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$\lambda-$ Core of a Sequence and Related Inequalities

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Abstract

The sequence spaces c^{λ} *and* cs^{λ} *have recently been introduced in [13] and [9], respectively, as the sets of all sequences whose* $\Lambda-$ *transforms are in the spaces* c *and* cs*, respectively. The main purpose of this study is to introduce the new type cores,* $\mathcal{K}_{\Lambda}-$ *core and* $S_{\Lambda}-$ *core, of a real valued sequence and also determine necessary and sufficient conditions for a matrix* A *to satisfy* $\mathcal{K}_{\Lambda}-core(Ax) \subseteq \mathcal{K}-core(x)$, $\mathcal{K}_{\Lambda}-core(Ax) \subseteq \sigma-core(x)$, $\mathcal{K}_{\Lambda}-core(Ax) \subseteq st-core(x)$, *and* $S_{\Lambda}-core(Ax) \subseteq \mathcal{K}-core(x)$, $S_{\Lambda}-core(Ax) \subseteq \sigma-core(x)$, $S_{\Lambda}-core(Ax) \subseteq st-core(x)$, *for all* $x \in \ell_{\infty}$.

Keywords: *Matrix transformatios, core of a sequence , Knopp's core theorem, invariant means, inequalities.*

1 Introduction

Let E be a subset of $N = \{0, 1, 2, ...\}$. The natural density δ of E is defined by $\delta(E) = \lim_n \frac{1}{n}|\{k \leq n : k \in E\}|$, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for every ε, $\delta\{k : |x_k - \ell| \geq \varepsilon\} = 0$, [7]. In this case, we write $st - \lim x = \ell$. By st and st_0, we denote the sets of statistically convergent and statistically null sequences. Fridy and Orhan [7] have introduced the notions of the statistically boundedness, statistical-limit superior ($st - \limsup$) and inferior ($st - \liminf$).

Let ℓ_{∞} and c be the Banach spaces of bounded and convergent sequences with the usual supremum norm respectively. Let σ be a one-to-one mapping from N into itself and T be an operator on ℓ_∞ defined by $Tx = x_{\sigma(k)}$. Then a continuous linear functional Φ on ℓ_∞ is said to be an invariant mean or a $\sigma-$mean if and only if (i) $\Phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k, (ii) $\Phi(e) = 1$, where $e = (1, 1, 1, ...)$, (iii) $\Phi(x) = \Phi(Tx)$ for all $x \in \ell_\infty$.

Throughout this paper we consider the mapping σ having no finite orbits, that is, $\sigma^p(k) \neq k$ for all positive integers $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ is pth iterate of σ at k. Thus, a $\sigma-$ mean extends the limit functional on c in the sense that $\Phi(x) = \lim x$ for all $x \in c$, [14]. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose $\sigma-$ means are equal. In the case $\sigma(k) = k+1$, a $\sigma-$mean often called a Banach limit and V_σ reduces to the set f of almost convergent sequences introduced by Lorentz [10]. The reader can refer to Raimi [16] for invariant means.

$$V_\sigma = \{x \in \ell_\infty : \lim_p t_{pn}(x) = s \;\text{ uniformly in }\; n,\; s = \sigma - \lim x\},$$

where

$$t_{pn}(x) = \frac{x_n + Tx_n + ... + T^p x_n}{p+1}, \quad t_{-1,n}(x) = 0.$$

We say that a bounded sequence $x = (x_k)$ is $\sigma-$convergent if and only if $x \in V_\sigma$. By $V_{0\sigma}$, we denote the space of $\sigma-$null sequences. It is well known [16] that $x \in \ell_\infty$ if and only if $(Tx - x) \in V_{0\sigma}$ and $V_\sigma = V_{0\sigma} \oplus Re$.

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ be a real number sequence. Then $Ax = ((Ax)_n) = (\sum_k a_{nk} x_k)$ denotes the $A-$transform of x. If X and Y are two sequence spaces, then we use $(X : Y)$ to denote the set of all matrices A such that Ax exists and $Ax \in Y$ for all $x \in X$. Troughout, $\sum_k$ will denote the summation from $k = 1$ to ∞.

If X and Y are equipped with the limits $X - \lim$ and $Y - \lim$, respectively, $A = (a_{nk}) \in (X : Y)$ and $Y - \lim_n (Ax)_n = X - \lim_k x_k$ for all $x = (x_k) \in X$, then we say A regularly transforms X into Y and write $A = (a_{nk}) \in (X : Y)_{reg}$. Let $\lambda = (\lambda_k)$ be a strictly increasing sequence of positive reals tending to infinity; that is $0 < \lambda_0 < \lambda_1 < \lambda_2 < ...,\quad \lim_{k\to\infty} \lambda_k = \infty$. We define the matrix $\Lambda = (\lambda_{nk})$ of weighted mean relative to the sequence λ by

$$\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k,\ n \in N$. With a direct calculation we derive the equality

$$(\Lambda x)_n = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k; \quad (n \in N).$$

Let us consider the following functionals defined on ℓ_∞:

$$\begin{aligned} l(x) &= \liminf_{k\to\infty} x_k, \qquad L(x) = \limsup_{k\to\infty} x_k, \\ q_\sigma(x) &= \limsup_{p\to\infty} \sup_{n\in N} \frac{1}{p+1} \sum_{i=0}^{p} x_{\sigma^i(n)}, \\ W(x) &= \inf_{z\in Z} L(x+z). \end{aligned}$$

Knopp's core (or $\mathcal{K}-$core) [3] and $\sigma-$core [12] of a real bounded sequence x were defined by the closed intervals $[l(x),\ L(x)]$ and $[-q_\sigma(-x), q_\sigma(x)]$, respectively, and also the inequalities $q_\sigma(Ax) \le L(x)$ (σ-core of $Ax \subseteq \mathcal{K}$-core of x), $q_\sigma(Ax) \le q_\sigma(x)$ (σ-core of $Ax \subseteq \sigma$-core of x), for all $x \in \ell_\infty$, was studied. Furthermore, we have that $q_\sigma(x) = W(x)$ for all $x \in \ell_\infty$ [12]. Several researchers studied on $\sigma-$core, (see [2,4–6,8,11,15]). Also, the textbook [1] containing the chapter titled " Core of a Sequence", reviewed the Knopp core, σ-core, $\mathcal{I}$-core, $\mathcal{F}_\mathcal{B}$-core.

Recently, Fridy and Orhan [7] introduced the notions of statistical boundedness, statistical limit superior (or briefly $st-\limsup$) and statistical limit inferior (or briefly $st-\liminf$), defined the statistical core (or briefly $st-core$) of a statistically bounded sequence is the closed interval $[st-\liminf x,\ st-\limsup x]$ and also determined necessary and sufficient conditions for a matrix A to yield $\mathcal{K}-\text{core}(Ax) \subseteq st-\text{core}(x)$ for all $x \in \ell_\infty$.

2 The Lemmas

In this section, we prove some lemmas which are needed in proving our main results and need the following lemma due to Das [6] for the proof of next theorem. In what follows we only consider that the inequality $\liminf_{n\to\infty}\left(\frac{\lambda_{n+1}}{\lambda_n}\right) > 1$ holds.

Lemma 2.1 *Let* $\|C\| = \|(c_{mk}(p))\| < \infty$ *and* $\lim_m \sup_p |c_{mk}(p)| = 0$. *Then, there is a* $y = (y_k) \in \ell_\infty$ *such that* $\|y\| \le 1$ *and*

$$\limsup_m \sup_p \sum_k c_{mk}(p) y_k = \limsup_m \sup_p \sum_k |c_{mk}(p)|.$$

Lemma 2.2 *[13] The inclusions* $c_0^\lambda \subset c^\lambda \subset \ell_\infty^\lambda$ *strictly hold.*

Corollary 2.3 *[13] The equalities* $c_0^\lambda = c_0,\ c^\lambda = c$ *and* $\ell_\infty^\lambda = \ell_\infty$ *hold if and only if* $\liminf_{n\to\infty}\left(\frac{\lambda_{n+1}}{\lambda_n}\right) > 1$.

Lemma 2.4 *[9] The inclusions* $cs^\lambda \subset c_0^\lambda$ *and* $bs^\lambda \subset \ell_\infty^\lambda$ *strictly hold.*

Lemma 2.5 *Let* $\|\Lambda\| < \infty$. *Then,* $A \in (\ell_\infty : c^\lambda)$ *if and only if*

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty, \tag{1}$$

$$\lim_m \frac{1}{\lambda_m} \sum_{n=0}^{m} (\lambda_n - \lambda_{n-1}) a_{nk} = \alpha_k \quad \text{for each } k, \tag{2}$$

$$\lim_m \sum_k \left| \frac{1}{\lambda_m} \sum_{n=0}^{m} (\lambda_n - \lambda_{n-1}) - \alpha_k \right| = 0. \tag{3}$$

Following is a result of Lemma 2.5.

Lemma 2.6 *Let* $\|\Lambda\| < \infty$. *Then,* $A \in (\ell_\infty : c_0^\lambda)$ *if and only if the conditions* (1) *and* (3) *of Lemma 2.5 hold with* $\alpha_k = 0$ *for all* $k \in N$.

Lemma 2.7 *Let* $\|\Lambda\| < \infty$. *Then,* $A \in (c : c^\lambda)_{reg}$ *if and only if the conditions* (1) *and* (2) *of Lemma 2.5 hold with* $\alpha_k = 0$ *for all* $k \in N$ *and*

$$\lim_m \sum_k \frac{1}{\lambda_m} \sum_{n=0}^{m} (\lambda_n - \lambda_{n-1}) a_{nk} = 1. \tag{4}$$

Lemma 2.8 *Let* $\|\Lambda\| < \infty$. *Then,* $A \in (V_\sigma : c^\lambda)_{reg}$ *if and only if*

$$A \in (c : c^\lambda)_{reg}, \tag{5}$$

$$A(T - I) \in (\ell_\infty : c_0^\lambda). \tag{6}$$

Lemma 2.9 *Let* $\|\Lambda\| < \infty$. *Then,* $A \in (st \cap \ell_\infty : c^\lambda)_{reg}$ *if and only if the condition* (5) *holds, and*

$$\lim_m \sum_{k \in E} \left| \frac{1}{\lambda_m} \sum_{n=0}^{m} (\lambda_n - \lambda_{n-1}) a_{nk} \right| = 0 \tag{7}$$

for every $E \subseteq N$ *with* $\delta(E) = 0$.

Proof. Suppose first that $A \in (st \cap \ell_\infty : c^\lambda)_{reg}$. Then, (5) follows from the fact that $c \subset st \cap \ell_\infty$. Now, for a given $x \in \ell_\infty$ and a subset E of N with $\delta(E) = 0$, let us define a sequence $y = (y_k)$ by

$$y_k = \begin{cases} x_k \ , & k \in E \\ 0 \ , & k \notin E. \end{cases}$$

By our assumption, since $y \in st_0 \cap \ell_\infty$, we have $Ay \in c_0^\lambda$. On the other hand, since $Ay = \sum_{k \in E} a_{nk} x_k$, the matrix $D = (d_{nk})$ defined by

$$d_{nk} = \begin{cases} a_{nk} \ , & k \in E \\ 0 \ , & k \notin E, \end{cases}$$

for all n, must be in $(\ell_\infty : c_0^\lambda)$. Thus, the necessity of (7) follows from Lemma 2.6.

Conversely, let (5) and (7) hold and let x be any sequence in $st \cap \ell_\infty$ with $st - \lim x = \ell$. Write $E = \{k : |x_k - \ell| \geq \varepsilon\}$ for any given $\varepsilon > 0$, so that $\delta(E) = 0$. Since $A \in (c : c^\lambda)_{reg}$, we have

$$\begin{aligned}\lim_m \sum_k \sum_{n=0}^m \lambda_{mn} a_{nk} x_k &= \lim_m \left(\sum_k \sum_{n=0}^m \lambda_{mn} a_{nk}(x_k - \ell) + \ell \sum_k \sum_{n=0}^m \lambda_{mn} a_{nk} \right) \\ &= \lim_m \sum_k \frac{1}{\lambda_m} \sum_{n=0}^m (\lambda_n - \lambda_{n-1}) a_{nk}(x_k - \ell) + \ell.\end{aligned}$$

On the other hand,

$$\left| \sum_k \frac{1}{\lambda_m} \sum_{n=0}^m (\lambda_n - \lambda_{n-1}) a_{nk}(x_k - \ell) \right| \leq \|x\| \sum_{k \in E} \frac{1}{\lambda_m} \left| \sum_{n=0}^m (\lambda_n - \lambda_{n-1}) a_{nk} \right| + \varepsilon \|\Lambda\| \|A\|,$$

the condition (7) implies that

$$\lim_m \sum_k \frac{1}{\lambda_m} \sum_{n=0}^m (\lambda_n - \lambda_{n-1}) a_{nk}(x_k - \ell) = 0. \tag{8}$$

Hence, $\lim \Lambda(Ax) = st - \lim x$; that is, $A \in (st \cap \ell_\infty : c^\lambda)_{reg}$, which completes the proof.

Lemma 2.10 *Let $\|\Lambda\| < \infty$. Then, $A \in (\ell_\infty : cs^\lambda)$ if and only if the condition* (1) *of the Lemma 2.5 holds and*

$$\lim_m \sum_{n=0}^m \frac{1}{\lambda_n} \sum_{i=0}^n (\lambda_i - \lambda_{i-1}) a_{ik} = \alpha_k \quad \textit{for each} \quad k, \tag{9}$$

$$\lim_m \sum_k \left| \sum_{n=0}^m \frac{1}{\lambda_n} \sum_{i=0}^n (\lambda_i - \lambda_{i-1}) a_{ik} - \alpha_k \right| = 0. \tag{10}$$

Lemma 2.11 *Let $\|\Lambda\| < \infty$. Then, $A \in (\ell_\infty : cs_0^\lambda)$ if and only if the conditions* (1) *and* (10) *hold with $\alpha_k = 0$ for all $k \in N$.*

Lemma 2.12 *Let $\|\Lambda\| < \infty$. Then, $A \in (c : cs^\lambda)_{reg}$ if and only if the conditions* (1) *and* (9) *hold with $\alpha_k = 0$ for all $k \in N$ and*

$$\lim_m \sum_k \sum_{n=0}^m \frac{1}{\lambda_n} \sum_{i=0}^n (\lambda_i - \lambda_{i-1}) a_{ik} = 1. \tag{11}$$

Lemma 2.13 *Let $\|\Lambda\| < \infty$. Then, $A \in (V_\sigma : cs^\lambda)_{reg}$ if and only if*

$$A \in (c, cs^\lambda)_{reg}, \tag{12}$$

$$A(T-I) \in (\ell_\infty, cs_0^\lambda). \tag{13}$$

Lemma 2.14 *Let $\|\Lambda\| < \infty$. Then, $A \in (st \cap \ell_\infty : cs^\lambda)_{reg}$ if and only if the condition* (12) *holds, and*

$$\lim_m \sum_{k\in E} \left| \sum_{n=0}^{m} \frac{1}{\lambda_n} \sum_{i=0}^{n} (\lambda_i - \lambda_{i-1}) a_{ik} \right| = 0 \tag{14}$$

for every $E \subseteq N$ with $\delta(E) = 0$.

3 $\mathcal{K}_\Lambda-$Core

In this section, we define the concept of $\mathcal{K}_\Lambda-$core and give some core theorems related to the space c^λ.

Definition 3.1 *Let $x \in \ell_\infty$. Then, $\mathcal{K}_\Lambda-$core of x is defined by the closed interval $[-L_\Lambda(-x),\ L_\Lambda(x)]$, where*

$$L_\Lambda(x) = \limsup_m \frac{1}{\lambda_m} \sum_{n=0}^{m} (\lambda_n - \lambda_{n-1}) x_n. \tag{15}$$

From the definition, it is easy to see that $\mathcal{K}_\Lambda-$core$(x) = \{\ell\}$ if and only if $\lim \Lambda_m(x) = \ell$, that is, $x \in c^\lambda$.

Theorem 3.2 *Let $\|\Lambda\| < \infty$. Then, $\mathcal{K}_\Lambda-$core$(Ax) \subseteq \mathcal{K}-$core$(x)$ for all $x \in \ell_\infty$ if and only if $A \in (c : c^\lambda)_{reg}$ and*

$$\lim_m \sum_k \frac{1}{\lambda_m} \left| \sum_{n=0}^{m} (\lambda_n - \lambda_{n-1}) a_{nk} \right| = 1. \tag{16}$$

Proof. Suppose first that $\mathcal{K}_\Lambda-$core$(Ax) \subseteq \mathcal{K}-$core$(x)$ for all $x \in \ell_\infty$. In this case, $L_\Lambda(Ax) \le L(x)$ for all $x \in \ell_\infty$. Then, one can easily see that

$$l(x) \le -L_\Lambda(-Ax) \le L_\Lambda(Ax) \le L(x).$$

If $x \in c$, then $l(x) = L(x) = \lim x$ and hence $-L_\Lambda(-Ax) = L_\Lambda(Ax) = \lim \Lambda(Ax) = \lim x$. This means that $A \in (c : c^\lambda)_{reg}$.

Now, let us define $C = (c_{mk})$ by

$$c_{mk} = \frac{1}{\lambda_m} \sum_{n=0}^{m} (\lambda_n - \lambda_{n-1}) a_{nk} \tag{17}$$

for all $k,\ m \in N$. Then, it is easy to see that the conditions of Lemma 2.1 are satisfied by the matrix C. Hence, there is a $y \in \ell_\infty$ such that $\|y\| \leq 1$ and

$$\limsup_m \sum_k c_{mk} y_k = \limsup_m \sum_k |c_{mk}|.$$

Therefore, by using the hypothesis, we can write

$$\begin{aligned} 1 &\leq \liminf_m \sum_k |c_{mk}| \leq \limsup_m \sum_k |c_{mk}| \\ &= \limsup_m \sum_k c_{mk} y_k = L_\Lambda(Ay) \leq L(y) \leq \|y\| \leq 1. \end{aligned}$$

This gives the necessity of (16).

Conversely, suppose that $A \in (c : c^\lambda)_{reg}$ and (16) holds for all $x \in \ell_\infty$. For any real number z, we write $z^+ := \max\{z, 0\}, z^- := \max\{-z, 0\}, |z| = z^+ + z^-, z = z^+ - z^-$ and $|z| - z = 2z^-$. Thus, for any given $\varepsilon > 0$, there is a $k_0 \in N$ such that $x_k < L(x) + \varepsilon$ for all $k > k_0$. Now, we can write

$$\begin{aligned} \sum_k c_{mk} x_k &= \sum_{k<k_0} c_{mk} x_k + \sum_{k \geq k_0} (c_{mk})^+ x_k - \sum_{k \geq k_0} (c_{mk})^- x_k \\ &\leq \|x\| \sum_{k<k_0} |c_{mk}| + (L(x) + \varepsilon) \sum_k |c_{mk}| + \|x\| \sum_k [|c_{mk}| - c_{mk}]. \end{aligned}$$

Therefore, by applying the operator $\limsup_m$ to the last inequality and using hypothesis, we have $L_\Lambda(Ax) \leq L(x) + \varepsilon$. Hence, the proof is completed, since ε is arbitrary and $x \in \ell_\infty$.

Theorem 3.3 *Let* $\|\Lambda\| < \infty$. *Then,* $\mathcal{K}_\Lambda-$*core*$(Ax) \subseteq \sigma-$*core*(x) *for all* $x \in \ell_\infty$ *if and only if* $A \in (V_\sigma : c^\lambda)_{reg}$ *and* (16) *hold.*

Proof. Let $\mathcal{K}_\Lambda-$core$(Ax) \subseteq \sigma-$core(x) for all $x \in \ell_\infty$. Then, since $L_\Lambda(Ax) \leq q_\sigma(x)$ and $q_\sigma(x) \leq L(x)$ for all $x \in \ell_\infty$, the necessity of (16) follows from Theorem 3.2.

Also, we can write that

$$-q_\sigma(-x) \leq -L_\Lambda(-Ax) \leq L_\Lambda(Ax) \leq q_\sigma(x)$$

i.e.,

$$\sigma - \liminf x \leq -L_\Lambda(-Ax) \leq L_\Lambda(Ax) \leq \sigma - \limsup x.$$

If x is chosen in V_σ, then $\sigma - \liminf x = \sigma - \limsup x = \sigma - \lim x$. Therefore, we have from the last inequality that $-L_\Lambda(-Ax) = L_\Lambda(Ax) = \lim \Lambda(Ax) = \sigma - \lim x$ and so, $A \in (V_\sigma : c^\lambda)_{reg}$.

Conversely, suppose that $A \in (V_\sigma : c^\lambda)_{reg}$ and (16) holds. In this case, since $c \subset V_\sigma$, by using Theorem 3.2, we have $L_\Lambda(Ax) \leq L(x)$ for all $x \in \ell_\infty$.

$$\inf_{z \in V_{0\sigma}} L_\Lambda(Ax + Az) \leq \inf_{z \in V_{0\sigma}} L(x + z) = W(x). \tag{18}$$

On the other hand, since $Az \in c_0^\lambda$ for $z \in V_{0\sigma}$, we can write that

$$\inf_{z\in V_{0\sigma}} L_\Lambda(Ax+Az) \geq L_\Lambda(Ax) + \inf_{z\in V_{0\sigma}} L_\Lambda(Az) = L_\Lambda(Ax). \tag{19}$$

Thus, combining the statements (18) and (19), we obtain that $L_\Lambda(Ax) \leq W(x)$ for all $x \in \ell_\infty$ which completes the proof, since $q_\sigma(x) = W(x)$, [12].

Theorem 3.4 *Let* $\|\Lambda\| < \infty$. *Then,* $\mathcal{K}_\Lambda-$*core*$(Ax) \subseteq st-$*core*(x) *for all* $x \in \ell_\infty$ *if and only if* $A \in (st \cap \ell_\infty : c^\lambda)_{reg}$ *and* (16) *hold.*

Proof. Assume that $\mathcal{K}_\Lambda-$core$(Ax) \subseteq st-$core(x) for all $x \in \ell_\infty$. Then, $L_\Lambda(Ax) \leq \beta(x)$ for all $x \in \ell_\infty$ where $\beta(x) = st - \limsup x$. Hence, since $\beta(x) = st - \limsup x \leq L(x)$ for all $x \in \ell_\infty$ (see [7]), we obtain (16) from Theorem 3.2. Furthermore, we can write that

$$-\beta(-x) \leq -L_\Lambda(-Ax) \leq L_\Lambda(Ax) \leq \beta(x)$$

i.e.,

$$st - \liminf x \leq -L_\Lambda(-Ax) \leq L_\Lambda(Ax) \leq st - \limsup x.$$

If $x \in st \cap \ell_\infty$, then $st - \liminf x = st - \limsup x = st - \lim x$. Thus, the last inequality implies that $st - \lim x = -L_\Lambda(-Ax) = L_\Lambda(Ax) = \lim \Lambda(Ax)$, that is, $A \in (st \cap \ell_\infty : c^\lambda)_{reg}$.

Conversely, assume that $A \in (st \cap \ell_\infty : c^\lambda)_{reg}$ and (16) hold. If $x \in \ell_\infty$, then $\beta(x)$ is finite. Let E be a subset of N defined by $E = \{k : x_k > \beta(x) + \varepsilon\}$ for any given $\varepsilon > 0$. Then it is obvious that $\delta(E) = 0$ and $x_k \leq \beta(x) + \varepsilon$ if $k \notin E$. Now, we can write that

$$\begin{aligned}
\sum_k c_{mk}x_k &= \sum_{k<k_0} c_{mk}x_k + \sum_{k\geq k_0} c_{mk}x_k = \sum_{k<k_0} c_{mk}x_k + \sum_{k\geq k_0} c^+_{mk}x_k - \sum_{k\geq k_0} c^-_{mk}x_k \\
&\leq \|x\| \sum_{k<k_0} |c_{mk}| + \sum_{\substack{k\geq k_0 \\ k\notin E}} c^+_{mk}x_k + \sum_{\substack{k\geq k_0 \\ k\in E}} c^+_{mk}x_k + \|x\| \sum_{k\geq k_0} (|c_{mk}| - c_{mk}) \\
&\leq \|x\| \sum_{k<k_0} |c_{mk}| + (\beta(x)+\varepsilon) \sum_{\substack{k\geq k_0 \\ k\notin E}} |c_{mk}| + \|x\| \sum_{\substack{k\geq k_0 \\ k\in E}} |c_{mk}| \\
&\quad + \|x\| \sum_{k\geq k_0} [|c_{mk}| - c_{mk}],
\end{aligned}$$

where $C = (c_{mk})$ is defined by (17). By applying the operator $\limsup_m$ to the last inequality and using hypothesis, it follows that $L_\Lambda(Ax) \leq \beta(x) + \varepsilon$. This completes the proof, since ε is arbitrary.

4 $S_\Lambda-$Core

In this section, the concept of $S_\Lambda-$core for $x \in \ell_\infty$ is defined and necessary and sufficient conditions for a matrix A to satisfy $S_\Lambda-$core$(Ax) \subseteq \mathcal{K}-$core$(x)$, $S_\Lambda-$core$(Ax) \subseteq \sigma-$core(x) and $S_\Lambda-$core$(Ax) \subseteq st-$core(x) for all $x \in \ell_\infty$ are determined.

Definition 4.1 *Let $x \in \ell_\infty$. Then, $S_\Lambda-$core of x is defined by the closed interval $[-M^*(-x), M^*(x)]$, where*

$$M^*(x) = \limsup_{m\to\infty} \sum_{n=0}^{m} \frac{1}{\lambda_n} \sum_{i=0}^{n} (\lambda_i - \lambda_{i-1}) x_i.$$

From the definition, it is easy to see that $S_\Lambda-$core$(x) = \ell$ if and only if $\lim_m \sum_{n=0}^{m} (\Lambda x)_n = \ell$, that is, $x \in cs^\lambda$.

Theorem 4.2 *Let $\|\Lambda\| < \infty$. Then, $S_\Lambda -$ core$(Ax) \subseteq \mathcal{K} -$ core(x) for all $x \in \ell_\infty$ if and only if $A \in (c : cs^\lambda)_{reg}$ and*

$$\lim_m \sum_k \left| \sum_{n=0}^{m} \frac{1}{\lambda_n} \sum_{i=0}^{n} (\lambda_i - \lambda_{i-1}) a_{ik} \right| = 1. \tag{20}$$

Theorem 4.3 *Let $\|\Lambda\| < \infty$. Then, $S_\Lambda -$ core$(Ax) \subseteq \sigma -$ core(x) for all $x \in \ell_\infty$ if and only if $A \in (V_\sigma : cs^\lambda)_{reg}$ and (20) hold.*

Theorem 4.4 *Let $\|\Lambda\| < \infty$. Then, $S_\Lambda -$ core$(Ax) \subseteq st -$ core(x) for all $x \in \ell_\infty$ if and only if $A \in (st \cap \ell_\infty : cs^\lambda)_{reg}$ and (20) hold.*

Since Theorem 4.2, 4.3 and 4.4 can be proved similarly with Theorem 3.2, 3.3 and 3.4, proofs of their are trivial.

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