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Greens' Function and Dual Integral Equations Method to Solve Heat Equation for Unbounded Plate

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Abstract

In this paper we present Greens function for solving non-homogeneous two dimensional non-stationary heat equation in axially symmetrical cylindrical coordinates for unbounded plate high with discontinuous boundary conditions. The solution of the given mixed boundary value problem is obtained with the aid of the dual integral equations (DIE), Greens function, Laplace transform (L-transform) , separation of variables and given in form of functional series.

Keywords: *Greens function, dual integral equations, mixed boundary conditions.*

1 Introduction

Several problems involving homogeneous mixed boundary value problems of a non-stationary heat equation and Helmholtz equation were discussed with details in monographs [1, 4-7]. In this paper we present the use of Greens function and DIE for solving non-homogeneous and non-stationary heat equation in axially symmetrical cylindrical coordinates for unbounded plate of a high h. The goal of this paper is to extend the applications DIE which is widely used for solving mathematical physics equations of the elliptic and parabolic types in different coordinate systems subject to mixed boundary conditions, to solve non-homogeneous heat conduction equation, related to both first and second mixed boundary conditions acted on the surface of unbounded cylindrical plate. The solution of the non-homogeneous mixed problem is based on earlier results received in [5, 7] and emphasizes that the weight function, unknown function, and free term depend on a parameter of a L- transform parameter. The DIE solution discussed below is reduced to some type of singular integral with unknown function kernel and free term depend on a parameter of L-transform. The obtained integral equation is solved by expanding its unknown function as a functional series of a Laplace transform parameter. The use of Green's function in the solution with unmixed boundary conditions with different coordinates and applications can be found for example in [3, 9].

2 Problem Formulations

Find a temperature distribution function$u = u(r, z, \tau)$, of unbounded solid plate $0 \le z < h$ $0 \le r < \infty$, $\tau > 0$ with heat generation function, the initial temperature distribution function is constant and satisfies mixed boundary value problem

$$u_t - a^2\nabla^2 u = g(r,t) \quad (1)$$

$$u(r,0,\tau) = f_1(r,\tau), \quad r \in \Omega \quad (2)$$

$$u_z(r,0,\tau) = f_2(r,\tau), \quad r \in \overline{\Omega} \quad (3)$$

The initial condition

$$u(r,z,0) = \theta(r,z,0) - T_0, \quad T_0 \text{ constant}$$

The functions $f_i(r,\tau), i = 1,2$ known and integrable functions, with respect to r, $0 < r < \infty$,and$\tau, \tau > 0$, $\Omega = \{r, 0 < r < R, z = 0\}$, $\overline{\Omega} = \{r, R < r < \infty, z = 0\}$, $\theta_z = \partial\theta / \partial z$, a is a heat diffusivity coefficient, constant, $g(r,t)$ is known function,(heat generating function).

On a surface $z = h$, a linear combination of unmixed boundary condition is given

$$\alpha_1 u + \alpha_2 u_z = 0 \quad (4)$$

Where α_1, α_2 constants. There are three particular cases of (4)

Case (i): If $\alpha_1 = 1, \alpha_2 = 0$, a long the boundary surface $z = h, u = u(r,h,\tau)$ is kept at zero temperature satisfies a homogeneous boundary conditions first kind

$$u(r,h,\tau) = 0, \quad r \ge 0 \quad (5)$$

Applying (L-transform) to the boundary-value problem (1)-(5) with respect to τ, then by separating variables for (1) under the assumption that r, z bounded at zero and infinity, we obtain the solution of the homogeneous part of the problem in form of improper integral

$$\overline{u_1}(r,z,s)=\int_0^\infty \overline{C}(p,s)J_0(pr)\frac{sh[(h-z)\sqrt{p^2+k}\,]}{ch[h\sqrt{p^2+k}\,]}dp \qquad (6)$$

Where $sh(x)$ and $ch(x)$ are sine and cosine hyperbolic functions, $J_0(pr)$ is a Bessel function first kind of order zero, s: parameter of L-transform [2] , a is a heat diffusivity coefficient (constant) $k=s/a$ and $\overline{C}(p,s)$ is an unknown function.

Substitution a mixed boundary conditions (1) and (2) into (6), yields the following a pair of DIE to determine the unknown function $\overline{C}(p,s)$:

$$\int_0^\infty \overline{C}(p,s)J_0(pr)th(h\sqrt{p^2+k}\,)dp=\overline{f}_1(r,s)\ ,r\in\Omega \qquad (7)$$

$$\int_0^\infty \overline{C}(p,s)\sqrt{p^2+k}\ J_0(pr)dp=f_2(r,s)\ ,r\in\overline{\Omega} \qquad (8)$$

where $thx=shx/chx$. Hyperbolic tangent function.

Now to solve DIE (7) and (8) ,the unknown function $\overline{C}(p,s)$ should be replaced by another unknown function $\overline{\varphi}(t,s)$ with the help of the relation [1]

$$\overline{C}(p,s)=\frac{p}{\sqrt{p^2+k}}\int_0^R \overline{\varphi}(t,s)\cos(t\sqrt{p^2+k}\,)dt \qquad (9)$$

Where $\overline{\varphi}(t,s)=\ L[\varphi(t,\tau)]$ differentiable with respect to a variable t and analytical with respect to a parameter s, also $\varphi(t,\tau)$ should be continuous or piecewise continuous in any interval $\tau_1<\tau<\tau_2$, then use the expansion [5]

$$\overline{\varphi}(t,s)=\exp(-R\sqrt{k}\,)\sum_{n=0}^{\infty}\varphi_n(t)s^{n/2-1} \qquad (10)$$

The inverse Laplace transform of (10) always exists [2]

$$L^{-1}\left\{\exp(-R\sqrt{k}\,)\sum_{n=0}^{\infty}\varphi_n(t)s^{n/2-1}\right\}=\frac{2}{\sqrt{\pi}}\exp(-\frac{R^2}{4a\tau})\sum_{n=0}^{\infty}\varphi_n(t)\frac{H_{n-1}(R/2\sqrt{a\tau}\,)}{2^n\tau^{n/2}}$$

Substitute (9) into (6), use (10) we obtain a general solution of (1) in L-transform image

$$\overline{u_1}(r,z,s)=\int_0^R \overline{\varphi}(t,s)\left(\int_0^\infty \frac{p}{\sqrt{a^2+k}}J_0(pr)\cos\left(t\sqrt{p^2+k}\right)\frac{ch((h-z)\sqrt{p^2+k})}{sh(h\sqrt{p^2+k})}dp\right)dt$$

$$=\frac{1}{2}\sum_n^\infty\sum_{\nu=0}^\infty\sum_{\mu=0}^\infty(-1)^{\nu+\mu-1}\int_0^R\varphi_n(t)s^{n/2}\frac{\exp(-c(R+\sqrt{a_\mu{}^2+r^2})}{\sqrt{a_\mu{}^2+r^2}}dt \qquad (11)$$

$$a_1=(2\nu+1)h-(h-z)-it,\ a_2=(2\nu+1)h+(h-z)-it,$$
$$a_3=(2\nu+1)h-(h-z)+it,\ a_4=(2\nu+1)h+(h-z)+it.$$

$\varphi_i(t), i=1,....,n$ are the solution of an integral equation of the second kind [5].

Apply the inverse L-transform to (11), a general solution of the homogeneous part of the problem (1)-(6) for the first particular case when $g(r,\tau)=0$

$$u_1(r,z,\tau)=\int_0^R\varphi_n(t)G_1(r,z,t,\tau,0)dt \qquad (12)$$

$$G_1(r,z,t,\tau,0)=\frac{a}{\sqrt{\pi}}\sum_{n=0}^\infty\sum_{\upsilon=0}^\infty\sum_{\mu=0}^\infty(-1)^{\nu+\mu-1}\exp\left(-\frac{\left(R+\sqrt{a_\mu{}^2+r^2}\right)^2}{4a\tau}\right)x$$
$$\frac{H_{n-1}((R+\sqrt{a_\mu{}^2+r^2})/2\sqrt{4a\tau}}{2^n\tau^{n/2}\sqrt{a_\mu{}^2+r^2}} \qquad (13)$$

Based on (12),(13), we construct the solution of the non-homogeneous part of the problem when $g(r,\tau)\neq 0$

$$u(r,z,\tau)=\int_0^R\varphi_n(t)G_1(r,z,t,\tau,0)dt+\frac{1}{a}\int_0^\tau\int_0^R g(t,\xi)G_1(r,z,t,\tau,\xi)dtd\xi \qquad (14)$$

Where $G_1(r,z,\tau,\xi)$ is the Green's function given by the expression

$$G_1(r,z,t,\tau,\xi)=\frac{a}{\sqrt{\pi}}\sum_{n=0}^\infty\sum_{\upsilon=0}^\infty\sum_{\mu=0}^\infty(-1)^{\nu+\mu-1}\exp\left(-\frac{\left(R+\sqrt{a_\mu{}^2+r^2}\right)^2}{4a(\tau-\xi)}\right)x$$
$$\frac{H_{n-1}((R+\sqrt{a_\mu{}^2+r^2})/2\sqrt{4a(\tau-\xi)})}{2^n(\tau-\xi)^{n/2}\sqrt{a_\mu{}^2+r^2}} \qquad (15)$$

Case (ii): The second particular case if $\alpha_1 = 0, \alpha_2 = 1,$ on a surface of cylindrical coordinates $z = h, r \geq 0$ in (4) a normal derivative is prescribed unmixed homogeneous boundary condition of the second kind (an insulated boundary condition) in L-transform field

$$\overline{\theta}_z(r,h,s) = 0 \ , r \geq 0 \tag{16}$$

$\overline{\theta}_z(r,h,s) = L[\theta_z(r,h,\tau)]$ .The general solution of the heat equation (2.1) when $g(r,\tau) = 0$ in L-transform with regard to (16), under the assumption that $\theta(r,z,\tau)$ is limited as $\sqrt{z^2 + r^2} \to \infty$ and $r = 0$, the medium is initially at constant temperature, by applying (L-transform) and separation of variables for (1), we have

$$\overline{u}_2(r,z,s) = \int_0^\infty \overline{C}(p,s) J_0(pr) \frac{ch\left((h-z)\sqrt{p^2+k}\right)}{sh\left(h\sqrt{p^2+k}\right)} dp \tag{17}$$

Use a mixed boundary conditions (2) and (3) (17) ,a pair DIE were obtained to determine the unknown function $\overline{C}(p,s)$

$$\left.\begin{aligned} &\int_0^\infty \overline{C}(p,s) J_0(pr) \coth(h\sqrt{p^2+k})dp = \overline{f}_1(r,s) \ , r \in \Omega \\ &\int_0^\infty \overline{C}(p,s)\sqrt{p^2+k}\, J_0(pr)dp = \overline{f}_2(r,s), r \in \overline{\Omega} \end{aligned}\right\} \tag{18}$$

where $\coth x = chx / shx$ hyperbolic cotangent function. The solution of the dual equations (18) is obtained with the help of equalities (9) and (10) in L-transform image. Repeat the same procedure of case (i), the general solution of the homogeneous part of the problem for the second particular case when $g(r,\tau) = 0$ is

$$\overline{u_2}(r,z,s) = \int_0^R \overline{\varphi}(t,s) \left(\int_0^\infty \frac{p}{\sqrt{a^2+c^2}} J_0(pr) \cos\left(t\sqrt{p^2+c^2}\right) \frac{ch[(h-z)\sqrt{p^2+s/a}]}{sh[h\sqrt{p^2+s/a}]} dp \right) dt$$

$$= \frac{1}{2i} \sum_n^\infty \sum_{\nu=0}^\infty \sum_{\mu=0}^\infty (-1)^{\mu-1} \int_0^R \varphi_n(t) s^{n/2} \frac{\exp(-k(R+\sqrt{a_\mu^2+r^2}))}{\sqrt{a_\mu^2+r^2}} dt \tag{19}$$

Applying the inverse L-transform for the homogeneous part of the problem (19) when $g(r,\tau) \neq 0$, the solution of the non-homogeneous mixed boundary value problem(1)-(5) and (16) is

$$u(r,z,\tau)=\int_0^R \varphi_n(t)G_2(r,z,t,\tau,0)dt+\int_0^\tau\int_0^R g(t,\xi)G_2(r,z,t,\tau,\xi)dtd\xi \qquad (20)$$

Where $G_2(r,z,\tau,\xi)$ is the Green's function given by the expression

$$G_2(r,z,t,\tau,\xi)=\frac{a}{i\sqrt{\pi}}\sum_{n=0}^{\infty}\sum_{\upsilon=0}^{\infty}\sum_{\mu=0}^{\infty}(-1)^{\mu-1}\exp\left(-\frac{\left(R+\sqrt{a_\mu{}^2+r^2}\right)^2}{4a(\tau-\xi)}\right)x$$
$$\frac{H_{n-1}((R+\sqrt{a_\mu{}^2+r^2})/2\sqrt{4a(\tau-\xi)})}{2^n(\tau-\xi)^{n/2}\sqrt{a_\mu{}^2+r^2}} \qquad (21)$$

where $a_\mu,\mu=1,2,3,4$ $i=\sqrt{-1}$ in (21) have the same values given in (13). Notice that at $z\to\infty, r\to\infty$, the general solution of the given problem for both two above particular cases is vanish in L-transform or in the original field, furthermore, satisfy the considered boundary-value problem. As $s\to 0$, and $\overline{g}(r,s)=0$ the solution of the non-homogeneous mixed boundary value problem mentioned above reduce to the known solutions [8]. If $g(r,\tau)\neq 0$, $h\to\infty$, the solutions (14), (20) correspond solutions of non-homogeneous mixed problem for a half-space [7].

Case (iii): Consider a more general case when $\alpha_1 u+\alpha_2 u_z=0$ If $\alpha_1=-1, \alpha_2=1$. Separating variables for (1) in L-transform image, with regard to (4) where r,z bounded at zero and infinity, we obtain the solution of the homogeneous part of the problem

$$\overline{u}_3(r,z,s)=$$
$$\int_0^\infty \overline{C}(p,s)J_0(pr)\left(\frac{(\sqrt{p^2+k}-1)\exp(-(2h-z)\sqrt{p^2+k})}{\sqrt{p^2+k}+1}+\exp(-z)\sqrt{p^2+k}\right)dp \qquad (22)$$

Use a mixed a boundary conditions (2), (3) to (22), we obtain a DIE for determination $\overline{C}(p,s)$

$$\int_0^\infty \overline{C}(p,s)J_0(pr)\overline{g}_1(p,s)dp=\overline{f}_1(r,s)\ , r\in\Omega \qquad (23)$$

$$\int_0^\infty \overline{C}(p,s)J_0(pr)\overline{g}_2(p,s)dp=\overline{f}_2(r,s)\ , r\in\overline{\Omega} \qquad (24)$$

Where

$$\overline{g}_1(p,s)=\frac{(\sqrt{p^2+k}-1)\exp(-2h\sqrt{p^2+k})}{\sqrt{p^2+k}+1}+1$$

$$\overline{g_2}(p,s)=\sqrt{p^2+k}\left(\frac{(\sqrt{p^2+k}-1)\exp(-2h\sqrt{p^2+k})}{\sqrt{p^2+k}+1}-1\right)$$

The DIE (23) and (24) very complicated to solve, since the known methods of solution (Hankel integral transform or substitution method cannot be used in this situation), however as $h\to\infty$, the above dual equations becomes

$$\int_0^\infty \overline{C}(p,s)J_0(pr)dp=\overline{f_1}(r,s)\ ,r\in\Omega \tag{25}$$

$$\int_0^\infty \overline{C}(p,s)\sqrt{p^2+k}\ J_0(pr)dp=\overline{f_2}(r,s)\ ,r\in\overline{\Omega} \tag{26}$$

(25) and (26) play very important role in the solution of a non stationary heat equation in a cylindrical coordinates for a semi-space and have exact solution discussed with much details in [1,4]. As $s\to 0$, and $g(r,\tau)=0$, all solutions of DIE mentioned above were reduced to the Laplace equation with mixed conditions of the first and for the second kind[8,10].

3 Conclusion

Finally the above technique involving Greens function and inhomogeneous mixed boundary value problem can be used for solving several inhomogeneous problems (heat equation, Helmholtz equation and Laplace equation) in cylindrical coordinates, spherical coordinates and other coordinate system under mixed boundary conditions of the first, the second and of the third kind .

References

[1] N. Abdelrazaq, The solution of heat equation with mixed boundary conditions, *Journal of Mathematics and Statistics*, Science Publications New York, 2(2) (2006), 346-350.

[2] R.M. Coita, *The Integral Transform Method in Thermal and Fluids Sciences and Engineering*, New York, Beqell House, Inc., (1998).

[3] D.G. Duffy, *Green's Functions with Application*s, Chapman& Hall, CRC, (2001).

[4] N.A. Hoshan, Exact solution of certain dual integral equations involving heat conduction equation, *Far East Journal of Applied Mathematics*, 35(1) (2009), 81-88.

[5] N.A. Hoshan, The dual integral equations method to solve heat conduction equation for unbounded plate, *Computational Mathematics and Modeling*, 21(2) (2010), 226-238.

[6] N.A. Hoshan, Integral transform method in some mixed problems, *International Journal of Mathematical Forum*, 4(40) (2009), 1977-1980.

[7] N.A. Hoshan and F. Al-Quran, Greens function in some mixed boundary value problems for heat equation, *International Journal of Academic Research*, 2(4) (2010), 100-103.
[8] B.N. Mandal and N. Mandal, *Advances in Dual Integral Equations*, London CRC, (1999).
[9] M.N. Ozisik, *Heat Conduction*, Wiley & Sons, New York, (2002).
[10] J.S. Uflyand, *Dual Equations in Mathematical Physics Equations*, Leningrad Nauka, (1977).