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Weighted Szeged Index of Generalized Hierarchical Product of Graphs

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Abstract

The Szeged index of a graph G, denoted by $Sz(G) = \sum_{uv=e\in E(G)} n_u^G(e)n_v^G(e)$. Similarly, the Weighted Szeged index of a graph G, denoted by $Sz_w(G) = \sum_{uv=e\in E(G)} \Big(d_G(u)+ d_G(v)\Big)n_u^G(e)n_v^G(e)$, where $d_G(u)$ is the degree of the vertex u in G. In this paper, the exact formulae for the weighted Szeged indices of generalized hierarchical product and Cartesian product of two graphs are obtained.

Keywords: *Generalized hierarchical product, Cartesian product, Szeged index, weighted Szeged index.*

1 Introduction

All the graphs considered in this paper are connected and simple. A vertex $x \in V(G)$ is said to be *equidistant* from the edge $e = uv$ of G if $d_G(u, x) = d_G(v, x)$, where $d_G(u, x)$ denotes the distance between u and x in G. The edges $e = uv$ and $f = xy$ of G are said to be *equidistant edges* if $min\{d_G(u, x), d_G(u, y)\} =$

$min\{d_G(v,x), d_G(v,y)\}$. The degree of the vertex u in G is denoted by $d_G(u)$.

For an edge $uv = e \in E(G)$, the number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v in G is denoted by $n_u^G(e)$; analogously, $n_v^G(e)$ is the number of vertices of G whose distance to the vertex v in G is smaller than the distance to the vertex u; the vertices equidistant from both the ends of the edge $e = uv$ are not counted.

Graph theory successfully provides the chemists with a variety of very useful tools, namely, different topological indices. A *topological index* of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [14]. Several types of such indices exist, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index.

The two topological indices, namely, the Szeged index of G, denoted by $Sz(G)$, and, weighted Szeged index of G, denoted by $Sz_w(G)$, are defined as follows:

$$Sz(G) = \sum_{e=uv\in E(G)} n_u^G(e)n_v^G(e),$$
$$Sz_w(G) = \sum_{e=uv\in E(G)} \Big(d_G(u)+d_G(v)\Big)n_u^G(e)n_v^G(e).$$

A graph G with a specified vertex subset $U \subseteq V(G)$ is denoted by $G(U)$. Barriere et al. [1, 2] defined a new product of graphs, namely, the generalized hierarchical product, as follows: Let G and H be two graphs with a nonempty vertex subset $U \subseteq V(G)$. Then the *generalized hierarchical product,* denoted by $G(U) \sqcap H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (g,h) and (g',h') are adjacent if and only if $g = g' \in U$ and $hh' \in E(H)$ or, $gg' \in E(G)$ and $h = h'$ (see Fig.1). The *Cartesian product,* $G \square H$, of graphs G and H has the vertex set $V(G \square H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \square H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$, see Fig.2.

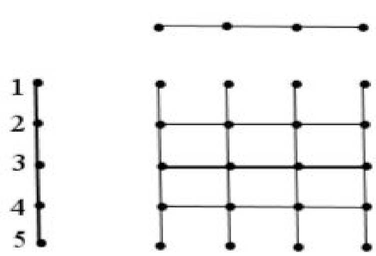

Fig.1. $P_5(U) \sqcap P_4$, where $U = \{2,3,4\}$

To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G \square H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G \square H$. But in the generalized hierarchical product, to each vertex $u \in U$, there is an isomorphic copy of H and to each vertex $v \in V(G)$, there is an isomorphic copy of G. In particular, if $U = V(G)$, then $G \square H = G(U) \sqcap H$.

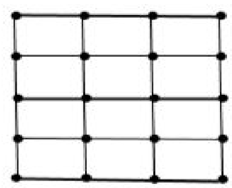

Fig.2. $P_5 \square P_4$

Let G and H be simple connected graphs with vertex sets $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively, and let U be a nonempty subset of G. Then $V(G(U) \sqcap H) = V(G) \times V(H)$; for our convenience, we write $V(G(U) \sqcap H) = \bigcup_{i=1}^{n} X_i$, where $X_i = \{u_i\} \times V(H)$; we may also write $V(G(U) \sqcap H) = \bigcup_{j=1}^{m} Y_j$, where $Y_j = V(G) \times \left\{v_j\right\}$. We denote the vertices of X_i by $\{(u_i, v_j) \mid 1 \leq j \leq m\}$ and the vertices of Y_j by $\{(u_i, v_j) | 1 \leq i \leq n\}$ and we call $X_i, 1 \leq i \leq n$, the i-th *layer* of $G(U) \sqcap H$ and $Y_j, 1 \leq j \leq m$, the j-th *column* of $G(U) \sqcap H$.

Weighted Szeged index of graph G has been introduced by Illić and Milosavljević [11]. They are given the upper and lower bounds for weighted vertex PI index of graph. Also the exact formula for weighted vertex PI index of Cartesian product of graphs is obtained in [11]. The Szeged index studied by Gutman [5], Gutman and Dobrynin [6] and Khadikar et. al. [9] is closely related to the Wiener index of a graph. Basic properties of Szeged index and its analogy to the Wiener index are discussed by Klavžar et. al.[8] . It is proved that for a tree T the Wiener index of T is equal to its Szeged index. Ashrafi et. al. [10] have explained the differences between Szeged and Wiener indices of graphs. The mathematical properties and chemical applications of Szeged index are well studied by Dobrynin et. al. [3], Gutman et. al. [4] and Randic et. al. [13]. Recently Pisanski and Randic [12] studied the measuring network bipartivity using Szeged index. In this paper, the exact formulae for the weighted Szeged indices of generalized hierarchical product and Cartesian product of two graphs are obtained.

2 Weighted Szeged Index of $G(U) \sqcap H$

Let $G = (V, E)$ be a graph and $U \subseteq V$. In $G(U)$, an *u - v path through U* is an u-v path in G containing some vertex $w \in U$(vertex w could be the vertex u or v). Let $d_{G(U)}(u, v)$ denote the length of a shortest u-v path through U in G. Note that, if one of the vertices u and v belongs to U, then $d_{G(U)}(u, v) = d_G(u, v)$. A vertex $x \in V(G(U))$ is said to be equidistant from $e = uv \in E(G(U))$ through U in $G(U)$, if $d_{G(U)}(u,x) = d_{G(U)}(v,x)$. For an edge e in $G(U)$, let $N_{G(U)}(e)$ denote the number of equidistant vertices of e through U in $G(U)$. Then $Sz(G(U))$ and $Sz_w(G(U))$ are defined as follows:

$$\begin{aligned} Sz(G(U)) &= \sum_{e=uv\in E(G)} \Big(n_u^{G(U)}(e) + n_v^{G(U)}(e)\Big) \\ Sz_w(G(U)) &= \sum_{e=uv\in E(G)} \Big(d_{G(U)}(u) + d_{G(U)}(v)\Big)\Big(n_u^{G(U)}(e) + n_v^{G(U)}(e)\Big). \end{aligned}$$

For an edge $e = uv \in E(G)$, let $T_G(e,u)$ be the set of vertices closer to u than v and $T_G(e,v)$ be the set of vertices closer to v than u. That is,

$$\begin{aligned} T_G(e,u) &= \{x \in V(G) | d_G(u,x) < d_G(v,x)\} \\ T_G(e,v) &= \{x \in V(G) | d_G(u,x) > d_G(v,x)\}. \end{aligned}$$

Similarly, for an edge $e = uv \in E(G(U))$,

$$\begin{aligned} T_{G(U)}(e,u) &= \{x \in V(G(U)) | d_{G(U)}(u,x) < d_{G(U)}(v,x)\} \\ T_{G(U)}(e,v) &= \{x \in V(G(U)) | d_{G(U)}(u,x) > d_{G(U)}(v,x)\}. \end{aligned}$$

The proof of the following lemma is left to the reader as it follows easily from the structure of $G(U) \sqcap H$. The lemma is used in the proof of the main theorem of this section.

Lemma 2.1. *Let G and H be graphs with $U \subseteq V(G)$. Then*
(i) $|V(G(U) \sqcap H)| = |V(G)|\,|V(H)|$, $|E(G(U) \sqcap H)| = |E(G)|\,|V(H)| + |E(H)|\,|U|$.
(ii) The degree of the vertex $(g,h) \in V(G(U) \sqcap H)$ is $d_{G(U)}(g) + \phi_U(g)d_H(h)$, where ϕ_U denote the characteristic function on the set U which is 1 on U and 0 outside U.
(iii) $d_{G(U)\sqcap H}((g,h)(g',h')) = \begin{cases} d_{G(U)}(g,g') + d_H(h,h'), \ if\ h \neq h', \\ d_G(g,g'), \ if\ h = h'. \end{cases}$ □

Next we compute the weighted Szeged index of the generalized hierarchical product of two connected graphs G and H.

Theorem 2.2. *Let G and H be two connected graphs with n,m vertices and p,q edges and let U be a nonempty subset of $V(G)$. Then* $Sz_w(G(U) \sqcap H) =$
$2n^2 Sz(H)\Big(\sum_{u_r\in U} d_{G(U)}(u_r)\Big) + n^2\,|U|\,Sz_w(H) + mSz_w(G) + m(m-1)^2 Sz_w(G(U)) + m(m-1)\sum_{u_iu_k\in E(G)} \Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n_{u_i}^G(e)n_{u_k}^{G(U)}(e) + n_{u_i}^{G(U)}(e)n_{u_k}^G(e)\Big) + 2q\sum_{u_iu_k\in E(G)} \Big(\phi_U(u_i) + \phi_U(u_k)\Big)n_{u_i}^G(e)n_{u_k}^G(e) + 2q(m-1)\sum_{u_iu_k\in E(G)} \Big(\phi_U(u_i) + \phi_U(u_k)\Big)\Big(n_{u_i}^G(e)n_{u_k}^{G(U)}(e) + n_{u_i}^{G(U)}(e)n_{u_k}^G(e)\Big)$
$+ 2q(m-1)^2 \sum_{u_iu_k\in E(G)} \Big(\phi_U(u_i) + \phi_U(u_k)\Big)n_{u_i}^{G(U)}(e)n_{u_k}^{G(U)}(e).$

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$, $V(H) = \{v_1, v_2, \ldots, v_m\}$ and let U be a nonempty subset of $V(G)$. For our convenience, we partition the edge set of $G(U) \sqcap H$ into two sets, $E_1 = \{(u_r, v_i)(u_r, v_k) \,|\, u_r \in U,\ v_iv_k \in E(H)\}$ and $E_2 = \{(u_r, v_i)(u_s, v_i) \,|\, u_ru_s \in E(G),\ v_i \in V(H)\}$, that is, $E_1 = \cup_{u_i \in U} E(\langle X_i \rangle)$ and $E_2 = \cup_{j=1}^{m} E(\langle Y_j \rangle)$.

Let $e = v_iv_k \in E(H)$ and let $v_j \in T_H(e; v_i)$. Then, for any $u_r \in U$ and $e' \in E_1 \subset E(G(U) \sqcap H)$, the distance of (u_r, v_i) to each vertex of Y_j, is less than its distance to the vertex (u_r, v_k) in $G(U) \sqcap H$. It can be observed that if some vertex $v_s \notin T_H(e, v_i)$, then all the vertices of the column Y_s are not in $T_{G(U)\sqcap H}(e'; (u_r, v_i))$ in $G(U) \sqcap H$. Also if v_r is equidistant to e in H, then every vertex of Y_r is equidistant to e'. Consequently, for the edge $e' \in E_1$ (of $G(U) \sqcap H$) corresponding to e (in H),

$$n_{(u_r,v_i)}^{G(U)\sqcap H}(e') = n\, n_{v_i}^{H}(e) \tag{1}$$

and similarly,

$$n_{(u_r,v_k)}^{G(U)\sqcap H}(e') = n\, n_{v_k}^{H}(e). \tag{2}$$

Hence for E_1 defined as above,

$$\begin{aligned}
& \sum_{(u_r,v_i)(u_r,v_k) = e' \in E_1} \Big(d_{G(U)\sqcap H}((u_r, v_i)) + d_{G(U)\sqcap H}((u_r, v_k))\Big) n_{(u_r,v_i)}^{G(U)\sqcap H}(e') n_{(u_r,v_k)}^{G(U)\sqcap H}(e') \\
= & \sum_{(u_r,v_i)(u_r,v_k) = e' \in E_1} \Big(d_{G(U)}(u_r) + d_H(v_i) + d_{G(U)}(u_r) + d_H(v_k)\Big)\Big(n^2\, n_{v_i}^{H}(e)\, n_{v_k}^{H}(e)\Big), \\
& \text{ by (1) and (2), where } e = v_iv_k \in E(H), \\
= & \; n^2 \sum_{u_r \in U} \sum_{v_iv_k = e \in E(H)} 2d_{G(U)}(u_r)\Big(n_{v_i}^{H}(e) n_{v_k}^{H}(e)\Big) \\
& + n^2 \sum_{u_r \in U} \sum_{v_iv_k = e \in E(H)} \Big(d_H(v_i) + d_H(v_k)\Big)\Big(n_{v_i}^{H}(e) n_{v_k}^{H}(e)\Big), \text{ since } |E_1| = |U|\,|E(H)|, \\
= & \; 2n^2 Sz(H)\Big(\sum_{u_r \in U} d_{G(U)}(u_r)\Big) + n^2\, |U|\; Sz_w(H).
\end{aligned} \tag{3}$$

Let $e = u_iu_k \in E(G(U))$. Then, for any $v_\ell \in V(H)$ $e' = (u_i, v_\ell)(u_k, v_\ell) \in E_2 \subset E(G(U) \sqcap H)$. If $u_j \in T_G(e, u_i)$ then $(u_j, v_\ell) \in T_{G(U)\sqcap H}(e', (u_i, v_\ell))$. Hence $\{(u_j, v_\ell) | u_j \in T_G(e, u_i)\} \subseteq T_{G(U)\sqcap H}(e', (u_i, v_r))$. If $\ell \neq s$ then since $d_{G(U)\sqcap H}((u_i, v_\ell), (u_j, v_s)) < d_{G(U)\sqcap H}((u_k, v_\ell), (u_j, v_s))$ if and only if $d_{G(U)}((u_i, v_j) + d_H(v_\ell, v_s) < d_{G(U)}((u_k, v_j) + d_H(v_\ell, v_s)$ if and only if $d_{G(U)}((u_i, v_j) < d_{G(U)}((u_k, v_j)$.

Therefore $\{(u_j, v_s) | u_j \in T_{G(U)}(e, u_i)\} \subseteq T_{G(U)\sqcap H}(e', (u_i, v_\ell))$.

Hence,

$$T_{G(U)\sqcap H}(e', (u_i, v_\ell)) = \Big|\{(u_j, v_\ell) | u_j \in T_G(e, u_i)\}\Big| + \Big|\{(u_j, v_s) | u_j \in T_{G(U)}(e, u_i)\}\Big|.$$

Consequently,

$$n_{(u_i,v_\ell)}^{G(U)\sqcap H}(e') = n_{u_i}^{G}(e) + (m-1) n_{u_i}^{G(U)}(e) \tag{4}$$

and similarly,

$$n^{G(U)\sqcap H}_{(u_k,v_\ell)}(e') = n^{G}_{u_k}(e) + (m-1)n^{G(U)}_{u_k}(e). \tag{5}$$

Hence for E_2 defined as above

$$\begin{aligned}
&\sum_{(u_i,v_\ell)(u_k,v_\ell)=e'\in E_2} \Big(d_{G(U)\sqcap H}((u_i,v_\ell)) + d_{G(U)\sqcap H}((u_k,v_\ell))\Big)\Big(n^{G(U)\sqcap H}_{(u_i,v_\ell)}(e')\, n^{G(U)\sqcap H}_{(u_k,v_\ell)}(e')\Big)\\
= &\sum_{v_\ell\in V(H)}\sum_{u_iu_k\in E(G)} \bigg(\Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n^{G}_{u_i}(e) + (m-1)n^{G(U)}_{u_i}(e)\Big)\\
&\Big(n^{G}_{u_k}(e) + (m-1)n^{G(U)}_{u_k}(e)\Big)\bigg)\\
&+\sum_{v_\ell\in V(H)}\sum_{u_iu_k\in E(G)} \bigg(d_H(v_\ell)\Big(\phi_U(u_i) + \phi_U(u_k)\Big)\Big(n^{G}_{u_i}(e) + (m-1)n^{G(U)}_{u_i}(e)\Big)\\
&\Big(n^{G}_{u_k}(e) + (m-1)n^{G(U)}_{u_k}(e)\Big)\bigg),\\
&\text{by (4) and (5), where } e = u_iu_k \in E(G(U))\\
= &\ S_1 + S_2, \text{ where } S_1 \text{ and } S_2 \text{ are the sums of the above terms, in order.}
\end{aligned} \tag{6}$$

We shall calculate S_1 and S_2 of (6) separately.

$$\begin{aligned}
S_1 = &\sum_{v_\ell\in V(H)}\sum_{u_iu_k\in E(G)} \Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n^{G}_{u_i}(e)n^{G}_{u_k}(e)\Big)\\
&+(m-1)\sum_{v_\ell\in V(H)}\sum_{u_iu_k\in E(G)} \Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n^{G}_{u_i}(e)n^{G(U)}_{u_k}(e) + n^{G(U)}_{u_i}(e)n^{G}_{u_k}(e)\Big)\\
&+(m-1)^2\sum_{v_\ell\in V(H)}\sum_{u_iu_k\in E(G)} \Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n^{G(U)}_{u_i}(e)n^{G(U)}_{u_k}(e)\Big)\\
= &\ m\sum_{u_iu_k\in E(G)} \Big(d_G(u_i) + d_G(u_k)\Big)n^{G}_{u_i}(e)n^{G}_{u_k}(e)\\
&+m(m-1)\sum_{u_iu_k\in E(G)} \Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n^{G}_{u_i}(e)n^{G(U)}_{u_k}(e) + n^{G(U)}_{u_i}(e)n^{G}_{u_k}(e)\Big)\\
&+m(m-1)^2\sum_{u_iu_k\in E(G)} \Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n^{G(U)}_{u_i}(e)n^{G(U)}_{u_k}(e)\Big)\\
= &\ mSz_w(G) + m(m-1)^2Sz_w(G(U))\\
&+m(m-1)\sum_{u_iu_k\in E(G)} \Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n^{G}_{u_i}(e)n^{G(U)}_{u_k}(e) + n^{G(U)}_{u_i}(e)n^{G}_{u_k}(e)\Big).
\end{aligned} \tag{7}$$

$$\begin{aligned}
S_2 = &\sum_{v_\ell\in V(H)}\sum_{u_iu_k\in E(G)} d_H(v_\ell)\Big(\phi_U(u_i) + \phi_U(u_k)\Big)\Big(n^{G}_{u_i}(e)n^{G}_{u_k}(e)\\
&+\ (m-1)\sum_{v_\ell\in V(H)}\sum_{u_iu_k\in E(G)} d_H(v_\ell)\Big(\phi_U(u_i) + \phi_U(u_k)\Big)\Big(n^{G}_{u_i}(e)n^{G(U)}_{u_k}(e) + n^{G(U)}_{u_i}(e)n^{G}_{u_k}(e)\Big)\\
&+\ (m-1)^2\sum_{v_\ell\in V(H)}\sum_{u_iu_k\in E(G)} d_H(v_\ell)\Big(\phi_U(u_i) + \phi_U(u_k)\Big)\Big(n^{G(U)}_{u_i}(e)n^{G(U)}_{u_k}(e)
\end{aligned} \tag{8}$$

Using (6) and the sums S_1 and S_2 in (7) and (8), respectively, we have,

$$\begin{aligned}
&Sz_w(G(U) \sqcap H)\\
&= 2n^2 Sz(H)\Big(\sum_{u_r \in U} d_{G(U)}(u_r)\Big) + n^2 |U|\ Sz_w(H) + mSz_w(G) + m(m-1)^2 Sz_w(G(U))\\
&+m(m-1) \sum_{u_i u_k \in E(G)} \Big(d_{G(U)}(u_i) + d_{G(U)}(u_k)\Big)\Big(n^G_{u_i}(e) n^{G(U)}_{u_k}(e) + n^{G(U)}_{u_i}(e) n^G_{u_k}(e)\Big)\\
&+2q \sum_{u_i u_k \in E(G)} \Big(\phi_U(u_i) + \phi_U(u_k)\Big)\Big(n^G_{u_i}(e) n^G_{u_k}(e)\\
&+2q(m-1) \sum_{u_i u_k \in E(G)} \Big(\phi_U(u_i) + \phi_U(u_k)\Big)\Big(n^G_{u_i}(e) n^{G(U)}_{u_k}(e) + n^{G(U)}_{u_i}(e) n^G_{u_k}(e)\Big)\\
&+2q(m-1)^2 \sum_{u_i u_k \in E(G)} \Big(\phi_U(u_i) + \phi_U(u_k)\Big)\Big(n^{G(U)}_{u_i}(e) n^{G(U)}_{u_k}(e).
\end{aligned}$$

□

In the above theorem, if we set $U = V(G)$, we obtain the following corollary.

Corollary 2.3. *[11] Let G and H be connected graphs. Then* $Sz_w(G \square H) = |V(H)|^3\ Sz_w(G) + |V(G)|^3\ Sz_w(H) + 4|V(H)|^2 |E(H)|\ Sz(G) + 4|V(G)|^2 |E(G)|\ Sz(H)$. □

Let $G_1, G_2, \ldots, G_n$ be graphs with vertex set $V(G_i)$ and edge set $E(G_i)$, $1 \le i \le n$. Denote by $\square_{i=1}^{n} G_i$ the Cartesian product of graphs $G_1, G_2, \ldots, G_n$. Clearly, $\left|V(\square_{i=1}^{n} G_i)\right| = \prod_{i=1}^{n} |V(G_i)|$. By induction on n, one can see that $\left|E(\square_{i=1}^{n} G_i)\right| = \prod_{i=1}^{n} |V(G_i)| \sum_{i=1}^{n} \frac{|E(G_i)|}{|V(G_i)|}$. In [8], S, Klavžar et al. have proved $Sz(\square_{i=1}^{n} G_i) = \sum_{i=1}^{n} Sz(G_i) \prod_{j=1, j\neq i}^{n} \left|V(G_j)\right|^3$.

Next we compute a similar result for the weighted Szeged index.

Theorem 2.4. *Let $G_1, G_2, \ldots, G_n$ be connected graphs. Then* $Sz_w(\square_{i=1}^{n} G_i) = \sum_{i=1}^{n} Sz_w(G_i) \prod_{j=1, j\neq i}^{n} \left|V(G_j)\right|^3 + 4 \sum_{i,j=1\ i\neq j}^{n} Sz(G_i) \left|V(G_j)\right|^2 \left|E(G_j)\right| \prod_{k=1, i\neq k\neq j}^{n} |V(G_k)|^3$.

Proof. The case $n = 2$ was proven in Theorem 2.2. We continue our argument by mathematical induction. Suppose that the results is valid for some n graphs.

$$
\begin{aligned}
& Sz_w\Big(\mathop{\Box}_{i=1}^{n+1} G_i\Big) = Sz_w\Big(\mathop{\Box}_{i=1}^{n} G_i \Box G_{n+1}\Big) \\
= & \left|V\Big(\mathop{\Box}_{i=1}^{n} G_i\Big)\right|^3 Sz_w(G_{n+1}) + |V(G_{n+1})|^3\, Sz_w\Big(\mathop{\Box}_{i=1}^{n} G_i\Big) \\
& +4\Big(\left|V\Big(\mathop{\Box}_{i=1}^{n} G_i\Big)\right|^2 \left|E\Big(\mathop{\Box}_{i=1}^{n} G_i\Big)\right| Sz(G_{n+1}) + |V(G_{n+1})|^2\, |E(G_{n+1})|\, Sz\Big(\mathop{\Box}_{i=1}^{n} G_i\Big)\Big) \\
= & Sz_w(G_{n+1}) \prod_{i=1}^{n} |V(G_i)|^3 + |V(G_{n+1})|^3 \sum_{i=1}^{n} Sz_w(G_i) \prod_{j=1, j\neq i}^{n} \left|V(G_j)\right|^3 \\
& +4 \sum_{i,j=1\, i\neq j}^{n} Sz(G_i) \left|V(G_j)\right|^2 \left|E(G_j)\right| \prod_{k=1, i\neq k\neq j}^{n} |V(G_k)|^3 \\
& +4\Big(Sz(G_{n+1}) \sum_{i=1}^{n} |V(G_i)|^2\, |E(G_i)| \prod_{j=1, j\neq i}^{n} \left|V(G_j)\right|^3 + \\
& |V(G_{n+1})|\, |E(G_{n+1})| \sum_{i=1}^{n} Sz(G_i) \prod_{j=1, j\neq i}^{n} \left|V(G_j)\right|^3 \Big) \\
= & \sum_{i=1}^{n+1} Sz_w(G_i) \prod_{j=1, j\neq i}^{n+1} \left|V(G_j)\right|^3 + 4\Big(\sum_{i,j=1,\, i\neq j}^{n} Sz(G_i) \left|V(G_j)\right|^2 \left|E(G_j)\right| \prod_{k=1,\, i\neq k\neq j}^{n+1} |V(G_k)|^3 \\
& + \sum_{i\leq j\leq n} Sz(G_i) \left|V(G_j)\right|^2 \left|E(G_j)\right| \prod_{k=1,\, i\neq k\neq j}^{n+1} |V(G_k)|^3 \Big) \\
= & \sum_{i=1}^{n} Sz_w(G_i) \prod_{j=1, j\neq i}^{n} \left|V(G_j)\right|^3 + 4 \sum_{i,j=1\, i\neq j}^{n} Sz(G_i) \left|V(G_j)\right|^2 \left|E(G_j)\right| \prod_{k=1, i\neq k\neq j}^{n} |V(G_k)|^3 .
\end{aligned}
$$

□

The proof of the following corollary directly follows from Theorem 2.4.

Corollary 2.5. *Let G be a connected graph. Then* $Sz_w(\Box G^n) = Sz_w(\mathop{\Box}_{i=1}^{n} G) = n\,|V(G)|^{3n-4}\,\{\,|V(G)|\; Sz_w(G) + 4(n-1)\,|E(G)|\, Sz(G)\}$. □

Example 2.6. *Suppose Q_n denotes the hypercube of dimension n. Then by Theorem 2.4,* $Sz_w(Q_n) = Sz_w(\Box K_2^n) = n^2\, 2^{(3n-2)}$.

Let us consider the graph G whose vertices are the $N-$tuples $b_1 b_2 \ldots b_N$ with $b_i \in \{0, 1, \ldots, n_i - 1\}, n_i \geq 2$, and two vertices be adjacent if the corresponding tuples differ in precisely one place; such a graph is called a *Hamming graph*. It is well-known fact that a graph G is a Hamming graph if and only if it can be written in the form $G = \mathop{\square}\limits_{i=1}^{N} K_{n_i}$ and so the Hamming graph is usually denoted by $H_{n_1 n_2 \ldots n_N}$. In the following lemma, the weighted Szeged index of a Hamming graph is computed.

It is easy to see that $Sz(K_n) = \frac{n(n-1)}{2}$ and $Sz_w(K_n) = n(n-1)^2$. The proof of the following lemma follows from Theorem 2.4.

Lemma 2.7. *Let G be a Hamming graph with above parameter. Then*

$$Sz_w(H_{n_1 n_2 \ldots n_N}) = \left(\sum_{i=1}^{N} \left(1 - \frac{1}{n_i}\right)^2 + \sum_{i,j=1,\ i \neq j}^{N} \frac{(n_i-1)(n_j-1)}{n_i^2} \right) \prod_{i=1}^{N} {n_i}^3. \qquad \square$$

Let C_n and P_n denote the cycle and path on n vertices, respectively. It is known that $Sz(C_n) = \frac{n^3}{4}$ when n is even, and $\frac{n(n-1)^2}{4}$ otherwise and $Sz(P_n) = \binom{n+1}{3}$; see [7]. It can be easily verified that $Sz_w(C_n) = n^3$ when n is even, and $n(n-1)^2$ otherwise and $Sz_w(P_n) = \frac{2(n-1)(n^2+n-3)}{3}$.

Using Theorems 2.2, 2.4 and $Sz_w(P_n)$,$Sz_w(C_n)$,$Sz(P_n)$ and $Sz(C_n)$, we obtain the exact weighted Szeged indices of the following graphs.

Example 2.8. *The graphs $L_n = P_n \square K_2$, $R = P_n \square C_m$, $S = C_m \square C_n$ and $T = P_m \square P_n$ are known as ladder, C_4 nanotubes, C_4 nanotorus and grid, respectively. The exact weigted Szeged indices of these graphs are given below.*

1. $Sz_w(L_n) = 14n^3 - 4n^2 - 24n + 16.$
2. $Sz_w(R) = \begin{cases} \frac{m^3}{3}(10n^3 - 3n^2 - 10n + 6) & \text{if } m \text{ is even} \\ \frac{2m^3}{3}(n^3 + n^2 - 5n + 3) + m(m-1)^2 n^2 (2n-1) & \text{if } m \text{ is odd.} \end{cases}$
3. $Sz_w(S) = \begin{cases} 2n^3 m(m-1)^2 + 2m^3 n(n-1)^2 & \text{if } m \text{ is odd } n \text{ is odd} \\ 2n^3 m(2m^2 - 2m + 1) & \text{if } m \text{ is odd } n \text{ is even} \\ 2m^3 n(2n^2 - 2n + 1) & \text{if } m \text{ is even } n \text{ is odd} \\ 4m^3 n^3 & \text{if } m \text{ is even } n \text{ is even.} \end{cases}$
4. $Sz_w(T) = \frac{2m^2(n-1)}{3}(2mn^2 + 2mn - 3m - n^2 - n) + \frac{2n^2(m-1)}{3}(2nm^2 + 2mn - 3n - m^2 - m).$
5. $Sz_w(C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}) = \begin{cases} k^2 \prod_{i=1}^{k} n_i^3 & \text{if each } n_i \text{ is even} \\ k \prod_{i=1}^{k} n_i^3 \sum_{i=1}^{k} (1 - \frac{1}{n_i})^2 & \text{if each } n_i \text{ is odd.} \end{cases}$

If each $n_i = n$, then $Sz_w(\square C_n^k) = \begin{cases} k^2 n^{3k} & \text{if each } n_i \text{ is even} \\ k^2 (n-1)^2 n^{3k-2} & \text{if each } n_i \text{ is odd.} \end{cases}$

Example 2.9. *Let $G = C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}$ and $H = C_{m_1} \square C_{m_2} \square \ldots \square C_{m_r}$, where n_i, $1 \leq i \leq k$ are even and m_j, $1 \leq j \leq r$ are odd. Using Theorem 2.2 and the above example we obtain the weighted Szeged index of the graph $G \square H$.*

$$Sz_w(G \square H) = \Big(\prod_{i=1}^{k} n_i^3 \Big) \Big(\prod_{j=1}^{r} m_j^3 \Big) \Big(k^2 + kr + (k+r) \sum_{i=1}^{r} r(1 - \frac{1}{m_i})^2 \Big).$$

If each $n_i = n \geq 3$ is even and $m_j = m \geq 3$ is odd, then $G = \square C_n^k$ and $H = \square C_m^r$ and $Sz_w(G \square H) = n^{3k} m^{3r} \Big(k^2 + kr + (k+r) r (1 - \frac{1}{m})^2 \Big)$.

Example 2.10. *Using Theorem 2.4, we obtain the exact weighted Szeged index of the grid graph $P_{n_1} \square P_{n_2} \square \dots \square P_{n_k}$.*

$$Sz_w(P_{n_1} \square P_{n_2} \square \dots \square P_{n_k}) = \frac{2}{3}\Big(\prod_{i=1}^{k} n_i^3\Big)\Big(\sum_{i=1}^{k} \frac{(n_i-1)(n_i^2+n_i-3)}{n_i^3} + \sum_{i,j=1, i\neq j}^{k} (1-\frac{1}{n_i})(1+\frac{1}{n_i})(1-\frac{1}{n_j})\Big).$$

If each $n_i = n$, then $Sz_w(\square P_n^k) = \frac{2k(n-1)n^{3(k-1)}}{3}((n^2+n-3)+(k-1)(n+1))$.

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