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# On the Generalized Solution for Composite Type Differential Equation

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**Abstract**

*In this paper, we study a boundary-value problem for a class of composite equation of a mixed–type problem in the space. The existence and uniqueness of the generalized solution is proved, the proof is based on an energy inequality and the density of the range of the operator generated by the problem.*

**Keywords**: *Energy inequality, operator, generalized solution.*

## 1 Introduction

In the rectangle $\Omega = (0,1)X(0,T)$ we consider the boundary-value problem.

$$l\,u = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) = f(t,x)$$

The initial conditions:

$$u(0,x) = \frac{\partial u}{\partial t}(0,x) = \frac{\partial^2 u}{\partial t^2}(0,x) = 0\,, \quad \forall x \in \Omega \tag{1-1}$$

and the boundary conditions:

$$\frac{\partial u}{\partial r} = \frac{\partial^3 u}{\partial r^3} = 0 \qquad \text{on } \; s$$

Where $s = \partial\Omega$ , r is the exterior point.

Analogous to the problem (1 – 1), we consider its dual problem. We denote by $l^*$ the formal dual of the operator $l$ which is defined with respect to the inner product in the space $L_2(Q)$ using

$$(lu, v) = (u, l^* v), \quad \forall u, v \in C_0^{3,4} \; (Q) \qquad (1-2)$$

The dual equations:

$$l^* v = \left(-\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) \left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2}\right) = g(t, x)$$

With the final conditions:

$$v(T, x) = \tfrac{\partial v}{\partial t}(T, x) = \tfrac{\partial^2 v}{\partial t^2}(T, x) = 0 \, , \quad \forall x \in \Omega \qquad (1-3)$$

and the boundary conditions:

$$\frac{\partial v}{\partial r} = \frac{\partial^3 v}{\partial r^3} = 0 \qquad \text{on } \; s$$

## 2 Functional Spaces

The domain $D(l)$ of the operator $l$ is $D(l) = H_0^{3,4}(\Omega)$, the subspace of the Sobolev space $H^{3,4}(\Omega)$ ,which consists of all the functions $u \in H^{3,4}(\Omega)$ satisfying the conditions of (1-1). The domain of $l^*$ is $D(l^*) = H_{\square}^{3,4}(\Omega)$ which consists of all the functions $v \in H^{3,4}(\Omega)$ satisfying the conditions of (1-3).
Let $H^{-3,-4}(\Omega)$ be the dual space of the space $H^{3,4}(\Omega)$ .

**Definition** *The solution of (1-1) is called the generalized solution of the operational equation*

$$lu = f \; , \quad u \in D(l) \qquad (2\text{-}1)$$

*and the solution of the problem (1 – 3) is called the generalized solution of the operational equation*

$$l^* v = g \; , \quad v \in D(l^*) \qquad (2\text{-}2)$$

Where $l$ , $l^*$ are extension of the operators $L, L^*$
Then we obtain:

$$(l^* v,u)=(v,Lu), \qquad \forall u\in D(l)\ , v\in H_0^{2,3}(\Omega)$$
$$(v,lu)=(L^* v,u), \qquad \forall u\in H_0^{2,3}(\Omega)\, v\in D(l^*)$$

# 3 A Priori Estimates

**Theorem** *For problems (1 – 1) and (1 – 3) we have the following a priori estimates:*

$$\|u\|_{2,3}\le c\|Lu\|_{-2,-3}\ ,\ \forall u\in D(L) \tag{3 – 1}$$
$$\|v\|_{2,3}\le c^*\|L^* v\|_{-2,-3}\ ,\ \forall v\in D(L^*) \tag{3 – 2}$$

Where the constants $c>0$ and $c^*>0$ are independent of $u$ and $v$.

**Proof.** Firstly we prove the inequality (3 –1) for the function $u\in D(l)$.
For $u\in D(l)$ we define the operator:

$$Mu=\Phi(t)\tfrac{\partial^2 u}{\partial t^2}-\Phi(t)\frac{\partial^3 u}{\partial x^2\partial t} \qquad \text{Where } \Phi(t)=(t-T)^2$$

Problem (1-1) can be written in the form:

$$lu=\frac{\partial^3 u}{\partial t^3}-\frac{\partial^3 u}{\partial t\partial x^2}-\frac{\partial^4 u}{\partial x^2\partial t^2}+\frac{\partial^4 u}{\partial x^4}=f(t,x)$$

Then we have:

$$\int_\Omega luMudtdx=\int_\Omega fMudtdx$$

We but

$$I_1=\int_\Omega\frac{\partial^3 u}{\partial t^3}(t-T)^2\frac{\partial^2 u}{\partial t^2}dtdx\ ,\ I_2=\int_\Omega\frac{\partial^3 u}{\partial t^3}(t-T)^2\frac{\partial^3 u}{\partial x^2\partial t}dtdx$$

Integrating by parts and using the conditions we obtain:

$$I_1=-\int_\Omega(t-T)(\frac{\partial^2 u}{\partial t^2})^2dtdx\ \ ,\ \ I_2=-\int_P(\frac{\partial^2 u}{\partial t\partial x})^2dtdx-\int_\Omega\frac{\partial^2\Omega}{\partial u^2}(t-T)^2\frac{\partial^4 u}{\partial x^2\partial t^2}dtdx$$

Also we but:

$$J_1=\int_\Omega\frac{\partial^3 u}{\partial t\partial x^2}(t-T)^2\frac{\partial^2 u}{\partial t^2}dtdx\quad ,\quad J_2=\int_\Omega(t-T)^2(\frac{\partial^3 u}{\partial x^2\partial t})^2\,dtdx$$

Integrating $J_1$ by parts and using the conditions (1-1), then we have:

$$, \quad J_1 = \int_\Omega (t-T) \ (\frac{\partial^2 u}{\partial x\, \partial t})^2 \, dtdx$$

Also we have:

$$K_1 = \int_\Omega \frac{\partial^4 u}{\partial x^2 \partial t^2} (t-T)^2 \frac{\partial^2 u}{\partial t^2} dtdx \ , \ K_2 = \int_\Omega \frac{\partial^4 \Omega}{\partial x^2 \partial t^2} (t-T)^2 \frac{\partial^3 u}{\partial x^2 \partial t} dtdx$$

Integrating the last integral by parts and using conditions (1-1) we have:

$$K_2 = -\int_\Omega (t-T)^2 (\frac{\partial^3 u}{\partial x^2 \partial t})^2 dtdx$$

Finally we but:

$$L_1 = \int_\Omega \frac{\partial^4 u}{\partial x^4} (t-T)^2 \frac{\partial^2 u}{\partial t^2} dtdx \quad , \ L_2 = \int_\Omega \frac{\partial^4 u}{\partial x^4} (t-T)^2 \frac{\partial^3 u}{\partial x^2 \partial t} dtdx$$

Integrating the last integral by parts and using conditions (1-1) we find that:

$$L_1 = -\int_\Omega \left\{ (t-T)^2 \left( \frac{\partial^3 u}{\partial t \partial x^2} \right)^2 + \left( \frac{\partial^2 u}{\partial x^2} \right)^2 \right\} dtdx \ . \ L_2 = \int_\Omega (t-T) \left( \frac{\partial^3 u}{\partial x^3} \right)^2 dtdx$$

Then we have:

$$\left( lu, (t-T)^2 \frac{\partial^2 u}{\partial t^2} - (t-T)^2 \frac{\partial^3 u}{\partial x^2 \partial t} \right) = \int_\Omega (T-t) \left( \frac{\partial^2 u}{\partial t^2} \right)^2 dxdt + \int_P (T-t) \left( \frac{\partial^2 u}{\partial t \partial x} \right)^2 dxdt + \int_\Omega \left( \frac{\partial^2 u}{\partial x^2} \right)^2 dxdt$$
$$+ \int_\Omega \left( \frac{\partial^2 u}{\partial x \partial t} \right)^2 dxdt + \int_\Omega (T-t) \left( \frac{\partial^3 u}{\partial x^2 \partial t} \right)^2 dxdt + \int_P (T-t) \left( \frac{\partial^3 u}{\partial x^3} \right)^2 dxdt$$

(3-3)

We use the following Poincare estimates:

$$\int_\Omega u^2 dxdt \le 4T^2 \int_\Omega \left( \frac{\partial u}{\partial t} \right)^2 dtdx \ , \quad \forall u \in D(l)$$
$$\int_\Omega \left( \frac{\partial\, u}{\partial t} \right)^2 dxdt \le 4T \int_\Omega (T-t) \left( \frac{\partial^2 u}{\partial t^2} \right)^2 dxdt \quad , \ \forall u \in D(l) \qquad (3\text{-}4)$$
$$\int_\Omega \left( \frac{\partial\, u}{\partial x} \right)^2 dxdt \le 4T \int_\Omega (t-t) \left( \frac{\partial^2 u}{\partial x \partial t} \right)^2 dxdt \quad , \ \forall u \in D(l)$$

Now apply $\varepsilon -$inequality to the left hand side of (3 – 3) and using inequalities (3 –4) we obtain (3 – 1) for $u \in D(L)$ .

# 4 Solvability of the Problem (Existence of Solution)

The uniqueness of the solution follows immediately from inequality (3-1). This inequality also ensures the closure of the range set $R(L)$ of the operator L.
We prove the existence of solution or we prove that for all functions $f \in H^{-2,-3}(\Omega)$ there is a unique solution of the problem (1-1).

To prove that $\overline{R(L)}$ equals the space $H^{-2,-3}(\Omega)$, we obtain the inclusion $\overline{R(L)} \subseteq R(L)$, and $R(L) = H^{-2,-3}(\Omega)$.

Let $\{f_k\}_{k\in N}$ be a Cauchy sequence in the space $H^{-2,-3}(\Omega)$, which consists of element of set $R(L)$. Then it corresponds to a sequence $\{u_n\}_{n\in N} \subseteq D(L)$ such that:

$$Lu_n = f_n \quad , \quad n \in N$$

From the energy inequality (3 – 1) we have

$$\|u_n - u_m\| \le c\|L(u_n - u_m)\| - c\|Lu_n - Lu_m\| = c\|f_n - f_m\| \le \varepsilon$$

Then $\{u_n\}_{n\in N}$ is also a Cauchy sequence in the space $H^{-2,-3}(\Omega)$, and converges to an element $u$ in $H^{2,3}(\Omega)$ .

Then we have: $Lu_n = f_n$ , $\lim Lu_n = \lim f_n$ then $Lu = f$ and $f \in R(L)$ this means that $f \in R(L)$ and $\overline{R(L)} \subseteq R(L)$ then we have that:

$\overline{R(L)} = R(L)$ and $R(L)$ is closed.

It remains to obtain the density of the set $R(L)$ in the space $H^{-2,-3}(\Omega)$ when $u \in D(L)$.
Therefore we establish an equivalent result which amounts to proving that $R^{\perp} = \{0\}$.
Let $v \in H^{-2,-3}(\Omega)$ be such that $(L\,u, v) = 0, \forall u \in D(L)$, that is

$(L^* v, u) = 0$ , $\forall u \in D(L)$. By the equality $(L^* v, u) = (v, Lu)$ , $\forall u \in D(L)$ we have $(v, Lu) = 0$ , $\forall u \in D(L), v \in H^{-2,-3}(\Omega)$ .

From $(L^{*}v,u)=0$ we have that $L^{*}v=0$ and by virtue of the inequality (3-2) we conclude that $v=0$ in the space $H^{-2,-3}(\Omega)$ when $u\in D(L)$.Then $R(L)$ is dense in $H^{-2,-3}(\Omega)$.

The inequality (3-2) can be proved in a similar way by using the operator:

$$M^{*}v=t^{2}\frac{\partial^{2}v}{\partial t^{2}}-t^{2}\frac{\partial^{3}v}{\partial x^{2}\partial t}$$

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