

$$\begin{aligned}
\sigma_r(T,X) &= \sigma_{co}(T,X)\backslash\sigma_p(T,X),\\
\sigma_c(T,X) &= \sigma(T,X)\backslash[\sigma_p(T,X) \cup \sigma_{co}(T,X)].
\end{aligned}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints are also useful.

Proposition 3.3. *[7, Proposition 1.3, p. 28] The following relations on the spectrum and subspectrum of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ hold:*

(a) $\sigma(T^*,X^*) = \sigma(T,X)$.

(b) $\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$.

(c) $\sigma_{ap}(T^*,X^*) = \sigma_{\delta}(T,X)$.

(d) $\sigma_{\delta}(T^*,X^*) = \sigma_{ap}(T,X)$.

(e) $\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$.

(f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$.

(g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to the defect spectrum and the point spectrum is dual to the compression spectrum. The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see Appell et al. [7]).

3.3 Goldberg's Classification of Spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(A) $R(T) = X$.

(B) $R(T) \neq \overline{R(T)} = X$.

(C) $\overline{R(T)} \neq X$.

and

(1) T^{-1} exists and is continuous.

(2) T^{-1} exists but is discontinuous.

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3. If an operator is in state C_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see Goldberg [13]).

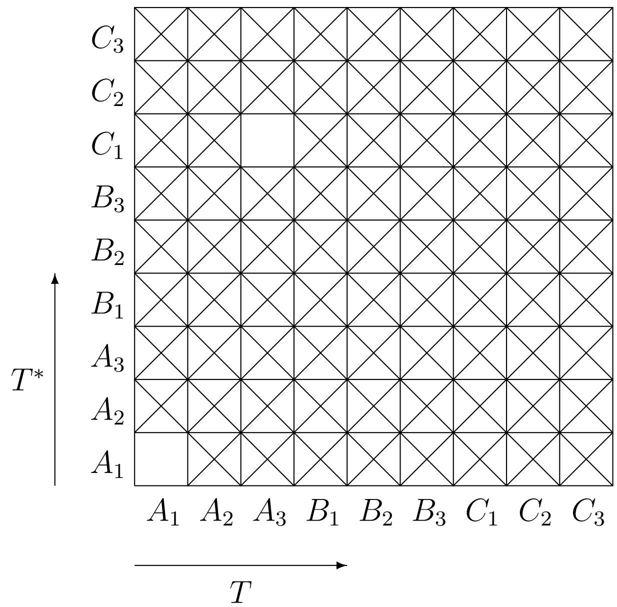

Table 1.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X

If α is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then $\alpha \in \rho(T, X)$. All scalar values of α not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $A_2\sigma(T, X) = \emptyset$, $A_3\sigma(T, X)$, $B_2\sigma(T, X)$, $B_3\sigma(T, X)$, $C_1\sigma(T, X)$, $C_2\sigma(T, X)$, $C_3\sigma(T, X)$. For example, if T_α is in a given state, C_2 (say), then we write $\alpha \in C_2\sigma(T, X)$.

By the definitions given above, we can illustrate the subdivision (2) in the following table:

		1	2	3
		T_α^{-1} exists and is bounded	T_α^{-1} exists and is unbounded	T_α^{-1} does not exist
A	$R(\alpha I - T) = X$	$\alpha \in \rho(T, X)$	–	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$
B	$\overline{R(\alpha I - T)} = X$	$\alpha \in \rho(T, X)$	$\alpha \in \sigma_c(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$
C	$\overline{R(\alpha I - T)} \neq X$	$\alpha \in \sigma_r(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$	$\alpha \in \sigma_r(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$

Table 1.2: Subdivision of spectrum of a linear operator

One can observe by the closed graph theorem that in the case A_2 cannot occur in a Banach space X. If we are not in the third column of Table 1.2, i.e., if α is not an eigenvalue of T, we may always consider the resolvent operator T_α^{-1} (on a possibly *thin* domain of definition) as *algebraic inverse* of $\alpha I - T$.

The forward difference operator Δ is represented by the matrix

$$\Delta = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Corollary 3.4. *$\Delta : h \to h$ is a bounded linear operator.*

4 On the Fine Spectrum of the Forward Difference Operator on the Hahn Space

In this section, we determine the spectrum and fine spectrum of the forward difference operator Δ on the Hahn space h and calculate the norm of the operator

Theorem 4.1. $\sigma(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| \leq 1\}$.

Proof. Let $|1 - \alpha| > 1$. Since $\Delta - \alpha I$ is triangle, $(\Delta - \alpha I)^{-1}$ exists and solving the matrix equation $(\Delta - \alpha I)x = y$ for x in terms of y gives the matrix $(\Delta - \alpha I)^{-1} = B = (b_{nk})$, where

$$b_{nk} = \begin{cases} \frac{1}{(1-\alpha)^{n+1}} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus, we observe that

$$\begin{aligned} \|(\Delta - \alpha I)^{-1}\|_{(h:h)} &= \sum_{n=1}^{\infty} n|b_{nk} - b_{n+1,k}| \\ &\leq \sum_{n=1}^{\infty} n|b_{nk}| + \sum_{n=1}^{\infty} n|b_{n+1,k}| \\ &= \sum_{n=1}^{\infty} \frac{n}{|1-\alpha|^{n+1}} + \sum_{n=1}^{\infty} \frac{n}{|1-\alpha|^{n+2}}. \end{aligned}$$

From the ratio test, we have

$$\|(\Delta - \alpha I)^{-1}\|_{(h:h)} < \infty,$$

that is, $(\Delta - \alpha I)^{-1} \in (h : h)$. But for $|1 - \alpha| \leq 1$,

$$\|(\Delta - \alpha I)^{-1}\|_{(h:h)} = \infty,$$

that is, $(\Delta - \alpha I)^{-1}$ is not in $B(h)$. This completes the proof. □

Theorem 4.2. $\sigma_p(\Delta, h) = \emptyset$.

Proof. Suppose that $\Delta x = \alpha x$ for $x \neq \theta$ in h. Then, by solving the system of linear equations

$$\begin{aligned} x_0 &= \alpha x_0, \\ -x_0 + x_1 &= \alpha x_1 \\ -x_1 + x_2 &= \alpha x_2 \\ &\vdots \\ -x_{n-1} + x_n &= \alpha x_n \\ &\vdots \end{aligned}$$

we find that if x_{n_0} is the first nonzero entry of the sequence $x = (x_n)$, then $\alpha = 1$. From the equality $-x_{n_0} + x_{n_0+1} = \alpha x_{n_0+1}$ we have x_{n_0} is zero. This contradicts the fact that $x_{n_0} \neq 0$, which completes the proof. □

Theorem 4.3. $\sigma_p(\Delta^*, h^*) = \{\alpha \in \mathbb{C} : |1 - \alpha| < 1\}$.

Proof. Suppose that $\Delta^* x = \alpha x$ for $x \neq \theta$ in $h^* \cong \sigma_\infty$. Then, by solving the system of linear equations

$$\begin{aligned} x_0 - x_1 &= \alpha x_0, \\ x_1 - x_2 &= \alpha x_1, \\ &\vdots \\ x_{n-1} - x_n &= \alpha x_n, \\ &\vdots \end{aligned}$$

we observe that $x_n = (1-\alpha)^n x_0$. Therefore, $\sup_n \frac{|x_0|}{n} \sum_{k=1}^{n} |1-\alpha|^k < \infty$ if and only if $|1-\alpha| < 1$. This step concludes the proof. $\square$

If $T \in B(h)$ with the matrix A, then it is known that the adjoint operator $T^* : h^* \to h^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space h^* of h is isometrically isomorphic to the Banach space σ_∞ of absolutely summable sequences normed by $\|x\| = \sum_{k=0}^{\infty} k|x_k - x_{k+1}|$.

Lemma 4.4. *[13, p. 59] T has a dense range if and only if T^* is one to one.*

Theorem 4.5. $\sigma_r(\Delta, h) = \sigma_p(\Delta^*, h^*)$.

Proof. For $|1-\alpha| < 1$, the operator $\Delta - \alpha I$ is triangle, so has an inverse. But $\Delta^* - \alpha I$ is not one to one by Theorem 4.3. Therefore by Lemma 4.4, $\overline{R(\Delta - \alpha I)} \neq h$ and this step concludes the proof. $\square$

Theorem 4.6. $\sigma_c(\Delta, h) = \{\alpha \in \mathbb{C} : |1-\alpha| = 1\}$.

Proof. For $|1-\alpha| < 1$, the operator $\Delta - \alpha I$ is triangle, so has an inverse but is unbounded. Also $\Delta^* - \alpha I$ is one to one by Theorem 4.3. By Lemma 4.4, $\overline{R(\Delta - \alpha I)} = h$. Thus, the proof is completed. $\square$

Theorem 4.7. $A_3\sigma(\Delta, h) = B_3\sigma(\Delta, h) = C_3\sigma(\Delta, h) = \emptyset$.

Proof. From Theorem 4.2 and Table 1.2., $A_3\sigma(\Delta, h) = B_3\sigma(\Delta, h) = C_3\sigma(\Delta, h) = \emptyset$ is observed. $\square$

Theorem 4.8. *$C_1\sigma(\Delta, h) = \emptyset$ and $\alpha \in \sigma_r(\Delta, h) \cap C_2\sigma(\Delta, h)$.*

Proof. We know $C_1\sigma(\Delta, h) \cup C_2\sigma(\Delta, h) = \sigma_r(\Delta, h)$ from Table 1.2. For $\alpha \in \sigma_r(\Delta, h)$, the operator $(\Delta - \alpha I)^{-1}$ is unbounded by Theorem 4.1. So $C_1\sigma(\Delta, h) = \emptyset$. This completes the proof. $\square$

Theorem 4.9. *The following results hold:*

(a) $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h)$.

(b) $\sigma_\delta(\Delta, h) = \sigma(\Delta, h)$.

(c) $\sigma_{co}(\Delta, h) = \{\alpha \in \mathbb{C} : |1-\alpha| < 1\}$.

Proof. (a) Since $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h) \backslash C_1\sigma(\Delta, h)$ from Table 1.2. and $C_1\sigma(\Delta, h) = \emptyset$ by Theorem 4.8, we have $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h)$.

(b) Since $\sigma_\delta(\Delta, h) = \sigma(\Delta, h) \backslash A_3\sigma(\Delta, h)$ from Table 1.2 and $A_3\sigma(\Delta, h) = \emptyset$ by Theorem 4.7, we have $\sigma_\delta(\Delta, h) = \sigma(\Delta, h)$.

(c) Since the equality $\sigma_{co}(\Delta, h) = C_1\sigma(\Delta, h) \cup C_2\sigma(\Delta, h) \cup C_3\sigma(\Delta, h)$ holds from Table 1.2, we have $\sigma_{co}(\Delta, h) = \{\alpha \in \mathbb{C} : |1-\alpha| < 1\}$ by Theorems 4.8 and 4.9. $\square$

The next corollary can be obtained from Proposition 2.1.

Corollary 4.10. *The following results hold:*

(a) $\sigma_{ap}(\Delta^*, \ell_1) = \sigma(\Delta, h)$.

(b) $\sigma_\delta(\Delta^*, \ell_1) = \{\alpha : |\alpha - (2-\delta)^{-1}| = (1-\delta)(2-\delta)\} \cup E$.

(c) $\sigma_p(\Delta^*, \ell_1) = \{\alpha \in \mathbb{C} : |\alpha - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)\} \cup S$.

5 Conclusion

Hahn [15] defined the space h and gave its some general properties. Goes and Goes [12] studied the functional analytic properties of the space h. The study on the Hahn sequence space was initiated by Rao [23] with certain specific purpose in Banach space theory. Also Rao [23] emphasized on some matrix transformations. Rao and Srinivasalu [24] introduced a new class of sequence space called the semi replete space. Rao and Subramanian [25] defined the semi Hahn space and proved that the intersection of all semi Hahn spaces is Hahn space. Balasubramanian and Pandiarani [8] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that $\beta-$ and $\gamma-$duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences. The sequence space h was introduced by Hahn [15] and Goes and Goes [12], and Rao [23, 24, 25] investigated some properties of the space h. Quite recently, Kirişci [17] has defined a new Hahn sequence space by using Cesàro mean, in [18].

The difference matrix Δ was used for determining the spectrum or fine spectrum acting as a linear operator on any of the classical sequence spaces c_0 and c, ℓ_1 and bv, ℓ_p for $(1 \leq p < \infty)$, respectively in [4], [10] and [2].

As a natural continuation of this paper, one can study the spectrum and fine spectrum of the Cesàro operator, Weighted mean operator or another known operators in the sequence space h.

Conflict of Interests

The authors declare that there are no conflict of interests regarding the publication of this paper.

References

[1] A.M. Akhmedov and F. Başar, The fine spectrum of the difference operator Δ over the sequence space bv_p, $(1 \leq p < \infty)$, *Acta Math. Sinica Eng. Ser.*, 23(10) (2007), 1757-1768.

[2] A.M. Akhmedov and F. Başar, On the fine spectra of the difference operator Δ over the sequence space ℓ_p, $(1 \leq p < \infty)$, *Demonstratio Math.*, 39(3) (2006), 585-595.

[3] A.M. Akhmedov and S.R. El-Shabrawy, On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c, *Comput. Math. Appl.*, 61(10) (2011), 2994-3002.

[4] B. Altay and F. Başar, On the fine spectrum of the difference operator Δ on c_0 and c, *Inform. Sci.*, 168(2004), 217-224.

[5] B. Altay and M. Karakuş, On the spectrum and fine spectrum of the Zweier matrix as an operator on some sequence spaces, *Thai J. Math.*, 3(2005), 153-162.

[6] M. Altun, On the fine spectra of triangular Toeplitz operators, *Appl. Math. Comput.*, 217(20) (2011), 8044-8051.

[7] J. Appell, E. Pascale and A. Vignoli, Nonlinear spectral theory, *De Gruyter Series in Nonlinear Analysis and Applications*, 10(2004), Walter de Gruyter, Berlin, New York.

[8] T. Balasubramanian and A. Pandiarani, The Hahn sequence spaces of fuzzy numbers, *Tamsui Oxf. J. Inf. Math. Sci.*, 27(2) (2011), 213-224.

[9] C. Coşkun, The spectra and fine spectra for p-Cesàro operators, *Turkish J. Math.*, 21(1997), 207-212.

[10] H. Furkan and K. Kayaduman, The fine spectra of the difference operator δ over the sequence spaces ℓ_1 and bv, *Internat. Math. Forum*, 1(24) (2006), 1153-1160.

[11] H. Furkan, H. Bilgiç and F. Başar, On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$, *Comput. Math. Appl.*, 60(7) (2010), 2141-2152.

[12] G. Goes and S. Goes, Sequences of bounded variation and sequences of Fourier coefficients I, *Math. Z.*, 118(1970), 93-102.

[13] S. Goldberg, *Unbounded Linear Operators*, Dover Publications, Inc. New York, (1985).

[14] M. Gonzàlez, The fine spectrum of the Cesàro operator in ℓ_p $(1 < p < \infty)$, *Arch. Math.*, 44(1985), 355-358.

[15] H. Hahn, Über folgen linearer operationen, *Monatsh. Math.*, 32(1) (1922), 3-88.

[16] V. Karakaya and M. Altun, Fine spectra of upper triangular double-band matrices, *J. Comput. Appl. Math.*, 234(2010), 1387-1394.

[17] M. Kirişci, A survey on the Hahn sequence space, *Gen. Math. Notes*, 19(2) (2013), 37-58.

[18] M. Kirişci, The Hahn sequence space defined by Cesàro mean, *Abstr. Appl. Anal.*, Article ID 817659(2013), 6 pages.

[19] M. Kirişci, p- Hahn sequence, *Far East J. Math. Sci.*, 90(1) (2014), 45-63.

[20] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons Inc., New York, Chichester, Brisbane, Toronto, (1978).

[21] B. de Malafosse, Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ, *Hokkaido Math. J.*, 31(2002), 283-299.

[22] J.I. Okutoyi, On the spectrum of C_1 as an operator on bv, *Commun. Fac. Sci. Univ. Ank. Ser. A_1*, 41(1992), 197-207.

[23] W.C. Rao, The Hahn sequence spaces I, *Bull. Calcutta Math. Soc.*, 82(1990), 72-78.

[24] W.C. Rao and T.G. Srinivasalu, The Hahn sequence spaces II, *Y.Y.U. J. Fac. Edu.*, 2(1996), 43-45.

[25] W.C. Rao and N. Subramanian, The Hahn sequence spaces III, *Bull. Malaysian Math. Sci. Soc.*, 25(2002), 163-171.

[26] J.B. Reade, On the spectrum of the Cesaro operator, *Bull. Lond. Math. Soc.*, 17(1985), 263-267.

[27] B.E. Rhoades, The fine spectra for weighted mean operators, *Pacific J. Math.*, 104(1) (1983), 219-230.

[28] P.D. Srivastava and S. Kumar, Fine spectrum of the generalized difference operator Δ_ν on sequence space ℓ_1, *Thai J. Math.*, 8(2) (2010), 7-19.

[29] P.D. Srivastava and S. Kumar, Fine spectrum of the generalized difference operator Δ_{uv} on sequence space ℓ_1, *Appl. Math. Comput.*, 218(11) (2012), 6407-6414.

[30] R.B. Wenger, The fine spectra of Hölder summability operators, *Indian J. Pure Appl. Math.*, 6(1975), 695-712.

[31] M. Yeşilkayagil and F. Başar, On the fine spectrum of the operator defined by the lambda matrix over the spaces of null and convergent sequences, *Abstr. Appl. Anal.*, Article ID 687393(2013), 13 pages.

[32] M. Yıldırım, On the spectrum and fine spectrum of the compact Rhally operators, *Indian J. Pure Appl. Math.*, 27(8) (1996), 779-784.