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On Almost Convergence and Difference Sequence Spaces of Order m with Core Theorems

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Abstract

This paper is mainly concerned with introducing the spaces $fs(\triangle^{(m)})$, $f(\triangle^{(m)})$ *and* $f_0(\triangle^{(m)})$ *that consist of all sequence whose* $\triangle^{(m)}$ *transforms are in the set of almost convergent sequence and series spaces. Certain topological properties of these new almost convergent sets have been investigated as well as* $\gamma-$ *and* $\beta-$*duals of the spaces* $fs(\triangle^{(m)})$ *and* $f(\triangle^{(m)})$*. In addition to that the non-existence of Schauder basis of the spaces* fs *and* $fs(\Delta^{(m)})$ *is shown. Furthermore, the characterization of certain matrix classes on/into the sets of generalized difference almost convergent sequence and series has exhaustively been examined. Finally, some identities and inclusion relations related to core theorems are established.*

Keywords: *Almost convergence, Schauder basis, Beta- and gamma -duals, Matrix mappings, Core theorems.*

1 Introduction

From summability theory perspective, the role played by algebraic, geometric and topologic properties of new Banach spaces which are matrix domains of triangle matrices in sequence spaces is very well-known.

While studies of certain inclusion relations between almost convergent spaces and some other spaces is very old, the examination of topologic properties of almost convergent sequence and series spaces and the studies connected with their duals are very recent (see [29], [30]). Matrix domains of the generalized difference matrix $B(r,s)$ and triple band matrix $B(r,s,t)$ in sets of almost null f_0 and almost convergent f sequences have been investigated by Başar and Kirişçi [3] and Sönmez [8], respectively. They have examined some algebraic and topologic properties of certain almost null and almost convergent spaces. Following these authors, Kayaduman and Şengönül have subsequently introduced some almost convergent spaces which are matrix domains of the Riesz matrix and Cesàro matrix of order 1 in sets of almost null f_0 and almost convergent f sequences (see, [4] and [5]). They have also studied some topological properties of these spaces, characterized some classes of matrix mappings and finally gave some core theorems. Quite recently, Karaisa and Karabıyık examined and studied some new almost sequence spaces which derived matrix domains of A^r matrix [36]. Note that, Karaisa and Özger introduced certain almost convergent sequence spaces which are related to almost convergent sequence and series spaces [37]. Further information on matrix domains of sequence spaces can be found in(see[1, 2, 21, 23, 26, 27, 28]).

The rest of this paper is organized, as follows: We give foreknowledge on main argument of this study and notations in the next section. In Section 3, we introduce almost convergent sequence and series spaces $fs(\triangle^{(m)})$ and $f(\triangle^{(m)})$ which are matrix domains of $\Delta^{(m)}$ matrix in the almost convergent sequence and series spaces fs and f, respectively, in addition to give certain theorems related to behavior of some sequences. In Section 4, we determine the beta- and gamma-duals of spaces $fs(\triangle^{(m)})$ and $f(\triangle^{(m)})$ and characterize classes $(\gamma : f(\triangle^{(m)}))$, $(f(\triangle^{(m)}) : \mu)$, $(\delta : fs(\triangle^{(m)}))$ and $(f(\triangle^{(m)}) : \theta)$; where $\gamma \in \{c(p), c_0(p), \ell_\infty(p), cs, bs, fs, f, c, \ell_\infty\}$, $\mu \in \{cs, bs, c, \ell_\infty\}$, $\delta \in \{cs, fs, bs\}$ and $\theta \in \{f, c, fs, \ell_\infty\}$. In the last section of paper; after comparing with related results in the existing literature, we note some original aspects of this study.

2 Notions, Notations and the Sets fs, f and f_0

In this section, we start with recalling the most important notations, definitions and make a few remarks on the meaning of the notions which are needed in this study.

By w, we shall denote the space of all real or complex valued sequences. We shall write ℓ_∞, c and c_0 for spaces of all bounded, convergent, and null sequences respectively. Also by bs and cs, we denote the spaces of all bounded and convergent series. Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from μ into γ, and we denote it by writing $A : \mu \to \gamma$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = \{(Ax)_n\}_{n\in\mathbb{N}}$ the $A-$ transform of x, is in γ; where

$$(Ax)_n = \sum_k a_{nk}x_k \quad (n \in \mathbb{N}). \tag{1}$$

The notation $(\mu : \gamma)$ denotes the class of all matrices A such that $A : \mu \to \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right hand side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \gamma$ for all $x \in \mu$. The matrix domain μ_A of an infinite matrix A in a sequence space μ is defined by

$$\mu_A = \{x = (x_k) \in \omega : Ax \in \mu\}. \tag{2}$$

A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x. The sequence space μ with a linear topology is called a *K-space* provided each of the maps $p_i : \mu \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A K-space μ is called an *FK-space* provided μ is a complete linear metric space. An FK-space whose topology is normable is called a *BK-space*. A sequence (b_k) in a normed space μ is called a *Schauder basis* for μ if and only if for each $x \in \mu$, there exists a unique sequence (α_k) of scalars such that $x = \sum_{k=0}^{\infty} \alpha_k b_k$.

The concept of statistically convergence for sequence real numbers was defined by Fast [7] and Steinhaus [6] independently in 1951. First we recall the following definitions:

Let K be a subset of $\mathbb{N}$. The natural density $\delta(K)$ of $K \subseteq \mathbb{N}$ is $\lim_n n^{-1}|\{k \leq n : k \in K\}|$ provided it exists, where $|E|$ denotes the cardinality of a set E. A sequence $x = (x_k)$ is called statistically convergent ($st-$convergent) to the number l, denoted $st - \lim x = l$, if every $\epsilon > 0$, $\delta(\{k : |x_k - l| \geq \epsilon\}) = 0$. We write S and S_0 to denote the sets of all statistically convergent sequences and statistically null sequences, respectively. The concepts of statistical boundedness, statistical limit superior(or briefly $st - \limsup$) and statistical limit inferior (or briefly $st - \liminf$) have been introduced by Fridy and Orhan in [13]. They have also studied on the notions of statistical core (or briefly $st-$core) of a statistically bounded sequence as closed interval $[st - \liminf, st - \limsup]$.

We list the following functionals on ℓ_∞:

$$\begin{aligned} l(x) &= \liminf_{k\to\infty} x_k, \quad L(x) = \limsup_{k\to\infty} x_k, \\ q_\sigma(x) &= \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \sum_{i=0}^{m} x_{\sigma^i(n)}, \\ L^*(x) &= \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \sum_{i=0}^{m} x_{n+i}. \end{aligned}$$

The $\sigma-$core of a real bounded sequence x is defined as the closed interval $[-q_\sigma(-x), q_\sigma(x)]$ and also the inequality $q_\sigma(Ax) \le q_\sigma(x)$ holds for all bounded sequences x. The Knopp-core (shortly $K-$core) of x is the interval $[l(x), L(x)]$ while Banach core (in short $B-$core) of x defined by the interval $[-L^*(-x), L^*(x)]$. In particular, when $\sigma(n) = n+1$ because of the equality $q_\sigma(x) = L^*(x)$, $\sigma-$core of x is reduced to the $B-$core of x (see [17, 14]). The necessary and sufficient conditions for an infinite matrix matrix A to satisfy the inclusion $K - core(Ax) \subseteq B - core(x)$ for each bounded sequences x obtained in [15].

We now focus on sets of almost convergent sequences. A continuous linear functional ϕ on ℓ_∞ is called a Banach limit if (i) $\phi(x) \ge 0$ for $x = (x_k), x_k \ge 0$ for every k, (ii) $\phi(x_{\sigma(k)}) = \phi(x_k)$ where σ is shift operator which is defined on ω by $\sigma(k) = k+1$ and (iii) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x are α [11], and denoted by $f - \lim x = \alpha$. In other words, $f - \lim x_k = \alpha$ uniformly in n if and only if

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} x_{k+n} \text{ uniformly in } n.$$

The characterization given above was proved by Lorentz in [11]. We denote the sets of all almost convergent sequences f and series fs by

$$f = \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} t_{mn}(x) = \alpha \text{ uniformly in } n \right\}$$

and

$$fs = \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \frac{x_j}{m+1} = l \text{ uniformly in } n \right\},$$

where

$$t_{mn}(x) = \sum_{k=0}^{m} \frac{1}{m+1} x_{k+n}, \; t_{-1,n} = 0$$

for all $m, n \in \mathbb{N}$. We know that the inclusions $c \subset f \subset \ell_\infty$ strictly hold. Because of these inclusions, norms $\|.\|_f$ and $\|.\|_\infty$ of the spaces f and ℓ_∞ are equivalent. So the sets f and f_0 are BK-spaces with the norm $\|x\|_f = \sup_{m,n} |t_{mn}(x)|$.

We note that, the γ- and β-duals of the set fs have been found by Başar and Kirişçi lately. Two basic results related to the space f: "f is a non-seperable closed subspace of $(\ell_\infty, \|.\|_\infty)$" and "Banach space f has no Shauder basis" are given in their paper [3].

Now, we give a theorem about non-existence of Schauder basis of the space fs.

Theorem 2.1 *The set fs has no Schauder basis.*

Proof: Let us define the matrix $S = (s_{nk})$ by $s_{nk} = 1$ $(0 \leq k \leq n)$, $s_{nk} = 0$ $(n > k)$. Then $x \in fs$ if and only if $(Sx)_n \in f$ for all n. As a result, since the set f has no basis fs has no basis too.

3 Difference Sequence Spaces

The difference spaces $c_0(\triangle), c(\triangle)$ and $\ell_\infty(\triangle)$ consisting of all sequences such that $\triangle^1 x = (x_k - x_{k+1})$ in sequence spaces c_0, c, ℓ_∞ which were introduced by Kızmaz [10]. Recently, the difference spaces bv_p consisting of sequences $x = (x_k)$ such that $(x_k - x_{k+1}) \in \ell_p$ have been studied in case $0 < p < 1$ by Altay and Başar [22], and in case $1 \leq p < \infty$ by Başar and Altay [20]. A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there exists subadditive function $h : X \longrightarrow \mathbb{R}$ such that $h(\theta) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \longrightarrow 0$ and $h(x_n - x) \longrightarrow 0$ imply $h(\alpha_n x_n - \alpha x) \longrightarrow 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X and $p = (p_k)$ be an arbitrary bounded sequence of positive reals. Then, linear spaces $c_0(p), c(p)$ and $\ell_\infty(p)$ were defined by Maddox [41] as follows:

$$
\begin{aligned}
c_0(p) &= \{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\}, \\
c(p) &= \{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = l \ \text{ for some } l \in \mathbb{R}\}, \\
\ell_\infty(p) &= \{x = (x_k) \in \omega : \sup_k |x_k|^{p_k} < \infty\}.
\end{aligned}
$$

Let μ denote any one of the classical sequence spaces c_0, c, ℓ_∞. In [25], Ahmad and Mursaleen defined paranormed spaces of difference sequences

$$\triangle\mu(p) = \{x = (x_k) \in \omega : \triangle^1 x = (x_k - x_{k+1}) \in \mu(p)\}.$$

The idea of difference sequences was generalized by Çolak and Et [16]. They defined the sequence spaces

$$\triangle^m \mu(p) = \{x = (x_k) \in \omega : \triangle^m x \in \mu(p)\},$$

where $m \in \mathbb{N}$ and $\triangle^m = \triangle^1(\triangle^{m-1})$. In [35], Polat and Başar introduced the spaces $e_0^r(\triangle^{(m)})$, $e_c^r(\triangle^{(m)})$ and $e_\infty^r(\triangle^{(m)})$ consisting of all sequences whose mth order differences are in Euler spaces e_0^r, e_c^r and e_∞^r respectively. Altay [24] studied the space $\ell_p(\triangle^{(m)})$ consisting of all sequences whose mth order differences are $p-$absolutely summable, which is a generalization of the spaces bv_p introduced by Başar and Altay [20] .

4 The Sets $f(\triangle^{(m)})$, $fs(\triangle^{(m)})$ and their Topological Properties

This section is devoted to examination of basic topologic properties of the sets $fs(\triangle^{(m)}))$ and $f(\triangle^{(m)})$. Most of the study given in this section is, by now, classical and not very difficult, but it is mandatory to give. Our main focus in this study is on the triangle matrix $\triangle^{(m)} = \{\delta_{nk}^{(m)}\}$ is defined by

$$\delta_{nk}^{(m)} = \begin{cases} (-1)^{n-k}\binom{m}{n-k} & (max\{0, n-m\} \leq k \leq n), \\ 0 & (0 \leq k < max\{0, n-m\} \text{ or } k > n). \end{cases}$$

We introduce sequence spaces $f(\triangle^{(m)})$, $f_0(\triangle^{(m)})$ and $fs(\triangle^{(m)})$ as the sets of all sequences such that their $\triangle^{(m)}-$ transforms are in spaces f, f_0 and fs, respectively, that is

$$f(\triangle^{(m)}) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{i=0}^{k} (-1)^{k-i} \binom{m}{i} x_{k-i+l} = \alpha \text{ uniformly in } l \right\},$$

$$f_0(\triangle^{(m)}) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{i=0}^{k} (-1)^{k-i} \binom{m}{i} x_{k-i+l} = 0 \text{ uniformly in } l \right\}$$

and

$$fs(\triangle^{(m)}) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \sum_{j=0}^{l+k} \sum_{i=0}^{j} (-1)^{j-i} \binom{m}{i} x_{j-i} = \beta \text{ uniformly in } l \right\}.$$

We can redefine the spaces $f_0(\triangle^{(m)})$, $f(\triangle^{(m)})$ and $fs(\triangle^{(m)})$ by notation of (2)

$$f_0(\triangle^{(m)}) = (f_0)_{\triangle^{(m)}},\ f(\triangle^{(m)}) = f_{\triangle^{(m)}} \text{ and } fs(\triangle^{(m)}) = (fs)_{\triangle^{(m)}}.$$

Define sequence $y = (y_k)$, which will be frequently used, as the $\triangle^{(m)}$-transform of a sequence $x = (x_k)$, i.e.

$$y_k(m) = \sum_{i=max\{0,n-m\}}^{k} (-1)^{k-i}\binom{m}{k-i}x_i = \sum_{i=0}^{k}(-1)^{k-i}\binom{m}{i}x_{k-i}. \quad (3)$$

Theorem 4.1 *Spaces $f(\triangle^{(m)})$ and $fs(\triangle^{(m)})$ have no Schauder basis.*

Proof: Since, it is known that the matrix domain μ_A of a normed sequence space μ has a basis if and only if μ has a basis whenever $A = (a_{nk})$ is a triangle [12, Remark 2.4]. The space f has no Schauder basis by [3, Corollary 3.3] we have $f(\triangle^{(m)})$ has no Schauder basis. Since the set fs has no basis in Theorem 2.1, $fs(\triangle^{(m)})$ has no Schauder basis.

Theorem 4.2 *The following statements hold.*

(i) *The sets $f(\triangle^{(m)})$ and $f_0(\triangle^{(m)})$ are linear spaces with co-ordinatewise addition and scalar multiplication which are BK-spaces with the norm*

$$\|x\|_{f(\triangle^{(m)})} = \sup_n \left|\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{j-i}\binom{m}{i}x_{j-i+k}\right|. \quad (4)$$

(ii) *The set $fs(\triangle^{(m)})$ is a linear space with co-ordinatewise addition and scalar multiplication which is a BK-space with the norm*

$$\|x\|_{fs(\triangle^{(m)})} = \sup_n \left|\sum_{k=0}^{n}\sum_{j=0}^{l+k}\sum_{i=0}^{j}(-1)^{j-i}\binom{m}{i}x_{j-i}\right|.$$

Proof: Since the second part can be similarly proved we only focus on the first part. Since the sets f and f_0 endowed with the norm $\|.\|_\infty$ are BK-spaces (see[18, Example 7.3.2(b)]) and the matrix $\triangle^{(m)} = (\delta_{nk}^{(m)})$ is normal, Theorem 4.3.2 of Wilansky [19, p.61] gives the fact that the spaces $f(\triangle^{(m)})$ and$f_0(\triangle^{(m)})$ are BK-spaces with the norm in (4).

Now, we may give the following theorem concerning isomorphisms between our spaces and the sets f, f_0 and fs.

Theorem 4.3 *The sequence spaces $f(\triangle^{(m)})$, $f_0(\triangle^{(m)})$ and $fs(\triangle^{(m)})$ are linearly isomorphic to the sequence spaces f, f_0 and fs, respectively. That is $f(\triangle^{(m)}) \cong f$, $f_0(\triangle^{(m)}) \cong f_0$ and $fs(\triangle^{(m)}) \cong fs$.*

Proof: To prove the fact that $f(\triangle^{(m)}) \cong f$, we should show the existence of a linear bijection between the spaces $f(\triangle^{(m)})$ and f. Consider the transformation T defined by notation of (2) from $f(\triangle^{(m)})$ to f by $x \mapsto y = Tx = \triangle^m x$. The linearity of T is clear. Further, it is clear that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y = (y_k) \in f(\triangle^{(m)})$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^{k} \binom{m+k-i-1}{k-i} y_i \text{ for each } k \in \mathbb{N}. \tag{5}$$

Whence

$$\begin{aligned} f(\triangle^{(m)}) - \lim x &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{i=0}^{k} (-1)^{k-i} \binom{m}{k-i} x_{k-i+l} \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{i=0}^{k} (-1)^{k-i} \binom{m}{k-i} \sum_{j=0}^{i} \binom{m+i-j-1}{i-j} y_{i-j+l} \\ &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} y_{n-k+l} \text{ uniformly in } l \\ &= f - \lim y \end{aligned}$$

which implies that $x \in f(\triangle^{(m)})$. As a result, T is surjective. Hence, T is a linear bijection which implies that the spaces $f(\triangle^m)$ and f are linearly isomorphic, as desired. Similarly, the isomorphisms $f_0(\triangle^{(m)}) \cong f_0$ and $fs(\triangle^{(m)}) \cong fs$ can be proved.

Theorem 4.4 *The inclusions $f_0(\triangle^{(m)}) \subset f_0(\triangle^{(m+1)})$, $f(\triangle^{(m)}) \subset f(\triangle^{(m+1)})$, $c(\triangle^{(m)}) \subset f(\triangle^m)$ and $f(\triangle^m) \subset \ell_\infty(\triangle^m)$ strictly hold.*

Proof: Let $x \in f_0(\triangle^m)$. Then, since the following inequality

$$\begin{aligned} \frac{1}{n+1} \left| \sum_{k=0}^{n} (\triangle^{(m+1)} x)_{k+l} \right| &= \frac{1}{n+1} \left| \sum_{k=0}^{n} \triangle^1 (\triangle^{(m)} x)_{k+l} \right| \\ &= \frac{1}{n+1} \left| \sum_{k=0}^{n} (\triangle^{(m)} x)_{k+l} - (\triangle^{(m)} x)_{l+k-1} \right| \\ &\le \frac{1}{n+1} \left| \sum_{k=0}^{n} (\triangle^m x)_{l+k} \right| + \frac{1}{n+1} \left| \sum_{k=0}^{n} (\triangle^{(m)} x)_{l+k-1} \right| \end{aligned}$$

trivially holds and tends to zero uniformly in l as $n \longrightarrow \infty$, $x \in f_0(\triangle^{(m+1)})$. This shows that the inclusion $f_0(\triangle^{(m)}) \subset f_0(\triangle^{(m+1)})$ holds. Further, let us consider sequence $x = \{x_k(m)\}$ defined $x_k(m) = \binom{m+k}{k}$ for all $k \in \mathbb{N}$. Then as $u = \triangle^{(m)}x = (1, 1, \dots 1, \dots) = e \notin f_0(\triangle^{(m)})$. On the other hand, $v = (\triangle^{(m+1)}x)_k = (\triangle^{(m)}x)_k - (\triangle^{(m)}x)_{k-1} = (0, 0, \dots) \in f_0(\triangle^{(m+1)})$.

We immediately observe that x is in $f_0(\triangle^{(m+1)})$ but not in $f_0(\triangle^{(m)})$. This shows that $x \in f_0(\triangle^{(m+1)}) \backslash f_0(\triangle^{(m)})$, hence the inclusion $f_0(\triangle^{(m)}) \subset f_0(\triangle^{(m+1)})$ is strict. Similarly, we can show that $f(\triangle^{(m)}) \subset f(\triangle^{(m+1)})$.

The validity of the inclusion $c(\triangle^{(m)}) \subset f(\triangle^{(m)}) \subset \ell_\infty(\triangle^{(m)})$ is easily by combining the definition of the sequence spaces $c(\triangle^{(m)})$, $f(\triangle^{(m)})$ and $\ell_\infty(\triangle^{(m)})$ and strict inclusions $c \subset f \subset \ell_\infty$ (see, [3]). This completes the proof.

5 Certain Matrix Mappings on the Sets $f(\triangle^{(m)})$ and $fs(\triangle^{(m)})$ and Some Duals

In present section, we shall characterize some matrix transformations between spaces of lamda almost convergent and almost convergent sequences in addition to paranormed and classical sequence spaces after giving $\beta-$ and γ duals of spaces $fs(\triangle^{(m)})$ and $f(\triangle^{(m)})$. We start with definition of $\beta-$ and $\gamma-$ duals.

If x and y are sequences and X and Y are subsets of ω, then we write $x \cdot y = (x_k y_k)_{k=0}^\infty$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$ and

$$M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a : a \cdot x \in Y \text{ for all } x \in X\}$$

for the multiplier space of X and Y; in particular, we use notations $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ for the $\beta-$ and γ–duals of X.

Lemma 5.1 *[9] $A = (a_{nk}) \in (f : \ell_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{6}$$

Lemma 5.2 *[9] $A = (a_{nk}) \in (f : c)$ if and only if (6) holds and there are $\alpha, \alpha_k \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ for all } k \in \mathbb{N},$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha,$$

$$\lim_{n\to\infty} \sum_k |\Delta(a_{nk} - \alpha_k)| = 0.$$

Theorem 5.3 *The $\gamma-$ dual of the space $f(\triangle^{(m)})$ is d_1, where*

$$d_1 = \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=1} \left| \sum_{j=k}^n \binom{m+j-k-1}{j-k} a_j \right| < \infty \right\}.$$

Proof: Take any sequence $a = (a_k) \in \omega$ and consider the following equality

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} y_j = \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j \right] y_k = (Cy)_n \tag{7}$$

for all $n \in \mathbb{N}$, where $C = \{c_{nk}\}$ is

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j, & 0 \le k \le n, \\ 0, & k > n \end{cases} \tag{8}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from (7) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in f(\triangle^{(m)})$ if and only if $Cy \in \ell_\infty$ whenever $y = (y_k) \in f$ where $C = \{c_{nk}\}$ is defined in (8). Therefore, with the help of Lemma 5.1 $(f(\triangle^{(m)}))^\gamma = d_1$.

Theorem 5.4 *The* $\beta-$ *dual of the space* $f(\triangle^{(m)})$ *is the intersection of sets*

$$\begin{aligned} d_2 &= \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j \ \ exists \right\}, \\ d_3 &= \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left[\sum_{j=k}^{n} \binom{m-j-1}{j} \right] \ \ exists \right\}, \\ d_4 &= \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{k} \triangle \left[\sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j - a_k \right] < \infty \right\}, \end{aligned}$$

where $a_k = \lim_{n\to\infty} \sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j$, *that is* $(f(\triangle^{(m)}))^\beta = d_1 \cap d_2 \cap d_3 \cap d_4$.

Proof: Let us take any sequence $a \in \omega$. By (7), $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in f(\triangle^{(m)})$ if and only if $Dy \in c$ whenever $y = (y_k) \in f$, where $C = \{c_{nk}\}$ defined in (8), we derive the consequence by Lemma 5.2 that $(f(\triangle^{(m)}))^\beta = d_1 \cap d_2 \cap d_3 \cap d_4$.

Theorem 5.5 *The* $\gamma-$ *dual of the space* $fs(\triangle^{(m)})$ *is the intersection of the sets*

$$\begin{aligned} c_1 &= \left\{ a = (a_k) \in \omega : \sup_{k} \sum_{k} \left| \triangle \left[\sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j \right] \right| < \infty \right\}, \\ c_2 &= \left\{ a = (a_k) \in \omega : \lim_{k\to\infty} \left(\sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j \right) = 0 \right\}, \end{aligned}$$

Namely, we have $[fs(\triangle^{(m)})]^\gamma = c_1 \cap c_2$.

Proof: We obtain from (7), $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in fs(\triangle^{(m)})$ if and only if $Cy \in \ell_\infty$ whenever $y = (y_k) \in fs$, where $C = \{c_{nk}\}$ is defined (8). Therefore, by Lemma 5.12(viii) $[fs(\triangle^{(m)}]^\gamma = c_1 \cap c_2$.

Theorem 5.6 *Define the set c_3 by*

$$c_3 = \left\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_k \left| \triangle^2 \left[\sum_{j=k}^{n} \binom{m+j-k-1}{j-k} a_j \right] \right| \ \textit{exists} \right\}.$$

Then, $[fs(\triangle^{(m)})]^\beta = c_1 \cap c_2 \cap c_3 \cap d_2$.

Proof: Let us take any sequence $a \in \omega$. By (7) $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in$ if and only if $Cy \in c$ whenever $y = (y_k) \in fs$. Since the column sequences of matrix $C = \{C_{nk}\}$ defined in (8) are convergent, we derive the consequence by Lemma 5.12(vii) that $[fs(\triangle^{(m)})]^\beta = c_1 \cap c_2 \cap c_3$.

For the sake of brevity following notations will be used:

$$\widetilde{a}(n,k,m) = \frac{1}{m+1} \sum_{i=0}^{m} \widetilde{a}_{n+i,k}, \qquad \tilde{a}(n,k) = \sum_{i=0}^{n} \tilde{a}_{ik},$$
$$c(n,k,m) = \frac{1}{m+1} \sum_{i=0}^{m} c_{n+i,k}, \qquad c(n,k) = \sum_{i=0}^{n} c_{ik},$$

where c_{nk} is defined in (8) and $\widetilde{a}_{nk} = \sum_{j=k}^{\infty} \binom{m+j-k-1}{j-k} a_{nj}$ for all $k, m, n \in \mathbb{N}$.

Assume that infinite matrices $\Upsilon = (\upsilon_{nk})$ and $\Omega = (\varpi_{nk})$ map the sequences $x = (x_k)$ and $y = (y_k)$ which are connected with relation (3) to the sequences $r = (r_n)$ and $s = (s_n)$, respectively, i.e.,

$$r_n = (\Upsilon x)_n = \sum_k \upsilon_{nk} x_k \ \text{ for all } \ n \in \mathbb{N}, \tag{9}$$

$$s_n = (\Omega y)_n = \sum_k \varpi_{nk} y_k \ \text{ for all } \ n \in \mathbb{N}. \tag{10}$$

One can easily conclude here that the method Υ is directly applied to the terms of the sequence $x = (x_k)$ while the method Ω is applied to the $\Delta^{(m)}$-transform of the sequence $x = (x_k)$. So, the methods Υ and Ω are essentially different.

Now suppose that matrix product $\Omega\Delta^{(m)}$ exists which is a much weaker assumption than conditions on the matrix Ω belonging to any matrix class, in general. It is not difficult to see that sequence in (10) reduces to the sequence in (9) as follows:

$$\begin{aligned}\left(\Omega y\right)_n &= \sum_k \sum_{j=0}^{m} (-1)^j \binom{m}{j} \varpi_{nk} x_{k-j} \\ &= \sum_k \sum_{j=k}^{\infty} (-1)^{j-k} \binom{m}{j-k} \varpi_{nj} x_k \\ &= \left(\Upsilon x\right)_n. \end{aligned} \tag{11}$$

Hence matrices $\Upsilon = (\upsilon_{nk})$ and $\Omega = (\varpi_{nk})$ are connected with the relations

$$\varpi_{nk} = \sum_{j=k}^{\infty} \binom{m+j-k-1}{j-k} \upsilon_{nj} \tag{12}$$

or

$$\upsilon_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} \binom{m}{j-k} \varpi_{nj} \text{ for all } k,n \in \mathbb{N}. \tag{13}$$

Note that the methods Υ and Ω are not necessarily equivalent since the order of summation may not be reversed.

We now give the following fundamental theorem connected with the matrix mappings on/into the almost convergent spaces $f(\Delta^{(m)})$ and $fs(\Delta^{(m)})$:

Theorem 5.7 *Let Y be any given sequence space and the matrices $\Upsilon = (\upsilon_{nk})$ and $\Omega = (\varpi_{nk})$ are connected with the relation (13). Then, $\Upsilon \in (f(\Delta^{(m)}) : Y)$ if and only if*

$$\Omega \in (f : Y) \textit{ and } (\upsilon_{nk})_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^{\beta} \textit{ for all } n \in \mathbb{N}. \tag{14}$$

Proof: Suppose that $\Upsilon = (\upsilon_{nk})$ and $\Omega = (\varpi_{nk})$ are connected with the relation (13) and let Y be any given sequence space and keep in mind that spaces $f(\Delta^{(m)})$ and f are norm isomorphic.

Let $\Upsilon \in (f(\Delta^{(m)}) : Y)$ and take any sequence $x \in f(\Delta^{(m)})$ and keep in mind that $y = \Delta^{(m)}x$. Then $(\upsilon_{nk})_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^{\beta}$ that is, (14) holds for all $n \in \mathbb{N}$ and $\Omega\Delta^{(m)}$ exists which implies that $(\varpi_{nk})_{k\in\mathbb{N}} \in \ell_1 = f^{\beta}$ for each $n \in \mathbb{N}$. Thus, Ωy exists for all $y \in f$. Hence by equality (11) we have $\Omega \in (f : Y)$.

On the other hand, assume that (14) holds and $\Omega \in (f : Y)$. Then, we have $(\varpi_{nk})_{k\in\mathbb{N}} \in \ell_1$ for all $n \in \mathbb{N}$ which gives together with $(\upsilon_{nk})_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^{\beta}$ for each $n \in \mathbb{N}$ that Υx exists. Then, equality $\Upsilon x = \Omega y$ in (11) again holds. Hence $\Upsilon x \in Y$ for all $x \in f(\Delta^{(m)})$, that is $\Upsilon \in (f(\Delta^{(m)}) : Y)$.

Theorem 5.8 *Let Y be any given sequence space and elements of infinite matrices $A = (a_{nk})$ and $C = (c_{nk})$ are connected with the relation*

$$c_{nk} = \sum_{j=k}^{m} (-1)^{j-k} \binom{m}{j} a_{n-j,k} \textit{ for all } k,n \in \mathbb{N}. \tag{15}$$

Then, $A=(a_{nk})\in(Y:f(\Delta^{(m)}))$ *if and only if* $C\in(Y:f)$.

Proof: Let $z=(z_k)\in Y$ and consider the following equality

$$\begin{aligned}\{\Delta^{(m)}(Az)\}_n &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}(Az)_{n-j}\\ &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}\sum_k a_{n-j,k}z_k\\ &= \sum_k\sum_{j=k}^{m}(-1)^{j-k}\begin{pmatrix} m \\ j\end{pmatrix}a_{n-j,k}z_k\end{aligned}$$

for all $n\in\mathbb{N}$. Whence, it can be seen from here that $Az\in f(\Delta^{(m)})$ whenever $z\in Y$ if and only if $Cz\in f$ whenever $z\in Y$. This completes the proof.

Theorem 5.9 *Let* Y *be any given sequence space and the matrices* $\Upsilon=(\upsilon_{nk})$ *and* $\Omega=(\varpi_{nk})$ *are connected with the relation (13). Then,* $\Upsilon\in(fs(\Delta^{(m)}):Y)$ *if and only if* $\Omega\in(fs:Y)$ *and* $(\upsilon_{nk})_{k\in\mathbb{N}}\in[fs(\Delta^{(m)})]^{\beta}$ *for all* $n\in\mathbb{N}$.

Proof: The proof is based on the proof of Theorem 5.7.

Theorem 5.10 *Let* Y *be any given sequence space and the elements of the infinite matrices* $A=(a_{nk})$ *and* $C=(c_{nk})$ *are connected with the relation (15). Then,* $A=(a_{nk})\in(Y:fs(\Delta^{(m)}))$ *if and only if* $C\in(Y:fs)$.

Proof: The proof is based on the proof of Theorem 5.8.

By Theorem 5.7, Theorem 5.8, Theorem 5.9 and Theorem 5.10 we have quite a few outcomes depending on the choice of space Y to characterize certain matrix mappings. Hence, by help of these theorems the necessary and sufficient conditions for classes $(f(\Delta^{(m)}):Y)$, $(Y:f(\Delta^{(m)}))$, $(fs(\Delta^{(m)}):Y)$ and $(Y:fs(\Delta^{(m)}))$ may be derived by replacing the entries of Υ and A by those of the entries of $\Omega=\Upsilon[\Delta^{(m)}]^{-1}$ and $C=\Delta^{(m)}A$, respectively; where the necessary and sufficient conditions on matrices Ω and C are read from the concerning results in the current literature.

Lemma 5.11 *Let* $A=(a_{nk})$ *be an infinite matrix. Then, the following statements hold:*

i $A\in(c_0(p):f)$ if and only if

$$\exists N>1\ni\quad\sup_{m\in\mathbb{N}}\sum_k|a(n,k,m)|\,N^{-1/p_k}<\infty\ \text{ for all }\ n\in\mathbb{N},\tag{16}$$

$$\exists\alpha_k\in\mathbb{C}\ \text{ for all }\ k\in\mathbb{N}\ni\quad\lim_{m\to\infty}a(n,k,m)=\alpha_k\ \text{uniformly in } n.\tag{17}$$

ii $A \in (c(p) : f)$ if and only if (16), (17) hold and

$$\exists \alpha \in \mathbb{C} \ni \quad \lim_{m \to \infty} \sum_k a(n,k,m) = \alpha \text{ uniformly in } n. \tag{18}$$

iii $A \in (\ell_\infty(p) : f)$ if and only if (16), (17) hold and

$$\exists N > 1 \ni \quad \lim_{m \to \infty} \sum_k |a(n,k,m) - \alpha_k| \, N^{1/p_k} = 0 \text{ uniformly in } n. \tag{19}$$

Lemma 5.12 *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold.*

i *[Duran, [31]] $A \in (\ell_\infty : f)$ if and only if (6) holds and*

$$f - \lim a_{nk} = \alpha_k \textit{ exists for each fixed } k, \tag{20}$$

$$\lim_{m \to \infty} \sum_k |a(n,k,m) - \alpha_k| = 0 \textit{ uniformly in } n. \tag{21}$$

ii *[King, [33]] $A \in (c : f)$ if and only if (6), (20) hold and*

$$f - \lim \sum_k a_{nk} = \alpha. \tag{22}$$

iii *[Başar and Çolak, [39]] $A \in (cs : f)$ if and only if (20) holds and*

$$\sup_{n \in \mathbb{N}} \sum_k |\Delta a_{nk}| < \infty \ . \tag{23}$$

iv *[Başar and Çolak, [39]] $A \in (bs : f)$ if and only if (20), (23) hold and*

$$\lim_k a_{nk} = 0 \textit{ exists for each fixed } n, \tag{24}$$

$$\lim_{q \to \infty} \sum_k \frac{1}{q+1} \sum_{i=0}^{q} |\Delta \left[a(n+i,k) - \alpha_k \right]| = 0 \textit{ uniformly in } n. \tag{25}$$

v *[Duran, [31]] $A \in (f : f)$ if and only if (6), (20), (22) hold and*

$$\lim_{m \to \infty} \sum_k |\Delta \left[a(n,k,m) - \alpha_k \right]| = 0 \textit{ uniformly in } n. \tag{26}$$

vi *[Başar, [38]] $A \in (fs : f)$ if and only if (20), (24), (26) and (25) hold.*

vii *[Öztürk, [32]] $A \in (fs : c)$ if and only if (5.2), (23), (24) hold and*

$$\lim_{n\to\infty} \sum_k \left|\Delta^2 a_{nk}\right| = \alpha, \tag{27}$$

viii *$A \in (fs : \ell_\infty)$ if and only if (23) and (24) hold.*

ix *[Başar and Solak, [34]] $A \in (bs : fs)$ if and only if (24), (25) hold and*

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta a(n,k)| < \infty \ , \tag{28}$$

$$f - \lim a(n,k) = \alpha_k \text{ exists for each fixed } k. \tag{29}$$

x *[Başar, [38]] $A \in (fs : fs)$ if and only if (25), (28), (29) hold and*

$$\lim_{q\to\infty} \sum_k \frac{1}{q+1} \sum_{i=0}^{q} \left|\Delta^2 \left[a(n+i,k) - \alpha_k\right]\right| = 0 \text{ uniformly in } n, \tag{30}$$

xi *[Başar and Çolak, [39]] $A \in (cs : fs)$ if and only if (28) and (29) hold.*

xii *[Başar, [40]] $A \in (f : cs)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a(n,k)| < \infty \ , \tag{31}$$

$$\sum_n a_{nk} = \alpha_k \text{ exists for each fixed } k, \tag{32}$$

$$\sum_n \sum_k a_{nk} = \alpha, \tag{33}$$

$$\lim_{m\to\infty} \sum_k |\Delta \left[a(n,k) - \alpha_k\right]| = 0. \tag{34}$$

Now we give our main results related to matrix mappings on/into spaces of almost convergent series $fs(\Delta^{(m)})$ and sequences $f(\Delta^{(m)})$.

Corollary 5.13 *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

i $A \in (fs(\Delta^{(m)}) : f)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and (20), (24) hold with $\widetilde{a}_{nk}$ instead of a_{nk}, (26) holds with $\widetilde{a}(n,k,m)$ instead of $a(n,k,m)$ and (25) holds with $\tilde{a}(n,k)$ instead of $a(n,k)$.

ii $A \in (fs(\Delta^{(m)}) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and (5.2), (23), (24) and (27) hold with $\widetilde{a}_{nk}$ instead of a_{nk}.

iii $A \in (fs(\Delta^{(m)}) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and (23) and (24) hold with $\widetilde{a}_{nk}$ instead of a_{nk}.

iv $A \in (fs(\Delta^{(m)}) : fs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^\beta$ for all $n \in \mathbb{N}$ and (25), (28), (29) and (30) hold with $\tilde{a}(n,k)$ instead of $a(n,k)$.

v $A \in (cs : fs(\Delta^{(m)}))$ if and only if (28) and (29) hold with $c(n,k)$ instead of $a(n,k)$.

vi $A \in (bs : fs(\Delta^{(m)}))$ if and only if (24) holds with c_{nk} instead of a_{nk}, (25), (28) and (29) hold with $c(n,k)$ instead of $a(n,k)$.

vii $A \in (fs : fs(\Delta^{(m)}))$ if and only if (25), (28), (29) and (30) hold with $c(n,k)$ instead of $a(n,k)$.

Corollary 5.14 *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

i $A \in (c(p) : f(\Delta^{(m)}))$ if and only if (16), (17) and (18) hold with $c(n,k,m)$ instead of $a(n,k,m)$.

ii $A \in (c_0(p) : f(\Delta^{(m)}))$ if and only if (16) and (17) hold with $c(n,k,m)$ instead of $a(n,k,m)$.

iii $A \in (\ell_\infty(p) : f(\Delta^{(m)}))$ if and only if (16), (17) and (19) hold with $c(n,k,m)$ instead of $a(n,k,m)$.

Corollary 5.15 *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

i $A \in (f(\Delta^{(m)}) : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^\beta$ *for all* $n \in \mathbb{N}$ *and (6) holds with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.

ii $A \in (f(\Delta^{(m)}) : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^\beta$ *for all* $n \in \mathbb{N}$ *and (6), (5.2), (5.2) and (5.2) hold with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.

iii $A \in (f(\Delta^{(m)}) : bs)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^\beta$ *for all* $n \in \mathbb{N}$ *and (28) holds with* $\tilde{a}(n,k)$ *instead of* $a(n,k)$.

iv $A \in (f(\Delta^{(m)}) : cs)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in [f(\Delta^{(m)})]^\beta$ *for all* $n \in \mathbb{N}$ *and (5.2)-(31) hold with* $\tilde{a}(n,k)$ *instead of* $a(n,k)$.

Corollary 5.16 *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

i $A \in (\ell_\infty : f(\Delta^{(m)}))$ *if and only if (6), (20) hold with* c_{nk} *instead of* a_{nk} *and (21) holds with* $c(n,k,m)$ *instead of* $a(n,k,m)$.

ii *$A \in (f : f(\Delta^{(m)}))$ if and only if (6), (20), (26) hold with $c(n,k,m)$ instead of $a(n,k,m)$ and (22) hold with c_{nk} instead of a_{nk}.*

iii *$A \in (c : f(\Delta^{(m)}))$ if and only if (6), (20) and (22) hold with c_{nk} instead of a_{nk}.*

iv *$A \in (bs : f(\Delta^{(m)}))$ if and only if (20), (23), (24) hold with c_{nk} instead of a_{nk} and (25) holds with $c(n,k)$ instead of $a(n,k)$.*

v *$A \in (fs : f(\Delta^{(m)}))$ if and only if (20), (24) hold with c_{nk} instead of a_{nk}, (26) holds with $c(n,k,m)$ instead of $a(n,k,m)$ and (25) holds with $c(n,k)$ instead of $a(n,k)$.*

vi *$A \in (cs : f(\Delta^{(m)}))$ if and only if (23) and (24) hold with c_{nk} instead of a_{nk}.*

Remark 5.17 *Characterization of the classes $(f(\Delta^{(m)}) : f_\infty)$, $(f_\infty : f(\Delta^{(m)}))$, $(fs(\Delta^{(m)}) : f_\infty)$ and $(f_\infty : fs(\Delta^{(m)}))$ is redundant since spaces of almost bounded sequences f_∞ and ℓ_∞ are equal.*

6 Some Core Theorems

Let us start with the definition of $\triangle^{(m)}-$core of x. Let $x \in \ell_\infty$ then $\triangle^{(m)}-$core of x defined by closed interval $[R(x), r(x)]$, where

$$\begin{aligned} R(x) &= \limsup_{n\to\infty} \sup_{l\in\mathbb{N}} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{i=0}^{k} (-1)^{k-i} \binom{m}{i} x_{k-i+l}, \\ r(x) &= \liminf_{n\to\infty} \sup_{l\in\mathbb{N}} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{i=0}^{k} (-1)^{k-i} \binom{m}{i} x_{k-i+l}. \end{aligned}$$

Hence, it is easy to see that $\triangle^{(m)}-$core of x is α if and only if $f(\triangle^{(m)})-\lim x = \alpha$.

Theorem 6.1 *$\triangle^{(m)} - core(Ax) \subseteq K - core(x), (R(Ax) \leq L(x))$ for all $x \in \ell_\infty$ if and only if $A \in (c : f(\triangle^{(m)}))_{reg}$ and*

$$\lim_{n\to\infty} \sup_{l\in\mathbb{N}} \sum_{i} \frac{1}{n+1} \left| \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{m}{j} a_{k-j+l,i} \right| = 1. \tag{35}$$

Proof: Assume that $\triangle^{(m)} - core(Ax)$. Let $x = (x_k)$ be a convergent sequences, so we have $L(x) = l(x)$. By given assumption, we have

$$l(x) \leq r(Ax) \leq R(Ax) \leq L(x).$$

Hence, we obtain the equalties $R(Ax) = r(Ax) = \lim x$ which imply that $A \in (c : f(\triangle^{(m)}))_{reg}$. Now, let us consider sequences $D = (d_{nk}(m))$ of infinite matrices defined by

$$d_{nk}(m) = \sum_{k=0}^{n} \frac{1}{n+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{m}{j} a_{k-j+l,i}.$$

Since $A \in (c : f(\triangle^{(m)}))_{reg}$, it is easy to see that conditions of Lemma 2 (see Das, [17]) are satisfied for the matrix sequence A. Hence there exists $y \in \ell_\infty$ such that $\|y\| \leq 1$ and

$$R(Ay) = \limsup_{n\to\infty} \sup_{l} \sum_{k} |d_{nk}(m)|.$$

Hence $x = e = (1, 1, 1 \ldots)$, by using hypothesis, we can write

$$\begin{aligned} 1 &= r(Ae) \leq \liminf_{n\to\infty} \sup_{l\in\mathbb{N}} \sum_{k} |d_{nk}(m)| \leq \limsup_{n\to\infty} \sup_{l\in\mathbb{N}} \sum_{k} |d_{nk}(m)| \\ &= R(Ay) \leq L(y) \leq \|y\| \leq 1 \end{aligned}$$

which proves necessity of (35).

On the other hand, let $A \in (c : f(\triangle^{(m)}))_{reg}$ and (35) hold for all $x \in \ell_\infty$. We define any real number μ we write $\mu^+ = \max\{0, \mu\}$ and $\mu^- = \max\{-\mu, 0\}$ then $|\mu| = \mu^+ + \mu^-$ and $\mu = \mu^+ - \mu^-$. Hence for any given $\epsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $x_k < L(x) + \epsilon$ for all $k > k_0$. Then, we can write

$$\begin{aligned} \sum_{k} d_{nk}(m) &= \sum_{k<k_0} d_{nk}(m)x_k + \sum_{k\geq k_0} (d_{nk}(m))^+ x_k - \sum_{k\geq k_0} (d_{nk}(m))^- x_k \\ &\leq \|x\|_\infty \sum_{k<k_0} |d_{nk}(m)| + [L(x)+\epsilon] \sum_{k\geq k_0} |d_{nk}(m)| \\ &\quad + \|x\|_\infty \sum_{k\geq k_0} [|d_{nk}(m)| - d_{nk}(m)]. \end{aligned}$$

Thus, by applying $\limsup_n \sup_{l\in\mathbb{N}}$ to the above equation and using our hypothesis, we have $R(x) \leq L(x) + \epsilon$. This completes the proof, since ϵ is arbitrary and $x \in \ell_\infty$.

Theorem 6.2 *$A \in (S \cap \ell_\infty : f(\triangle^{(m)}))_{reg}$ if and only if $A \in (c : f(\triangle^{(m)}))_{reg}$ and*

$$\lim_{n\to\infty} \sum_{i\in E} \frac{1}{n+1} \left| \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{m}{j} a_{k-j+l,i} \right| = 0 \tag{36}$$

for every $E \subseteq \mathbb{N}$ with $\delta(E) = 0$.

Proof: Firstly, suppose that $A \in (S \cap \ell_\infty : f(\triangle^{(m)}))_{reg}$. Then $A \in (c : f(\triangle^{(m)}))_{reg}$ immediately follows from the fact that $c \subseteq S \cap \ell_\infty$. Now define sequence $z = (z_k)$ for all $x \in \ell_\infty$ as

$$z_k = \begin{cases} x_k, & k \in E, \\ 0, & k \notin E, \end{cases}$$

where E any subset of $\mathbb{N}$ with $\delta(E) = 0$. By our assumption, since $z \in S_0$, we have $Az \in f(\triangle^{(m)})$. On the other hand, since $Az = \sum_{k\in E} a_{nk} z_k$, the matrix $C = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} a_{nk}, & k \in E, \\ 0, & k \notin E \end{cases}$$

for all n, must belong to the class $(\ell_\infty : f(\triangle^{(m)}))$. Hence, the necessity (36) follows from Corollary (5.16)(i).
Conversely, let $A \in (c : f(\triangle^{(m)}))_{reg}$ and (36) holds. Let x be any sequence in $S \cap \ell_\infty$ with $st - \lim x = s$ and write $E = \{i : |x_i - s| \geq \epsilon\}$ for any given $\epsilon > 0$, so that $\delta(E) = 0$. Since $A \in (c : f(\triangle^{(m)}))_{reg}$ and $f(\triangle^{(m)}) - \lim \sum_k a_{nk} = 1$, we have

$$\begin{aligned} f(\triangle^{(m)}) - \lim(Ax) &= f(\triangle^{(m)}) - \lim \left(\sum_k a_{nk}(x_k - s) + s \sum_k a_{nk} \right) \\ &= f(\triangle^{(m)}) - \lim \sum_k a_{nk}(x_k - s) + s \\ &= \lim_{n\to\infty} \sup_{l\in\mathbb{N}} \sum_i \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{m}{j} a_{l+k-j,i}(x_i - s) + s. \end{aligned}$$

On the other hand, since we have

$$\begin{aligned} \left| \sum_i \frac{1}{n+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{m}{j} a_{k-j+l,i}(x_i - s) \right| &\leq \frac{\|x\|_\infty}{n+1} \sum_{i\in E} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{m}{j} a_{k-j+l,i} \right| \\ &\quad + \epsilon \|A\| \end{aligned}$$

condition (36) implies that

$$\lim_{n\to\infty} \sum_i \frac{1}{n+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{m}{j} a_{k-j+l,i}(x_i - s) = 0 \qquad \text{uniformly } l.$$

Therefore, $f(\triangle^{(m)}) - \lim(Ax) = st - \lim x$ that is $A \in (S \cap \ell_\infty : f(\triangle^{(m)}))_{reg}$ which completes the proof.

Theorem 6.3 *$\triangle^{(m)} - core(Ax) \subseteq st - core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (S \cap \ell_\infty : f(\triangle^{(m)}))_{reg}$ and (35) holds.*

Proof: Assume that the inclusion $\triangle^{(m)} - core(Ax) \subseteq st - core(x)$ holds for each bounded sequence x. Then, $R(Ax) \leq st - \sup(x)$ for all $x \in \ell_\infty$. Hence one may easily see that the following inequalities hold:

$$st - \inf(x) \leq r(Ax) \leq R(Ax) \leq st - \sup(x).$$

If $x \in (S \cap \ell_\infty)$, then we have $st - \inf(x) = st - \sup(x) = st - \lim x$. Which implies that $st - \sup(x) = r(x) = R(x) = f(\triangle^{(m)}) - \lim(Ax)$ that is $A \in (c : f(\triangle^{(m)}))_{reg}$.

On the other hand, let $A \in (S \cap \ell_\infty : f(\triangle^{(m)}))_{reg}$ and (35) hold for all $x \in \ell_\infty$. Then $st - \sup(x)$ is finite. Let E be a subset of $\mathbb{N}$ defined by $E = \{p : x_k > st - \sup(x) + \epsilon\}$ for a given $\epsilon > 0$. Then obvious that $\delta(E) = 0$ and $x_k \leq st - \sup(x) + \epsilon$, if $p \notin E$.

We define any real number μ we write $\mu^+ = \max\{0, \mu\}$ and $\mu^- = \max\{-\mu, 0\}$ then $|\mu| = \mu^+ + \mu^-$ and $\mu = \mu^+ - \mu^-$, then for fixed positive integer i_0 we can write

$$\begin{aligned}
\sum_i d_{ni}(m) &= \sum_{i<i_0} d_{ni}(m)x_i + \sum_{\substack{i\geq i_0 \\ i\in E}} (d_{ni}(m))^+ x_i \\
&+ \sum_{\substack{i> i_0 \\ i\notin E}} (d_{ni}(m))^- x_i - \sum_{i\geq i_0} (d_{ni}(m))^+ x_i \\
&\leq \|x\|_\infty \sum_{i<i_0} |d_{ni}(m)| + [st - \sup(x) + \epsilon] \sum_{\substack{i\geq i_0 \\ i\notin E}} |d_{ni}(m)| \\
&+ \|x\|_\infty \sum_{\substack{i\geq i_0 \\ i\in E}} |d_{ni}(m)| + \|x\|_\infty \sum_{i\geq i_0} [|d_{ni}(m)| - d_{ni}(m)].
\end{aligned}$$

Whence, applying $\limsup_n \sup_{l\in\mathbb{N}}$ to the above equation and using the hypothesis, we obtain $R(x) \leq st - \sup(x) + \epsilon$. This completes the proof since ϵ is arbitrary and $x \in \ell_\infty$.

7 Conclusion

Although the concept of almost convergence was defined by Lorentz [11], in 1948, neither the algebraic structure nor the topological structure of the space f has been essentially studied until Başar and Kirişçi [3, Section 3]. Two basic results related to the space f: "f is a non-seperable closed subspace of $(\ell_\infty, \|.\|_\infty)$" and "Banach space f has no Shauder basis" are given in their paper. One of the nice parts of their paper was to find beta- and gamma duals of set of almost convergent series fs. As a generalization of the spaces Başar and Kirişçi, the sequence space $f(B)$ which is matrix domain of the triple band matrix $B(r, s, t)$ in space f has recently been examined by Sönmez.

In the present paper, we study the domains of the triangle matrix $\Delta^{(m)}$ in almost convergent sequence spaces f and f_0 and series space fs. Nevertheless, the present results does not compare with the results obtained by Sönmez [8] and Başar and Kirişçi [3]. Corollaries 5.13 and 5.14 have a special importance to characterize the matrix classes $(\gamma : f(\Delta^{(m)}))$, $(fs(\Delta^{(m)}) : \delta)$ and $(\eta : fs(\Delta^{(m)}))$ where $\gamma \in \{c(p), c_0(p), \ell_\infty(p)\}$, $\delta \in \{f, fs, c, \ell_\infty\}$ and $\eta \in \{cs, fs, bs\}$. Finally, we should mention that some of the original aspects of this study are to give some core theorems connected with the matrix classes on/into almost convergent sequence space $f(\Delta^{(m)})$ besides the beta and gamma duals of the set $fs(\Delta^{(m)})$. In addition to that the non-existence of Schauder basis of spaces fs and $fs(\Delta^{(m)})$ was shown, which is an important result.

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