Gen. Math. Notes, Vol. 24, No. 1, September 2014, pp. 98-108

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On Characterization of Inextensible Flows of Dual Curves According to Dual Darboux Frame

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(Received: 30-5-14 / Accepted: 12-7-14)

Abstract

In this paper, we study inextensible flows dual curves according to dual Darboux frame. Necessary and sufficient conditions for an inelastic dual curve flow are expressed as a partial differential equation involving the dual geodesic curvature.

Keywords: *Dual Darboux Frame, Inextensible flows.*

1 Introduction

Recently, the study of the motion of inelastic curves has an important role. The time evolution of a curve represented by its corresponding flow. The flow of a curve is said to be inextensible if, firstly its arc length is preserved and secondly its intrinsic curvature is preserved. Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve. Some movement in nature is inspired to examine flow of curves as snake and elephant's trunk movement. For example, both Chirikjian and Burdick [5] and Mochiyama et al. [9] study the shape control hyper-redundant, or snake-like robots. Inextensible curve and surface flows emerge many problems in computer vision [14] and computer animation [10].

Particularly, inextensible time evolution of curves and surfaces is examined mathematically. Significant methods of this article developed by Gage and Hamilton [11], and Grayson [13] for studying the shrinking of closed plane curves to a circle via the heat equation. In [12] Gage also studies area preserving evolution of inelastic plane curves. Another related study is that [6], which considers less restrictive mappings that locally preserve volume only. In [2, 3] Know et al. study evolution of inelastic plane curves, and inextensible flows of curves and developable surfaces.

In this paper, we study inextensible flows dual curves according to dual Darboux frame. Necessary and sufficient conditions for an inelastic dual curve flow are expressed as a partial differential equation involving the dual geodesic curvature.

2 Preliminaries

Let $D = IR \times IR = \left\{ \overline{a} = (a, a^*) : a, a^* \in IR \right\}$ be the set of the pairs (a, a^*). For $\overline{a} = (a, a^*)$, $\overline{b} = (b, b^*) \in D$ the following operations are defined on *D*:

Equality:	$\overline{a} = \overline{b} \Leftrightarrow a = b,\ a^* = b^*$
Addition:	$\overline{a} + \overline{b} \Leftrightarrow (a + b,\ a^* + b^*)$
Multiplication:	$\overline{a}\overline{b} \Leftrightarrow (ab,\ ab^* + a^* b)$

The element $\varepsilon = (0,1) \in D$ satisfies the relationships

$$\varepsilon \neq 0, \quad \varepsilon^2 = 0, \quad \varepsilon 1 = 1\varepsilon = \varepsilon \tag{2.1}$$

Let consider the element $\overline{a} \in D$ of the form $\overline{a} = (a, 0)$. Then the mapping $f : D \to IR,\ \ f(a, 0) = a$ is a isomorphism. So, we can write $a = (a, 0)$. By the multiplication rule we have that

$$\overline{a} = (a, a^*) = a + \varepsilon a^*$$

Then $\overline{a} = a + \varepsilon a^*$ is called dual number and ε is called dual unit. Thus the set of dual numbers is given by

$$D = \left\{ \overline{a} = a + \varepsilon a^* : a, a^* \in IR,\ \varepsilon^2 = 0 \right\} \tag{2.2}$$

The set D forms a commutative group under addition. The associative laws hold for multiplication. Dual numbers are distributive and form a ring over the real number field [4].

Dual function of dual number presents a mapping of a dual numbers space on itself. Properties of dual functions were thoroughly investigated by Dimentberg [4]. He derived the general expression for dual analytic (differentiable) function as follows

$$f(\bar{x}) = f(x+\varepsilon x^*) = f(x) + \varepsilon x^* f'(x), \tag{2.3}$$

where $f'(x)$ is derivative of $f(x)$ and $x, x^* \in IR$. This definition allows us to write the dual forms of some well-known functions as follows

$$\begin{cases} \cos(\bar{x}) = \cos(x+\varepsilon x^*) = \cos(x) - \varepsilon x^* \sin(x), \\ \sin(\bar{x}) = \sin(x+\varepsilon x^*) = \sin(x) + \varepsilon x^* \cos(x), \\ \sqrt{\bar{x}} = \sqrt{x+\varepsilon x^*} = \sqrt{x} + \varepsilon \dfrac{x^*}{2\sqrt{x}}, \ (x>0). \end{cases} \tag{2.4}$$

Let $D^3 = D \times D \times D$ be the set of all triples of dual numbers, i.e.,

$$D^3 = \left\{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) : \bar{a}_i \in D,\ i = 1,2,3 \right\} \tag{2.5}$$

Then the set D^3 is called dual space. The elements of D^3 are called dual vectors. Similar to the dual numbers, a dual vector $\tilde{a}$ may be expressed in the form $\tilde{a} = \vec{a} + \varepsilon \vec{a}^* = (\vec{a}, \vec{a}^*)$, where $\vec{a}$ and $\vec{a}^*$ are the vectors of IR^3. Then for any vectors $\tilde{a} = \vec{a} + \varepsilon \vec{a}^*$ and $\tilde{b} = \vec{b} + \varepsilon \vec{b}^*$ of D^3, the scalar product and the vector product are defined by

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon \left(\left\langle \vec{a}, \vec{b}^* \right\rangle + \left\langle \vec{a}^*, \vec{b} \right\rangle \right), \tag{2.6}$$

and

$$\tilde{a} \times \tilde{b} = \vec{a} \times \vec{b} + \varepsilon \left(\vec{a} \times \vec{b}^* + \vec{a}^* \times \vec{b} \right), \tag{2.7}$$

Respectively, where $\left\langle \vec{a}, \vec{b} \right\rangle$ and $\vec{a} \times \vec{b}$ are the inner product and the vector product of the vectors $\vec{a}$ and $\vec{a}^*$ in IR^3, respectively.

The norm of a dual vector $\tilde{a}$ is given by

$$\|\tilde{a}\| = \|\vec{a}\| + \varepsilon \frac{\left\langle \vec{a}, \vec{a}^* \right\rangle}{\|\vec{a}\|}, \quad (\vec{a} \neq 0). \tag{2.8}$$

A dual vector $\tilde{a}$ with norm $1+\varepsilon 0$ is called dual unit vector. The set of dual unit vectors is given by

$$\tilde{S}^2 = \left\{ \tilde{a} = (a_1, a_2, a_3) \in D^3 : \left\langle \tilde{a}, \tilde{a} \right\rangle = 1 + \varepsilon 0 \right\}, \tag{2.9}$$

and called dual unit sphere[8,15].

In the Euclidean 3-space IR^3, an oriented line L is determined by a point $p \in L$ and a unit vector $\vec{a}$. Then, one can define $\vec{a}^* = \vec{p} \times \vec{a}$ which is called moment vector. The value of $\vec{a}^*$ does not depend on the point p, because any other point q in L can be given by $\vec{q} = \vec{p} + \lambda \vec{a}$ and then $\vec{a}^* = \vec{p} \times \vec{a} = \vec{q} \times \vec{a}$. Reciprocally, when such a pair $(\vec{a}, \vec{a}^*)$ is given, one recovers the line L as $L = \left\{ (\vec{a} \times \vec{a}^*) + \lambda \vec{a} : \vec{a}, \vec{a}^* \in E^3, \lambda \in IR \right\}$, written in parametric equations. The vectors $\vec{a}$ and $\vec{a}^*$ are not independent of one another and they satisfy the following relationships

$$\left\langle \vec{a}, \vec{a} \right\rangle = 1, \quad \left\langle \vec{a}, \vec{a}^* \right\rangle = 0 \tag{2.10}$$

The components a_i, a_i^* $(1 \le i \le 3)$ of the vectors $\vec{a}$ and $\vec{a}^*$ are called the normalized Plucker coordinates of the line L. We see that the dual unit vector $\tilde{a} = \vec{a} + \varepsilon \vec{a}^*$ corresponds to the line L. This correspondence is known as E. Study Mapping: There exists a one-to-one correspondence between the vectors of dual unit sphere $\tilde{S}^2$ and the directed lines of the space IR^3. By the aid of this correspondence, the properties of the spatial motion of a line can be derived. Hence, the geometry of ruled surface is represented by the geometry of dual curves lying on the dual unit sphere $\tilde{S}^2$.

The angle $\overline{\theta} = \theta + \varepsilon \theta^*$ between two dual unit vectors $\tilde{a}, \tilde{b}$ is called *dual angle* and defined by

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \cos \overline{\theta} = \cos \theta - \varepsilon \theta^* \sin \theta. \tag{2.11}$$

By considering the E. Study Mapping, the geometric interpretation of dual angle is that, θ is the real angle between the lines L_1, L_2 corresponding to the dual unit vectors $\tilde{a}, \tilde{b}$, respectively, and θ^* is the shortest distance between those lines [9].

Let now $(\tilde{x})$ be a dual curve represented by the dual vector $\tilde{e}(u) = \vec{e}(u) + \varepsilon \vec{e}^*(u)$. The unit vector $\vec{e}$ draws a curve on the real unit sphere S^2 and is called the (real) indicatrix of $(\tilde{x})$. We suppose throughout that it is not a single point. We take the parameter u as the arc-length parameter s of the real indicatrix and denote the differentiation with respect to s by primes. Then we have $\left\langle \vec{e}', \vec{e}' \right\rangle = 1$. The vector

$\vec{e}' = \vec{t}$ is the unit vector parallel to the tangent of the indicatrix. The equation $\vec{e}^*(s) = \vec{p}(s) \times \vec{e}(s)$ has infinity of solutions for the function $\vec{p}(s)$. If we take $\vec{p}_o(s)$ as a solution, the set of all solutions is given by $\vec{p}(s) = \vec{p}_o(s) + \lambda(s)\vec{e}(s)$, where λ is a real scalar function of s. Therefore we have $\langle \vec{p}', \vec{e}' \rangle = \langle \vec{p}_o', \vec{e}' \rangle + \lambda$. By taking $\lambda = \lambda_o = -\langle \vec{p}_o', \vec{e}' \rangle$ we see that $\vec{p}_o(s) + \lambda_o(s)\vec{e}(s) = \vec{c}(s)$ is the unique solution for $\vec{p}(s)$ with $\langle \vec{c}', \vec{e}' \rangle = 0$. Then, the given dual curve $(\tilde{x})$ corresponding to the ruled surface

$$\varphi_e = \vec{c}(s) + v\vec{e}(s) \tag{2.12}$$

may be represented by

$$\tilde{e}(s) = \vec{e} + \varepsilon \vec{c} \times \vec{e} \tag{2.13}$$

where

$$\langle \vec{e}, \vec{e} \rangle = 1, \ \langle \vec{e}', \vec{e}' \rangle = 1, \ \langle \vec{c}', \vec{e}' \rangle = 0 . \tag{2.14}$$

Then we have

$$\|\tilde{e}'\| = \vec{t} + \varepsilon \det(\vec{c}', \vec{e}, \vec{t}) = 1 + \varepsilon\Delta \tag{2.15}$$

where $\Delta = \det(\vec{c}', \vec{e}, \vec{t})$. The dual arc-length $\bar{s}$ of the dual curve $(\tilde{x})$ is given by

$$\bar{s} = \int_0^s \|\tilde{e}'(u)\| du = \int_0^s (1 + \varepsilon\Delta) du = s + \varepsilon \int_0^s \Delta du \tag{2.16}$$

From (16) we have $\bar{s}' = 1 + \varepsilon\Delta$. Therefore, the dual unit tangent to the curve $\tilde{e}(s)$ is given by

$$\frac{d\tilde{e}}{d\bar{s}} = \frac{\tilde{e}'}{\bar{s}'} = \frac{\tilde{e}'}{1 + \varepsilon\Delta} = \tilde{t} = \vec{t} + \varepsilon(\vec{c} \times \vec{t}) \tag{2.17}$$

Introducing the dual unit vector $\tilde{g} = \tilde{e} \times \tilde{t} = \vec{g} + \varepsilon \vec{c} \times \vec{g}$ we have the dual frame $\{\tilde{e}, \tilde{t}, \tilde{g}\}$ which is known as dual geodesic trihedron or dual Darboux frame of φ_e (or $(\tilde{e})$). Also, it is well known that the real orthonormal frame $\{\vec{e}, \vec{t}, \vec{g}\}$ is called the geodesic trihedron of the indicatrix $\vec{e}(s)$ with the derivations

$$\vec{e}' = \vec{t}, \ \vec{t}' = \gamma\vec{g} - \vec{e}, \ \vec{g}' = -\gamma\vec{t} \tag{2.18}$$

where γ is called the conical curvature [1]. Similar to (2.18), the derivatives of the vectors of the dual frame $\{\tilde{e}, \tilde{t}, \tilde{g}\}$ are given by

$$\frac{d\tilde{e}}{d\bar{s}} = \tilde{t}, \ \frac{d\tilde{t}}{d\bar{s}} = \bar{\gamma}\tilde{g} - \tilde{e}, \ \frac{d\tilde{g}}{d\bar{s}} = -\bar{\gamma}\tilde{t} \tag{2.19}$$

where

$$\bar{\gamma} = \gamma + \varepsilon(\delta - \gamma\Delta), \quad \delta = \langle \vec{c}', \vec{e} \rangle \tag{2.20}$$

and the dual darboux vector of the frame is $\tilde{d} = \overline{\gamma}\tilde{e} + \tilde{g}$. From the definition of Δ and (2.20) we also have

$$\vec{c}\,' = \delta\vec{e} + \Delta\vec{g} \tag{2.21}$$

The dual curvature of dual curve (ruled surface) $\tilde{e}(s)$ is

$$\overline{R} = \frac{1}{\sqrt{(1+\overline{\gamma}^2)}} \tag{2.22}$$

The unit vector $\tilde{d}_o$ with the same sense as the Darboux vector $\tilde{d} = \overline{\gamma}\tilde{e} + \tilde{g}$ is given by

$$\tilde{d}_o = \frac{\overline{\gamma}}{\sqrt{(1+\overline{\gamma}^2)}}\tilde{e} + \frac{1}{\sqrt{(1+\overline{\gamma}^2)}}\tilde{g} \tag{2.23}$$

Then, the dual angle between $\tilde{d}_o$ and $\tilde{e}$ satisfies the followings

$$\cos\overline{\rho} = \frac{\overline{\gamma}}{\sqrt{(1+\overline{\gamma}^2)}}, \quad \sin\overline{\rho} = \frac{1}{\sqrt{(1+\overline{\gamma}^2)}} \tag{2.24}$$

where $\overline{\rho}$ is the dual spherical radius of curvature. Hence $\overline{R} = \sin\overline{\rho}$, $\overline{\gamma} = \cot\overline{\rho}$ [7].

3 Inextensible Flows of Dual Curves according to Dual Darboux Frame

Throughout this study, we assume that $\tilde{x} : [0,1]\times[0,\omega] \to \mathbb{D}^3$ is a one parameter family of smooth dual curves in dual space $\mathbb{D}^3$. Let $\overline{u}$ be the curve parametrization variable, $0 \le \overline{u} \le 1$.

The arclenght of $\tilde{x}$ is given by

$$\overline{s} = \int_0^u \left|\frac{\partial\tilde{x}}{\partial\overline{u}}\right| d\overline{u} = \int_0^u \overline{v}\, d\overline{u} \tag{3.1}$$

where

$$\left|\frac{\partial\tilde{x}}{\partial\overline{u}}\right| = \left|\left\langle \frac{\partial\tilde{x}}{\partial\overline{u}}, \frac{\partial\tilde{x}}{\partial\overline{u}} \right\rangle\right|^{1/2}. \tag{3.2}$$

and we assume that s^* is a parameter in terms of s. The operator $\dfrac{\partial}{\partial\overline{s}}$ is given in terms of $\overline{u}$ by

$$\frac{\partial}{\partial\overline{s}} = \frac{1}{\overline{v}}\frac{\partial}{\partial\overline{u}}$$

where

$$\overline{v}=\left|\frac{\partial\tilde{x}}{\partial\overline{u}}\right|.$$

The arclenght parameter is $d\overline{s}=\overline{v}\,d\overline{u}$.

Any flow of $\tilde{x}$ can be represented as

$$\frac{\partial\tilde{x}}{\partial\overline{t}}=\overline{f}_1\tilde{e}+\overline{f}_2\tilde{t}+\overline{f}_3\tilde{g}\ . \tag{3.3}$$

The arclength is given up to a constant by

$$\overline{s}(\overline{u},\overline{t})=\int_0^u\overline{v}\,d\overline{u}\ .$$

In the dual space the requirement that the curve not be exposed to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial\overline{t}}\overline{s}(\overline{u},\overline{t})=\int_0^{\overline{u}}\frac{\partial\overline{v}}{\partial\overline{t}}d\overline{u}=0 \tag{3.4}$$

for $\overline{u}\in[0,1]$.

Definition 3.1: *A dual curve evolution* $\tilde{x}(\overline{u},\overline{t})$ *and* $\dfrac{\partial\tilde{x}}{\partial\overline{t}}$ *flow of this curve in* $\mathbb{D}^3$ *are said to be inextensible if*

$$\frac{\partial}{\partial\overline{t}}\left|\frac{\partial\tilde{x}}{\partial\overline{u}}\right|=0\ .$$

Lemma 3.2: *Let* $\dfrac{\partial\tilde{x}}{\partial\overline{u}}$ *be a smooth flow of the dual curve* $\tilde{x}$. *The flow is inextensible if and only if*

$$\begin{aligned}\frac{\partial v}{\partial t}&=\frac{\partial f_2}{\partial u}+f_1v-\gamma f_3v\\ \frac{\partial f_2^*}{\partial u}&=\gamma f_3^*v+\gamma^* f_3v+\gamma f_3v^*-f_1^*v-f_1\,v^*.\end{aligned} \tag{3.5}$$

Proof: Suppose that $\dfrac{\partial\tilde{x}}{\partial\overline{u}}$ be a smooth flow of the curve $\tilde{x}$. Considering definition of $\tilde{x}$, we have

$$\overline{v}^2=\left\langle\frac{\partial\tilde{x}}{\partial\overline{u}},\frac{\partial\tilde{x}}{\partial\overline{u}}\right\rangle \tag{3.6}$$

$\frac{\partial}{\partial \overline{u}}$ and $\frac{\partial}{\partial \overline{t}}$ commute since and are independent coordinates. Then, by differentiating of formula (3.6) we get

$$2\overline{v}\frac{\partial \overline{v}}{\partial \overline{t}} = \frac{\partial}{\partial \overline{t}}\left\langle \frac{\partial \tilde{x}}{\partial \overline{u}}, \frac{\partial \tilde{x}}{\partial \overline{u}} \right\rangle.$$

On the other hand, changing $\frac{\partial}{\partial \overline{u}}$ and $\frac{\partial}{\partial \overline{t}}$ we

$$\overline{v}\frac{\partial \overline{v}}{\partial \overline{t}} = \left\langle \frac{\partial \tilde{x}}{\partial \overline{u}}, \frac{\partial}{\partial \overline{u}}\left(\frac{\partial \tilde{x}}{\partial \overline{t}}\right) \right\rangle$$

From equation (3.3), we obtain

$$\overline{v}\frac{\partial \overline{v}}{\partial \overline{t}} = \left\langle \frac{\partial \tilde{x}}{\partial \overline{u}}, \frac{\partial}{\partial \overline{u}}(\overline{f}_1\tilde{e} + \overline{f}_2\tilde{t} + \overline{f}_3\tilde{g}) \right\rangle.$$

By the formula of dual darboux, we have

$$\frac{\partial \overline{v}}{\partial \overline{t}} = \left\langle \tilde{t}, \left(\frac{\partial \overline{f}_1}{\partial \overline{u}} - \overline{f}_2\overline{v}\right)\tilde{e} + \left(\overline{f}_1\overline{v} + \frac{\partial \overline{f}_2}{\partial \overline{u}} - \overline{\gamma}\,\overline{f}_3\overline{v}\right)\tilde{t} + \left(\frac{\partial \overline{f}_3}{\partial \overline{u}} + \overline{f}_2\overline{\gamma v}\right)\tilde{g} \right\rangle.$$

If above equation is separated real and dual part then we have

$$\frac{\partial v}{\partial t} = \left(f_1 v + \frac{\partial \overline{f}_2}{\partial u} - \gamma f_3 v\right)$$

and

$$\frac{\partial v^*}{\partial t} = \frac{\partial f_2^*}{\partial u} - \gamma f_3^* v - \gamma^* f_3 v - \gamma f_3 v^* + f_1^* v + f_1 v^*.$$

respectively. Desired expression is obtained with definition 3.1.

Theorem 3.3: *Let* $\frac{\partial \tilde{x}}{\partial \overline{u}} = \overline{f}_1\tilde{e} + \overline{f}_2\tilde{t} + \overline{f}_3\tilde{g}$ *be a smooth flow of the curve* $\tilde{x}$*. The flow is inextensible if and only if*

$$\begin{aligned} \frac{\partial f_2}{\partial s} &= \gamma f_3 + f_1 \\ \frac{\partial f_2^*}{\partial s} &= \gamma f_3^* + \gamma^* f_3 + f_1^* \end{aligned} \tag{3.7}$$

Proof: Let $\frac{\partial \tilde{x}}{\partial \overline{u}}$ be extensible. From Eq. (3.4), we have

$$\frac{\partial}{\partial \bar{t}}\bar{s}(\bar{u},\bar{t})=\int_0^{\bar{u}}\frac{\partial \bar{v}}{\partial \bar{t}}d\bar{u}=\int_0^{\bar{u}}(\frac{\partial \bar{f}_2}{\partial \bar{u}}+\bar{f}_1\bar{v}-\bar{\gamma}\bar{f}_3\bar{v})d\bar{u}=0 \tag{3.8}$$

$\forall \bar{u}\in[0,1]$. Substituting (3.5) in (3.8) and this expression separating dual and real part complete the proof of the theorem. We suppose that $\bar{v}=1$ and the local coordinate $\bar{u}$ corresponds to the curve arc length $\bar{s}$. Now we give following lemma that necessary.

Lemma 3.4:

$$\frac{\partial \tilde{e}}{\partial \bar{t}}=\left(\frac{\partial \bar{f}_1}{\partial \bar{s}}-\bar{f}_2\right)\tilde{e}+\left(\bar{f}_2\bar{\gamma}+\frac{\partial \bar{f}_3}{\partial \bar{s}}\right)\tilde{g}\,, \tag{3.9}$$

$$\frac{\partial \tilde{t}}{\partial \bar{t}}=-\tilde{e}+\bar{\psi}\tilde{g}\,, \tag{3.10}$$

$$\frac{\partial \tilde{g}}{\partial \bar{t}}=-\left(\bar{f}_2\bar{\gamma}+\frac{\partial \bar{f}_3}{\partial \bar{s}}\right)\tilde{e}-\bar{\psi}\tilde{t}\,, \tag{3.11}$$

where $\bar{\psi}=\left\langle\frac{\partial \tilde{t}}{\partial \bar{t}},\tilde{g}\right\rangle$.

Proof: Considering definition of $\tilde{x}$, we have

$$\frac{\partial \tilde{e}}{\partial \bar{t}}=\frac{\partial}{\partial \bar{t}}\frac{\partial \tilde{x}}{\partial \bar{s}}=\frac{\partial}{\partial \bar{s}}(\bar{f}_1\tilde{e}+\bar{f}_2\tilde{t}+\bar{f}_3\tilde{g}).$$

Using the Darboux equations, we get

$$\frac{\partial \tilde{e}}{\partial \bar{t}}=\left(\frac{\partial \bar{f}_1}{\partial \bar{s}}-\bar{f}_2\right)\tilde{e}+\left(\bar{f}_1+\frac{\partial \bar{f}_2}{\partial \bar{s}}-\bar{\gamma}\bar{f}_3\right)\tilde{t}+\left(\bar{f}_2\bar{\gamma}+\frac{\partial \bar{f}_3}{\partial \bar{s}}\right)\tilde{g}. \tag{3.12}$$

Substituting (3.7) in (3.12), we have

$$\frac{\partial \tilde{e}}{\partial \bar{t}}=\left(\frac{\partial \bar{f}_1}{\partial \bar{s}}-\bar{f}_2\right)\tilde{e}+\left(\bar{f}_2\bar{\gamma}+\frac{\partial \bar{f}_3}{\partial \bar{s}}\right)\tilde{g}.$$

Now differentiate the dual Darboux frame by t:

$$0=\frac{\partial}{\partial \overline{t}}\langle \tilde{e},\tilde{t}\rangle=\left\langle \frac{\partial \tilde{e}}{\partial \overline{t}},\tilde{t}\right\rangle+\left\langle \tilde{e},\frac{\partial \tilde{t}}{\partial \overline{t}}\right\rangle=1+\left\langle \tilde{e},\frac{\partial \tilde{t}}{\partial \overline{t}}\right\rangle,$$
$$0=\frac{\partial}{\partial \overline{t}}\langle \tilde{e},\tilde{g}\rangle=\left\langle \frac{\partial \tilde{e}}{\partial \overline{t}},\tilde{g}\right\rangle+\left\langle \tilde{e},\frac{\partial \tilde{g}}{\partial \overline{t}}\right\rangle=\overline{f}_2\overline{\gamma}+\frac{\partial \overline{f}_3}{\partial \overline{s}}+\left\langle \tilde{e},\frac{\partial \tilde{g}}{\partial \overline{t}}\right\rangle,$$
$$0=\frac{\partial}{\partial \overline{t}}\langle \tilde{t},\tilde{g}\rangle=\left\langle \frac{\partial \tilde{t}}{\partial \overline{t}},\tilde{g}\right\rangle+\left\langle \tilde{t},\frac{\partial \tilde{g}}{\partial \overline{t}}\right\rangle=\overline{\psi}+\left\langle \tilde{t},\frac{\partial \tilde{g}}{\partial \overline{t}}\right\rangle.$$

Considering $\left\langle \frac{\partial \tilde{t}}{\partial \overline{t}},\tilde{t}\right\rangle=\left\langle \frac{\partial \tilde{g}}{\partial \overline{t}},\tilde{g}\right\rangle=0$ and from above statement, we obtain

$$\frac{\partial \tilde{t}}{\partial \overline{t}}=-\tilde{e}+\overline{\psi}\tilde{g},$$
$$\frac{\partial \tilde{g}}{\partial \overline{t}}=-\left(\overline{f}_2\overline{\gamma}+\frac{\partial \overline{f}_3}{\partial \overline{s}}\right)\tilde{e}-\overline{\psi}\tilde{t},$$

where $\overline{\psi}=\left\langle \frac{\partial \tilde{t}}{\partial \overline{t}},\tilde{g}\right\rangle$.

The following theorem states the conditions on the dual geodesic curvature for the dual curve flow $\tilde{x}(\overline{s},\overline{t})$ to be inextensible.

Theorem 3.5: *Suppose* $\frac{\partial \tilde{x}}{\partial \overline{u}}=\overline{f}_1\tilde{e}+\overline{f}_2\tilde{t}+\overline{f}_3\tilde{g}$ *is inextensible. Then, the following system of partial differential equations holds:*

$$\frac{\partial \overline{\gamma}}{\partial \overline{t}}=\overline{f}_2\overline{\gamma}+\frac{\partial \overline{f}_3}{\partial \overline{s}}+\frac{\partial \overline{\psi}}{\partial \overline{s}}. \tag{3.13}$$

Proof: Using (3.11), we have

$$\begin{aligned}\frac{\partial}{\partial \overline{s}}\frac{\partial \tilde{g}}{\partial \overline{t}}&=\frac{\partial}{\partial \overline{s}}\left[-\left(\overline{f}_2\overline{\gamma}+\frac{\partial \overline{f}_3}{\partial \overline{s}}\right)\tilde{e}-\overline{\psi}\tilde{t}\right]\\&=\left(\frac{\partial(-\overline{f}_2\overline{\gamma})}{\partial \overline{s}}+\frac{\partial^2 \overline{f}_3}{\partial \overline{s}^2}\right)\tilde{e}-\left(\overline{f}_2\overline{\gamma}+\frac{\partial \overline{f}_3}{\partial \overline{s}}\right)\tilde{t}\\&\quad-\frac{\partial \overline{\psi}}{\partial \overline{s}}\tilde{t}-\overline{\psi}(\tilde{e}+\overline{\gamma}\tilde{g}).\end{aligned}$$

Therefore from dual Darboux frame

$$\frac{\partial}{\partial \overline{s}}\frac{\partial \tilde{g}}{\partial \overline{t}}=\frac{\partial}{\partial \overline{s}}(-\overline{\gamma}\tilde{t})=-\frac{\partial \overline{\gamma}}{\partial \overline{s}}\tilde{t}-\overline{\gamma}(-\tilde{e}+\overline{\psi}\tilde{g}).$$

Thus from two equations we have (3.13).

4 Conclusion

Inextensible time evolutions of curves and surfaces have an important role in computer vision, robotics and physical science. In this paper inextensible flows of dual curves according to dual darboux frame have given by considering important role of dual geometry. Since dual geometry have a significant role robotics, mechanism and dynamics, this study can be shed light on these areas.

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