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Distance Based Graph Invariants of Graph Operations

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Abstract

In this paper, the exact formulae for the Harary indices of join, disjunction, symmetric difference, strong product of graphs are obtained. Also, the Schultz and modified Schultz indices of join and strong product of graphs are computed. We apply some of our results to compute the Harary, Schultz and modified Schultz indices of fan graph, wheel graph, open fence and closed fence graphs.

Keywords: *Harary index, Graph operations.*

1 Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$.

A *sum* $G+H$ of two graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ is the graph on the vertex set $V(G)\cup V(H)$ and the edge set $E(G)\cup E(H)\cup\{uv \mid u\in V(G),\ v\in V(H)\}$. Hence, the sum of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs. The sum of two graphs is sometimes also called a *join,* and is denoted by $G\nabla H$.

The *disjunction* $G\vee H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and (u_1,v_1) is adjacent with (u_2,v_2) whenever $u_1u_2\in E(G)$ or $v_1v_2\in E(H)$. The *symmetric difference* $G\oplus H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ and $E(G\oplus H)=\{(u_1,v_1)(u_2,v_2)\,|\,u_1u_2\in E(G)$ or $v_1v_2\in E(H)$ but not both$\}$.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [6]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index; for other related topological indices see [10].

Let G be a connected graph. Then *Wiener index* of G is defined as $W(G)=\frac{1}{2}\sum\limits_{u,\,v\,\in\,V(G)} d_G(u,v)$ with the summation going over all pairs of distinct vertices of G. Similarly, the *Harary index* of G is defined as $H(G)=\frac{1}{2}\sum\limits_{u,\,v\,\in\,V(G)}\frac{1}{d_G(u,v)}$.

The Harary index of a graph G has been introduced independently by Plavsic et al. [8] and by Ivanciuc et al. [7] in 1993. Its applications and mathematical properties are well studied in [1, 2, 3, 9]. Zhou et al. [4] have obtained the lower and upper bounds of the Harary index of a connected graph. Very recently, Xu et al. [5] have obtained lower and upper bounds for the Harary index of a connected graph in relation to $\chi(G)$, chromatic number of G and $\omega(G)$, clique number of G and characterized the extremal graphs that attain the lower and upper bounds of Harary index. Also, Feng et. al. [2] have given a sharp upper bound for the Harary indices of graphs based on the matching number, that is, the size of a maximum matching. Various topological indices on tensor product, Cartesian product and strong product have been studied various authors, see [11, 12, 13, 14, 15].

Dobrynin and Kochetova [16] and Gutman [17] independently proposed a vertex-degree-weighted version of Wiener index called Schultz molecular topological index or Degree distance, which is defined for a connected graph G as $W_+(G)=\frac{1}{2}\sum\limits_{u,v\in V(G)}(d_G(u)+d_G(v))d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. In [18] it has been demonstrated that the Wiener index and the Schultz index are closely mutually related for certain classes of molecular graphs. Similarly, the *modified Schultz molecular topological index* is defined as $W_*(G)=\frac{1}{2}\sum\limits_{u,v\in V(G)} d_G(u)d_G(v)d_G(u,v)$.

The *first and second Zagerb index* are defined as $M_1(G) = \sum_{u\in V(G)} d_G(u)^2$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$. In fact, one can rewrite the first Zagreb index as $M_1(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have appilications in QSPR and QSAR studies as well, see [19]. Ashrafi et al. [20, 21] defined the *first Zagreb coindex* and *second Zagerb coindex* as $\overline{M}_1(G) = \sum_{uv\notin E(G)} (d_G(u) + d_G(v))$ and $\overline{M}_2(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v)$, respectively.

A path, cycle and complete graph on n vertices are denoted by P_n, C_n and K_n, respectively. In this paper, the exact formulae for the Harary indices of join, disjunction, symmetric difference, strong product of graphs are obtained. Also, the Schultz and modified Schultz indices of join and strong product of graphs are computed. We apply some of our results to compute the Harary, Schultz and modified Schultz indices of fan graph, wheel graph, open fence and closed fence graphs.

2 Harary indices of composite graphs

In this section, first we compute the Harary indices of join, disjunction and symmetric difference of two connected graphs. The proof of the following lemma follows easily from the definitions of join, disjunction, symmetric difference of two graphs.

Lemma 2.1.. *Let G and H be two graphs. Then*

(i) $d_{G+H}(u,v) = \begin{cases} 0, & \text{if } u = v \\ 1, & \text{if } uv \in E(G) \text{ or } uv \in E(H) \text{ or } (u \in V(G) \text{ and } v \in V(H)) \\ 2, & \text{otherwise.} \end{cases}$

(ii) $d_{G+H}(x) = \begin{cases} d_G(x) + |V(H)|, & \text{if } x \in V(G) \\ d_H(x) + |V(G)|, & \text{if } x \in V(H) \end{cases}$

(iii) $d_{G\vee H}((u,x),(v,y)) = \begin{cases} 0, & \text{if } u = v \text{ and } x = y \\ 1, & \text{if } uv \in E(G) \text{ or } xy \in E(H) \\ 2, & \text{otherwise.} \end{cases}$

(iv) $d_{G\oplus H}((u,x),(v,y)) = \begin{cases} 0, & \text{if } u = v \text{ and } x = y \\ 1, & \text{if } uv \in E(G) \text{ or } xy \in E(H) \text{ but not both} \\ 2, & \text{otherwise.} \end{cases}$ □

Theorem 2.2.. *Let G_1 and G_2 be graphs with n and m vertices, respectively. Then* $H(G_1 + G_2) = mn + \frac{1}{2}(|E(G_1)| + |E(G_2)|) + \frac{1}{4}(n(n-1) + m(m-1))$.

Proof. Set $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_m\}$. Then by Lemma

2.1., we have

$$\begin{aligned}
H(G_1+G_2) &= \frac{1}{2}\sum_{u,v\in V(G_1+G_2)} \frac{1}{d_{G_1+G_2}(u,v)} \\
&= \frac{1}{2}\Big(\sum_{uv\in E(G_1)} \frac{1}{d_{G_1+G_2}(u,v)} + \sum_{uv\notin E(G_1)} \frac{1}{d_{G_1+G_2}(u,v)} + \sum_{uv\in E(G_2)} \frac{1}{d_{G+H}(u,v)} \\
&\quad + \sum_{uv\in E(G_2)} \frac{1}{d_{G_1+G_2}(u,v)} + \sum_{u\in V(G_1),\ v\in V(G_2)} \frac{1}{d_{G_1+G_2}(u,v)}\Big) \\
&= |E(G_1)| + \frac{1}{2}\Big(\frac{n(n-1)}{2} - |E(G_1)|\Big) + |E(G_2)| \\
&\quad + \frac{1}{2}\Big(\frac{m(m-1)}{2} - |E(G_2)|\Big) + mn \\
&= mn + \frac{1}{2}\big(|E(G_1)| + |E(G_2)|\big) + \frac{1}{4}(n(n-1) + m(m-1)).
\end{aligned}$$

□

Using Theorem 2.2., we have the following corollary.

Corollary 2.3.. *Let G be graph on n vertices. Then $H(G + K_m) = mn + \frac{1}{2}|E(G)| + \frac{1}{4}\Big(n(n-1) + m(m-1)\Big)$.* □

Using Corollary 2.3., we compute the formula for Harary indices of fan and wheel graphs, $P_n + K_1$ and $C_n + K_1$, see Figs. 1a and 1b.

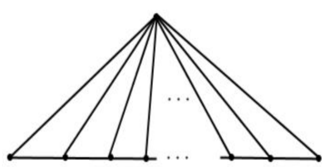

Fig. 1a. Fan graph

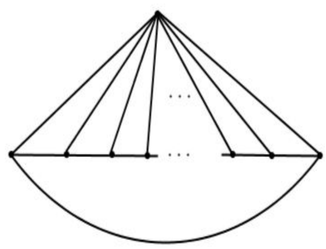

Fig.1b. Wheel graph

Example 1.
(i) $H(P_n + K_1) = \frac{1}{4}(n^2 + 5n - 2)$.
(ii) $H(C_n + K_1) = \frac{1}{4}(n^2 + 5n)$.

Theorem 2.4.. *Let G_1 and G_2 be graphs with n and m vertices, respectively. Then $H(G_1 \vee G_2) = \frac{m^2}{2}|E(G_1)| + \frac{m^2}{2}|E(G_2)| - |E(G_1)|\,|E(G_2)| + \frac{1}{4}mn(mn-1)$.*

Proof. Set $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_m\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G_1 \vee G_2$. Then by Lemma 2.1., we have

$$\begin{aligned}
H(G_1 \vee G_2) &= \frac{1}{2} \sum_{x_{ij},\, x_{kp} \in V(G_1 \vee G_2)} \frac{1}{d_{G_1 \vee G_2}(x_{ij}, x_{kp})} \\
&= \frac{1}{2} \sum_{x_{ij} \in V(G_1 \vee G_2)} \Big\{ \Big(md(u_i) + nd(v_j) - d(u_i) + d(v_j) \Big) \\
&\quad + \frac{1}{2} \Big(mn - md(u_i) - nd(v_j) + d(u_i)d(v_j) - 1 \Big) \Big\} \\
&= \frac{1}{2} \sum_{x_{ij} \in V(G_1 \vee G_2)} \Big(\frac{1}{2} md(u_i) + \frac{1}{2} nd(v_j) - \frac{1}{2} d(u_i)d(v_j) + \frac{1}{2}(mn-1) \Big) \\
&= \frac{m^2}{2} |E(G_1)| + \frac{m^2}{2} |E(G_2)| - |E(G_1)|\,|E(G_2)| + \frac{1}{4} mn(mn-1).
\end{aligned}$$

□

Using similar argument as Theorem 2.4., one can prove the following result:

Theorem 2.5.. *Let G_1 and G_2 be graphs with n and m vertices, respectively. Then $H(G_1 \oplus G_2) = \frac{m^2}{2} |E(G_1)| + \frac{m^2}{2} |E(G_2)| - |E(G_1)|\,|E(G_2)| + \frac{1}{4} mn(mn-1)$.* □

3 Harary index of strong product of graphs

In this section, we obtain the Harary index of $G \boxtimes K_r$.

Theorem 3.1.. *Let G be a connected graph with n vertices and m edges. Then $H(G \boxtimes K_r) = r^2 H(G) + \frac{1}{2} nr(r-1)$.*

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \boxtimes K_r$. One can see that for any pair of vertices $x_{ij}, x_{kp} \in V(G \boxtimes K_r)$, $d_{G \boxtimes K_r}(x_{ij}, x_{ip}) = 1$ and $d_{G \boxtimes K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.

$$\begin{aligned}
H(G \boxtimes K_r) &= \frac{1}{2} \sum_{x_{ij},\, x_{kp} \in V(G \boxtimes K_r)} \frac{1}{d_{G \boxtimes K_r}(x_{ij}, x_{kp})} \\
&= \frac{1}{2} \Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} \frac{1}{d_{G \boxtimes K_r}(x_{ij}, x_{ip})} + \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{j=0}^{r-1} \frac{1}{d_{G \boxtimes K_r}(x_{ij}, x_{kj})} \\
&\quad + \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} \frac{1}{d_{G \boxtimes K_r}(x_{ij}, x_{kp})} \Bigg) \\
&= \frac{1}{2} \Big(nr(r-1) + 2rH(G) + 2r(r-1)H(G) \Big) \\
&= r^2 H(G) + \frac{1}{2} nr(r-1).
\end{aligned}$$

□

By direct calculations we obtain expressions for the values of the Harary indices of P_n and C_n. $H(P_n) = n\Big(\sum_{i=1}^{n} \frac{1}{i}\Big) - n$ and $H(C_n) = \begin{cases} n\Big(\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 1, & \text{if } n \text{ is even} \\ n\Big(\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big), & \text{if } n \text{ is odd.} \end{cases}$

As an application we present formulae for Harary indices of open and closed fences, $P_n \boxtimes K_2$ and $C_n \boxtimes K_2$, see Fig. 2.

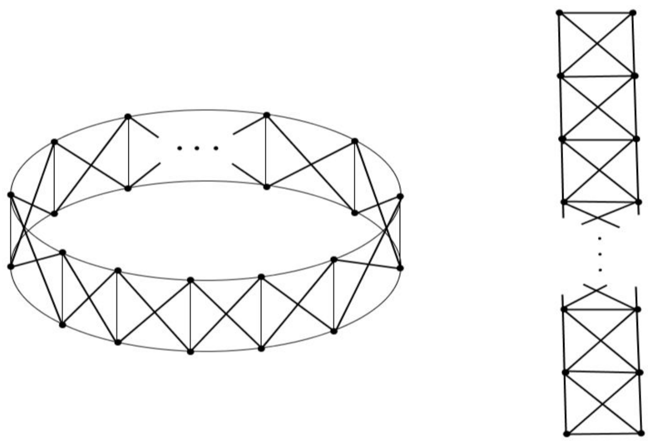

Fig. 2. Closed and open Fence graphs

By using Theorem 3.1., $H(C_n)$ and $H(P_n)$, we obtain the exact Harary indices of the following graphs.

Example 2.

(i) $H(P_n \boxtimes K_2) = n\Big(4\sum_{i=1}^{n} \frac{1}{i} - 3\Big)$.

(ii) $H(C_n \boxtimes K_2) = \begin{cases} n\Big(1 + 4\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 4, & \text{if } n \text{ is even} \\ n\Big(1 + 4\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big), & \text{if } n \text{ is odd.} \end{cases}$

4 Schultz and modified Schultz indices of join and strong product of graphs

In this section, we obtain the Schultz and modified Schultz indices of $G_1 + G_2$ and $G \boxtimes K_r$.

Theorem 4.1.. *Let G_1 and G_2 be graphs with n and m vertices, respectively. Then* $W_+(G_1 + G_2) = M_1(G) + M_1(H) + 2(\overline{M}_1(G) + \overline{M}_1(H)) + nm(3n + 3m - 4)$.

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$. By Lemma 2.1., we have

$$\begin{aligned}
W_+(G + H) &= \frac{1}{2} \sum_{u,v \in V(G+H)} \Big(d_{G+H}(u) + d_{G+H}(v)\Big) d_{G+H}(u, v) \\
&= \frac{1}{2}\Big(\sum_{uv \in E(G)} (d_G(u) + m + d_G(v) + m) + 2 \sum_{uv \notin E(G)} (d_G(u) + m + d_G(v) + m) \\
&\quad + \sum_{uv \in E(H)} (d_H(u) + n + d_H(v) + n) + 2 \sum_{uv \in E(H)} (d_H(u) + n + d_H(v) + n) \\
&\quad + \sum_{u \in V(G),\ v \in V(H)} (d_G(u) + m + d_H(v) + n)\Big) \\
&= M_1(G) + M_1(H) + 2(\overline{M}_1(G) + \overline{M}_1(H)) + nm(3n + 3m - 4).
\end{aligned}$$

□

Using similar arguments as Theorem 4.1., one can prove the following result:

Theorem 4.2.. *Let G_1 and G_2 be graphs with n and m vertices p and q edges, respectively. Then* $W_*(G_1 + G_2) = M_2(G) + M_2(H) + mM_1(G) + nM_1(H) + 2\overline{M}_2(G) + 2\overline{M}_2(H) + 2m\overline{M}_1(G) + 2n\overline{M}_1(H) + mn(3mn - m - n) + 4pq - m^2p - n^2q + 2mn(p + q)$.
□

Using Theorems 4.1. and 4.2., we have the following corollaries.

Corollary 4.3.. *Let G be graph on n vertices. Then* $W_+(G + K_m) = M_1(G) + 2\overline{M}_1(G) + m(m-1)^2 + nm(3n + 3m - 4)\big)$. □

Corollary 4.4.. *Let G be graph on n vertices and p egdes. Then* $W_*(G + K_m) = M_2(G) + mM_1(G) + 2\overline{M}_2(G) + 2m\overline{M}_1(G) + mn(3mn - m - n) + mp(2n - m) + \frac{1}{2}m(m - 1)(4p - n^2 + 2mn + 3m^2 - 4m + 1)$. □

One can observe that $M_1(C_n) = 4n,\ n \geq 3$, $M_1(P_1) = 0$, $M_1(P_n) = 4n - 6,\ n > 1$ and $M_1(K_n) = n(n-1)^2$. Similarly, $\overline{M_1}(K_n) = 0$, $\overline{M_1}(P_n) = 2(n-2)^2$ and $\overline{M_1}(C_n) = 2n(n-3)$.

By direct calculations we obtain the second Zagreb indices and coindices of P_n and C_n. $M_2(P_n) = 4(n-2)$, $M_2(C_n) = 4n$, $\overline{M}_2(P_n) = 2n^2 - 10n + 13$, and $\overline{M}_2(C_n) = 2n(n-3)$.

Using Corollaries 4.3. and 4.4., we compute the formulae for Schultz and modified Schultz indices of fan and wheel graphs.

Example 3.
(i) $W_+(P_n + K_1) = 7n^2 - 13n + 10$.
(ii) $W_+(C_n + K_1) = n(7n - 9)$.
(iii) $W_*(P_n + K_1) = 12n^2 - 32n + 29$.
(iv) $W_*(C_n + K_1) = 12n^2 - 18n$.

Theorem 4.5.. *Let G be a connected graph with n vertices and m edges. Then* $W_+(G \boxtimes K_r) = r\Big(r^2 W_+(G) + 2r(r-1)W(G) + 2r(r-1)m + n(r-1)^2\Big)$.

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \boxtimes K_r$. The degree of the vertex x_{ij} in $G \boxtimes K_r$ is $d_G(u_i) + d_{K_r}(v_j) + d_G(u_i)d_{K_r}(v_j)$, that is $d_{G\boxtimes K_r}(x_{ij}) = rd_G(u_i) + (r-1)$. One can observe that for any pair of vertices x_{ij}, $x_{kp} \in V(G \boxtimes K_r)$, $d_{G\boxtimes K_r}(x_{ij}, x_{ip}) = 1$ and $d_{G\boxtimes K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.

$$\begin{aligned}
W_+(G \boxtimes K_r) &= \frac{1}{2} \sum_{x_{ij},\, x_{kp} \in V(G\boxtimes K_r)} \Big(d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kp})\Big) d_{G\boxtimes K_r}(x_{ij}, x_{kp}) \\
&= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \Big(d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{ip})\Big) d_{G\boxtimes K_r}(x_{ij}, x_{ip}) \\
&\quad + \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{j=0}^{r-1} \Big(d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kj})\Big) d_{G\boxtimes K_r}(x_{ij}, x_{kj}) \\
&\quad + \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \Big(d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kp})\Big) d_{G\boxtimes K_r}(x_{ij}, x_{kp}) \Bigg) \\
&= \frac{1}{2}\{A_1 + A_2 + A_3\}, \qquad (4.1)
\end{aligned}$$

where A_1, A_2 and A_3 are the sums of the terms of the above expression, in order.

We shall obtain A_1 to A_3 of (4.1), separately.

$$\begin{aligned}
A_1 &= \sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \Big(d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{ip})\Big) d_{G\boxtimes K_r}(x_{ij}, x_{ip}) \\
&= \sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \Big(2d_G(u_i) + 2(r-1) + 2(r-1)d_G(u_i)\Big) \\
&= 4r^2(r-1)m + 2nr(r-1)^2 \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}
A_2 &= \sum_{j=0}^{r-1}\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\Big(d_{G\boxtimes K_r}(x_{ij})+d_{G\boxtimes K_r}(x_{kj})\Big)d_{G\boxtimes K_r}(x_{ij},x_{kj})\\
&= \sum_{j=0}^{r-1}\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\Big(d_G(u_i)+(r-1)d_G(u_i)+d_G(u_k)+(r-1)d_G(u_k)+2(r-1)\Big)d_G(u_i,u_k)\\
&= r\sum_{j=0}^{r-1}\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\Big(d_G(u_i)+d_G(u_k)\Big)d_G(u_i,u_k)+\sum_{j=0}^{r-1}\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}2(r-1)d_G(u_i,u_k)\\
&= 2r^2W_+(G)+4r(r-1)W(G). \qquad (4.3)
\end{aligned}$$

$$\begin{aligned}
A_3 &= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0,\\ j\neq p}}^{r-1}\Big(d_{G\boxtimes K_r}(x_{ij})+d_{G\boxtimes K_r}(x_{kp})\Big)d_{G\boxtimes K_r}(x_{ij},x_{kp})\\
&= r^2(r-1)\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\Big(d_G(u_i)+d_G(u_k)\Big)d_G(u_i,u_k)+2r(r-1)^2\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}d_G(u_i,u_k)\\
&= 2r^2(r-1)W_+(G)+4r(r-1)^2W(G). \qquad (4.4)
\end{aligned}$$

Using (4.2), (4.3) and (4.4) in (4.1), we have

$$W_+(G\boxtimes K_r) = r\Big(r^2W_+(G)+2r(r-1)W(G)+2r(r-1)m+n(r-1)^2\Big).$$

□

Using similar arguments as Theorem 4.5., one can prove the following result:

Theorem 4.6.. *Let G be a connected graph with n vertices and m edges. Then* $W_*(G\boxtimes K_r) = r\Big(r^3W_*(G)+r^2(r-1)W_+(G)+r(r-1)^2W(G)+\frac{1}{2}r^2(r-1)^2M_1(G)+\frac{1}{2}(r-1)^3n+2r(r-1)^2m\Big)$. □

One can see that $W(P_n)=\frac{n(n^2-1)}{6}$ and $W(C_n)=\begin{cases}\frac{n^3}{8}, & \text{if } n \text{ is even}\\ \frac{n(n^2-1)}{8}, & \text{if } n \text{ is odd.}\end{cases}$

Similarly, $W_+(P_n)=\frac{1}{3}n(n-1)(2n-1)$, $W_*(P_n)=\frac{1}{3}(n-1)(2n^2-4n+3)$ and $W_+(C_n)=W_*(C_n)=4W(C_n)$.

As an application we present formulae for Schultz and modified Schultz indices of open and closed fence graphs.

Example 4.

(i) $W_+(P_n\boxtimes K_2)=\frac{1}{3}(4n(n-1)(5n-1)+30n-24)$.

$(ii)\ W_+(C_n \boxtimes K_2) = \begin{cases} 5n(n^2+2) & \text{n is even} \\ 5n(n^2+1) & \text{n is odd.} \end{cases}$

$(i)\ W_*(P_n \boxtimes K_2) = \frac{1}{3}\Big(2(n-1)(25n^2-35n+24)+75n-96\Big).$

$(ii)\ W_*(C_n \boxtimes K_2) = \begin{cases} \frac{25}{2}n(n^2+2), & \text{if } n \text{ is even} \\ \frac{25}{2}n(n^2+1), & \text{if } n \text{ is odd.} \end{cases}$

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