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On the Classical Sets of Sequences with Fuzzy b-Metric

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Abstract

Since the utilization of Zadeh's Extension Principle is quite difficult in practice, we prefer the idea of using the sum of the series of level sets. In this paper we present some classical sets of sequences of fuzzy numbers with respect to the notion of fuzzy b-metric. Also, we introduce the completeness of such spaces and derive the relationships between these sets and their classical forms. In addition, we use our results corresponding with series not only directly improve and generalize some results in metric spaces and b-metric spaces, and also expand and complement some previous results in fuzzy metric spaces with the level sets.

Keywords: *Set of the sequences of fuzzy numbers, fuzzy level sets, b-metric, complete metric space.*

1 Introduction

Some problems, particularly the problem of the convergence of measurable functions with respect to a measure, lead to a generalization of notion of a metric. As a continuation of metric notion Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle in metric spaces which is a very popular tool for solving problems in nonlinear analysis. In [7], Czerwik introduced b-metric spaces as a generalization of metric spaces. Also he proved the contraction mapping principle in b-metric spaces which generalizes the famous Banach contraction principle. Later, several papers have devoted to the fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces have been obtained (see [2, 4, 5, 15, 16, 17, 23, 24, 30, 31, 36, 39, 43]). In recent investigations, the fixed point in nonconvex analysis, especially in an ordered normed space, occupies a prominent place in many aspects.

Especially, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and studied fixed point theory in the so-called bistructural spaces. For more details on fixed point results, its applications, comparison of different contractive conditions and related results in ordered metric spaces, the reader may refer to [1, 3, 6, 9, 10, 11, 12, 13, 21, 27, 29, 40, 42, 49] and the references mentioned therein.

By $\omega(F)$, we denote the set of all sequences of fuzzy numbers. We define the classical sets $\ell_\infty(b,\lambda)$, $c(b,\lambda)$, $c_0(b,\lambda)$ and $\ell_q(b,\lambda)$ with respect to the b-metric consisting of the b-bounded, b-convergent, b-null and absolutely q-th order of b-summable sequences of fuzzy numbers, as follows:

$$\begin{aligned}
\ell_\infty(b,\lambda) &:= \left\{u=(u_k)\in\omega(F):\sup_{k\in\mathbb{N}} D^b(u_k,\overline{0})<\infty\right\},\\
c(b,\lambda) &:= \left\{u=(u_k)\in\omega(F):\exists l\in E^1 \ni {}^b\lim_{k\to\infty} D^b(u_k,l)=0\right\},\\
c_0(b,\lambda) &:= \left\{u=(u_k)\in\omega(F):\ {}^b\lim_{k\to\infty} D^b(u_k,\overline{0})=0\right\},\\
\ell_q(b,\lambda) &:= \left\{u=(u_k)\in\omega(F):\sum_k D^b(u_k,\overline{0})^q<\infty\right\},\quad (1\le q<\infty)
\end{aligned}$$

where the distance function D^b on the space E^1 of fuzzy numbers by means of the b-metric ρ defined by

$$D^b(u,v):=\sup_{\lambda\in[0,1]}\rho([u]_\lambda,[v]_\lambda):=\sup_{\lambda\in[0,1]}\max\{(d(u_\lambda^-,v_\lambda^-)^p,(d(u_\lambda^+,v_\lambda^+)^p\}$$

with $s=2^{p-1}$ where $u,v\in E^1$ and $p\in\mathbb{R}$ with $p>1$. Since D^b is a b-metric, denotes fuzzy b-metric, by means of ρ based on λ and Hausdorff metric d.

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. It can be shown that the sets $\ell_\infty(b,\lambda)$, $c(b,\lambda)$ and $c_0(b,\lambda)$ are b-complete with the b-metric $D_\infty^{b,\lambda}$ defined by

$$D_\infty^{b,\lambda}(u,v):=\sup_{k\in\mathbb{N}} D^b(u_k,v_k)$$

with $s=2^{p-1}$ and $u=(u_k)$, $v=(v_k)$ are the elements of the sets $c(b,\lambda)$, $c_0(b,\lambda)$ or $\ell_\infty(b,\lambda)$.

On the other hand, we can introduce and prove that the set $\ell_q(b,\lambda)$ is b-complete with $D_q^{b,\lambda}$ defined by

$$D_q^{b,\lambda}(u,v) := \left\{\sum_k \left[D^b(u_k,v_k)\right]^q\right\}^{1/q}, \quad (1 \leq q < \infty)$$

with $s = 2^{q-1}$ where $u = (u_k)$, $v = (v_k)$ are the points of $\ell_p(b,\lambda)$.

Mursaleen and Başarır [28] have recently introduced some new sets of sequences of fuzzy numbers generated by a non-negative regular matrix A some of which are reduced to the Maddox spaces $\ell_\infty(p,F)$, $c(p,F)$, $c_0(p,F)$ and $\ell(p,F)$ of sequences of fuzzy numbers for the special cases of that matrix A. Altın, Et and Çolak [51] have recently defined the concepts of lacunary statistical convergence and lacunary strongly convergence of generalized difference sequences of fuzzy numbers. They have also given some relations related to these concepts and showed that lacunary $\triangle^m$-statistical convergence and lacunary strongly $\triangle^m_{(p)}$-convergence are equivalent for $\triangle^m$-bounded sequences of fuzzy numbers. Quite recently; Talo and Başar [33] have extended the main results of Başar and Altay [8] to the fuzzy numbers. Also, Talo and Başar [35] have recently studied the normed quasilinearity of the classical sets $\ell_\infty(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$ of sequences of fuzzy numbers and derived some related results. Furthermore, Talo and Başar [34] have introduced the sets $\ell_\infty(F,f)$, $c(F,f)$, $c_0(F,f)$ and $\ell_p(F,f)$ of sequences of fuzzy numbers defined by a modulus function and given some topological properties of the sets together with some inclusion relations. Finally, Kadak and Başar [44, 45, 46, 47] have presented some new notions about the power series and Fourier series of fuzzy numbers on fuzzy level sets. The main purpose of the present paper is to study the corresponding sets $\ell_\infty(b,\lambda)$, $c(b,\lambda)$, $c_0(b,\lambda)$ and $\ell_q(b,\lambda)$ of sequences of fuzzy numbers via b-metric to the classical spaces ℓ_∞, c, c_0 and ℓ_p of sequences with real or complex terms.

The rest of this paper is organized, as follows:

In section 2, some required definitions and consequences related with the b-metric, sequences and series of fuzzy numbers are given. The most relevant and recent literature is also reported in Section 2. Section 3 is terminated with the condensation of the results on the sum of the series of the fuzzy sets given by M. Stojaković and Z. Stojaković in [26]. Additionally, an example on b-convergence of series of fuzzy numbers is also presented in this section. Furthermore, some notions i.e uniformly convergent, continuity and boundedness are established via fuzzy b-metric and the completeness of such sequence spaces of fuzzy numbers via b-metric are presented.

2 Preliminaries, Background and Notation

2.1 Fuzzy Level Sets

A *fuzzy number* is a fuzzy set on the real axis, i.e. a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following four conditions:

(i) u is normal, i.e. there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e. $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, (cf. Zadeh [19]), where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

λ-*level set* $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda := \begin{cases} \{t \in \mathbb{R} : u(t) \geq \lambda\} & , \quad (0 < \lambda \leq 1), \\ \overline{\{t \in \mathbb{R} : u(t) > \lambda\}} & , \quad (\lambda = 0). \end{cases}$$

The set $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1 & , \quad (x = r), \\ 0 & , \quad (x \neq r). \end{cases}$$

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on E^1 by

$$\begin{aligned} u + v = w \quad &\Leftrightarrow \quad [w]_\lambda = [u]_\lambda + [v]_\lambda \text{ for all } \lambda \in [0,1] \\ &\Leftrightarrow \quad w^-(\lambda) = u^-(\lambda) + v^-(\lambda), w^+(\lambda) = u^+(\lambda) + v^+(\lambda). \end{aligned}$$

Further

$$[ku]_\lambda = k[u]_\lambda \text{ for all } \lambda \in [0,1]$$

and

$$uv = w \Leftrightarrow [w]_\lambda = [u]_\lambda [v]_\lambda \text{ for all } \lambda \in [0,1],$$

where it is immediate that

$$w^-(\lambda) = \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

and

$$w^+(\lambda) = \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

for all $\lambda \in [0,1]$. Let W be the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e. $A := [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A,B) := \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can easily be observed that d is a metric on W (cf. Diamond and Kloeden [37]) and (W,d) is a complete metric space, (cf. Nanda [41]).

$$A \preceq B \text{ if and only if } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B}.$$

The partial ordering relation $\preceq$ on E^1 is defined as follows:

$$u \preceq v \Leftrightarrow [u]_\lambda \preceq [v]_\lambda \Leftrightarrow u^-(\lambda) \leq v^-(\lambda) \text{ and } u^+(\lambda) \leq v^+(\lambda) \text{ for all } \lambda \in [0,1].$$

Definition 2.1 *(Triangular Fuzzy Number)* [18, Definition, p. 137] *We can define the triangular fuzzy number u as $u = (u_1, u_2, u_3)$ whose membership function μ is interpreted as follows;*

$$\mu(x) = \begin{cases} \frac{x-u_1}{u_2-u_1} & , \quad u_1 \leq x \leq u_2, \\ \frac{u_3-x}{u_3-u_2} & , \quad u_2 \leq x \leq u_3, \\ 0 & , \quad x < u_1, \quad x > u_3. \end{cases}$$

Then, the result $[u]_\lambda := [u^-(\lambda), u^+(\lambda)] = [(u_2 - u_1)\lambda + u_1, -(u_3 - u_2)\lambda + u_3]$ holds for each $\lambda \in [0,1]$.

2.2 b-Metric

Consistent with [7] and [43], the following definitions and results will be needed in the sequel.

Definition 2.2 *[14] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is a b-metric if, for all $x,y,z \in X$, the following conditions are satisfied:*

(b_1) $d(x,y) = 0$ *iff* $x = y$,

(b_2) $d(x,y) = d(y,x)$,

(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric if (and only if) $s = 1$. We present an easy example to show that in general a b-metric need not be a metric.

Example 2.3 *[2] Let (X, d) be a metric space, and $d^b(x, y) = (d(x, y)^p)$, where $p \geq 1$ is a real number. Then d^b is a b-metric with $s = 2^{p-1}$.*

However, (X, d) is not necessarily a metric space. For example, if $X = \mathbb{R}$ is the set of real numbers and $d^b(x, y) = |x - y|$ is the usual Euclidean metric, then $d^b(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, but it is not a metric on $\mathbb{R}$.

Definition 2.4 *[22] Let (X, ρ) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*

(a) *b-convergent if and only if there exists $x \in X$ such that $\rho(x_n, x) \to 0$, as $n \to \infty$. In this case, we write ${}^b\lim_{n\to\infty} x_n = x$.*

(b) *b-Cauchy if and only if $\rho(x_n, x_m) \to 0$, as $n, m \to \infty$.*

Definition 2.5 *[22] In a b-metric space (X, ρ) the following assertions hold:*

(a) *A b-convergent sequence has a unique limit.*

(b) *Each b-convergent sequence is b-Cauchy.*

(c) *In general, a b-metric is not continuous.*

Also very recently N. Hussain et al. have presented an example of a b-metric which is not continuous (see Example 3 in [30]).

Definition 2.6 *[22]*

(a) *The b-metric space (X, ρ) is b-complete iff every b-Cauchy sequence in X is b-convergent.*

(b) *Let (X, ρ) be a b-metric space. If Y is a nonempty subset of X, then the closure Y of Y is the set of limits of all b-convergent sequences of points in Y, i.e.,*

$$\overline{Y} := \left\{x \in X : \text{there exists a sequence } \{x_n\} \text{ in } Y \text{ so that } {}^b\lim_{n\to\infty} x_n = x\right\}.$$

Taking into account of the above definition, we have the following concepts.

(c) *Let (X, ρ) be a b-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\} \in Y$ which b-converges to an element x, we have $x \in Y$.*

(d) *Let (X, ρ) and (X', ρ') be two b-metric spaces. Then a function $f : X \to X'$ is b-continuous at a point $x \in X$ if and only if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x;$\{f(x_n)\}$ is b-convergent to $f(x)$.*

Lemma 2.7 *[2] Let (X, d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}, \{y_n\}$ are b-convergent to x, y, respectively. Then we have,*

$$\frac{1}{s^2}\ d(x, y) \leq {}^b\liminf_{n\to\infty} d(x_n, y_n) \leq {}^b\limsup_{n\to\infty} d(x_n, y_n) \leq s^2\ d(x, y).$$

In particular, if $x = y$, then we have ${}^b\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s}\ d(x, z) \leq {}^b\liminf_{n\to\infty} d(x_n, z) \leq {}^b\limsup_{n\to\infty} d(x_n, z) \leq s^2\ d(x, z).$$

Further, we have the following basic definitions with respect to an arbitrary b-metric on real which is essential in the text.

Definition 2.8 (Uniform b-convergence) *Let $\{f_n(x)\}$ be a sequence of real-valued functions defined on a set $A \subseteq \mathbb{R}$ and ρ be arbitrary b-metric on $\mathbb{R}$. If $\{f_n(x)\}$ b-converges pointwise on a set A, then we can define $f : A \to \mathbb{R}$ by*

$${}^b\lim_{n\to\infty} f_n(x) = f(x) \quad or \quad f_n(x) \xrightarrow{\rho} f(x) \quad for\ each \quad x \in A.$$

In other words, $\{f_n(x)\}$ b-converges to f on A if and only if for each $x \in A$ and for an arbitrary $\epsilon > 0$, there exists an integer $N = N(\epsilon, x)$ such that $\rho(f_n(x), f(x)) < \epsilon$ whenever $n > N$. The integer N in the definition of pointwise convergence may, in general, depend on both $\epsilon > 0$ and $x \in A$. If, however, one integer can be found that works for all points in A, then the b-convergence is said to be uniform. That is, a sequence of functions $\{f_n(x)\}$ uniformly b-converges to f on a set A if for each $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\rho(f_n(x), f(x)) < \epsilon \quad or \quad f_n(x) \xrightarrow{u\rho} f(x) \quad whenever \quad n > N(\epsilon) \quad and\ for\ all \quad x \in A.$$

Definition 2.9 *Let ρ be arbitrary b-metric on $\mathbb{R}$ with $s \geq 1$ then the following basic definitions can be given as follows:*

(i) *(b-Limit) Suppose $f:\mathbb{R}\to\mathbb{R}$ is defined on the real line and $c, L \in \mathbb{R}$. It is said the b-limit of f, as x approaches c with respect to b-metric ρ, is L and written ${}^b\lim_{x\to c} f(x) = L$. On the other hand*

$$^b\lim_{x\to c} f(x) = L \Leftrightarrow \forall\epsilon > 0, \exists\delta > 0 \ni \rho(f(x), L) \le \epsilon \textit{ for all } \ \rho(x, c) < \delta.$$

(ii) *(b-Continuity) A function $f:\mathbb{R}\to\mathbb{R}$ is said to be b-continuous at c if it is both defined at c and its value at c equals the b-limit of f as x approaches c with respect to b-metric ρ, denoted by ${}^b\lim_{x\to c} f(x) = f(c)$.*

(iii) *(b-Boundedness) A sequence $(x_n) \in \omega$ is called b-bounded if and only if the set which consists the terms of the sequence (x_n) is a bounded set. That is to say that a sequence $(x_n) \in \omega$ is said to be b-bounded if and only if there exist a number $m > 0$ such that $\rho(x_n, 0) \le m$ for all $n \in \mathbb{N}$.*

By taking the value $s = 1$ for an arbitrary b-metric ρ the notions in Definition 2.9 are reduced to those of classical limit, continuity and boundedness.

3 Main Results

Lemma 3.1 *Let d be usual metric. Then the distance function D^b defined by*

$$D^b(u,v) := \sup_{\lambda\in[0,1]} \rho([u]_\lambda, [v]_\lambda) := \sup_{\lambda\in[0,1]} \max\{(d(u_\lambda^-, v_\lambda^-)^p, (d(u_\lambda^+, v_\lambda^+)^p\} \qquad (1)$$

is a fuzzy b-metric for $u = [u]_\lambda, v = [v]_\lambda \in E^1$. Furthermore the function ρ is also b-metric denoted by fuzzy usual b-metric with $s = 2^{p-1}$ where $p > 1$ is a real number.

Proof 3.2 *One can easily show by a routine verification that ρ and D^b satisfy b-metric axioms in Definition 2.2. So, we prove only the axiom for D^b.*

Let $u = [u]_\lambda$, $v = [v]_\lambda$ and $w = [w]_\lambda \in E^1$. By taking into account the conditions $\max\{a+b, c+d\} \le \max\{a, c\} + \max\{b, d\}$ *for all $a, b, c, d > 0$ and $(a+b)^p \le 2^{p-1}(a^p + b^p)$ from convexity of the function x^p for $p \ge 1$. We immediately deduce that*

(b_3) $D^b(u,w)$

$$\begin{aligned}
&= \sup_{\lambda\in[0,1]} \max\{(d(u_\lambda^-,w_\lambda^-)^p,(d(u_\lambda^+,w_\lambda^+)^p\} \\
&\leq \sup_{\lambda\in[0,1]} \max\left\{[d(u_\lambda^-,v_\lambda^-)+d(v_\lambda^-,w_\lambda^-)]^p,[d(u_\lambda^+,v_\lambda^+)+d(v_\lambda^+,w_\lambda^+)]^p)\right\} \\
&\leq 2^{p-1}\sup_{\lambda\in[0,1]} \max\left\{d(u_\lambda^-,v_\lambda^-)^p+d(v_\lambda^-,w_\lambda^-)^p,d(u_\lambda^+,v_\lambda^+)^p+d(v_\lambda^+,w_\lambda^+)^p\right\} \\
&\leq 2^{p-1}\big(\sup_{\lambda\in[0,1]} \max\left\{d(u_\lambda^-,v_\lambda^-)^p,d(u_\lambda^+,v_\lambda^+)^p\right\} \\
&+ \sup_{\lambda\in[0,1]} \max\left\{d(v_\lambda^-,w_\lambda^-)^p,d(v_\lambda^+,w_\lambda^+)^p\right\}\big) \\
&= 2^{p-1}(D^b(u,v)+D^b(v,w)).
\end{aligned}$$

Therefore, (E^1,D^b) is b-metric space on E^1 with $s=2^{p-1}$ where $p>1$ is a real number.

As a result of Bolzano-Weierstrass Theorem for every bounded infinite sequence of fuzzy numbers in [48] we may give the next corollary based on fuzzy level set completeness.

Corollary 3.3 *The set (E^1,D^b) is b-complete metric space on E^1 with $s=2^{p-1}$ where $p>1$ is a real number.*

Example 3.4 *Let ρ be b-metric in Lemma 3.1. Consider the membership functions $u(x)$ and $v(x)$ defined by triangular form as*

$$u(x)=\begin{cases} 2x-1 & , \quad 1/2\leq x\leq 2/3, \\ 3-2x & , \quad 2/3<x\leq 1, \\ 0 & , \quad otherwise, \end{cases} \qquad v(x)=\begin{cases} 12x-1 & , \quad 1/12\leq x\leq 1/6, \\ 3-12x & , \quad 1/6\leq x\leq 1/4, \\ 0 & , \quad otherwise, \end{cases}$$

for all $k\in\mathbb{N}$. It is trivial that $u_\lambda^-=\frac{\lambda+1}{2}$, $v_\lambda^-=\frac{\lambda+1}{12}$ and $u_\lambda^+=\frac{3-\lambda}{2}$, $v_\lambda^+=\frac{3-\lambda}{12}$ for all $\lambda\in[0,1]$. Therefore we can calculate the distance between two fuzzy numbers u and v with respect to ρ

$$\begin{aligned}
D^b(u,v) &= \sup_{\lambda\in[0,1]} \max\{(d(u_\lambda^-,v_\lambda^-)^p,(d(u_\lambda^+,v_\lambda^+)^p\} \\
&= \sup_{\lambda\in[0,1]} \max\{|u_\lambda^- - v_\lambda^-|^p,|u_\lambda^+ - v_\lambda^+|^p\} \\
&= \sup_{\lambda\in[0,1]} \max\left\{\left|\frac{5(\lambda+1)}{12}\right|^p,\left|\frac{5(3-\lambda)}{12}\right|^p\right\}=(5/4)^p.
\end{aligned}$$

If we choose $p=1$, then the distance $D^b(u,v)=5/4$ which is equal to the usual fuzzy distance. If we take $p=3$ then $s=4$ and $D^b(u,v)=(5/4)^3$. Furthermore one can conclude that the fuzzy distance based on b-metric ρ of two fuzzy numbers depends on the choice of s.

To avoid undue repetition in the statements, we give the next proposition without proof which is concerning Lemma 3.1. Now, we may give:

$$D^b(u,\overline{0}) = \sup_{\lambda\in[0,1]} \max\{|u_\lambda^-|^p, |u_\lambda^+|^p\} = \max\{|u_0^-|^p, |u_0^+|^p\}. \tag{2}$$

Proposition 3.5 *Let $u, v, w, z \in E^1$ and $k \in \mathbb{R}$. Then,*

(i) $D^b(ku, kv) = |k|^p D^b(u, v)$ *with* $p \geq 1$.

(ii) $D^b(u + v, w + v) = D^b(u, w)$.

(iii) $D^b(u + v, w + z) \leq s[D^b(u, w) + D^b(v, z)]$ *with* $s \geq 1$.

(iv) $s|D^b(u,\overline{0}) - D^b(v,\overline{0})| \leq D^b(u, v) \leq s[D^b(u,\overline{0}) + D^b(v,\overline{0})]$ *for all* $s \geq 1$.

Following Matloka [27], we give some definitions concerning the sequences of fuzzy numbers with respect to the fuzzy b-metric below, which are needed in the text.

Lemma 3.6 *(cf. [32]) The following statements hold:*

(i) $D^b(uv,\overline{0}) \leq D^b(u,\overline{0})D^b(v,\overline{0})$ *for all* $u, v \in E^1$.

(ii) *If* $u_k \to u$, *as* $k \to \infty$ *then* $D^b(u_k,\overline{0}) \to D^b(u,\overline{0})$, *as* $k \to \infty$; *where* $(u_k) \in \omega(F)$.

Proof 3.7 *(i) It is clear that the inequalities* $|u_\lambda^-|^p \leq D^b(u,\overline{0})$ *and* $|u_\lambda^+|^p \leq D^b(u,\overline{0})$ *hold for all* $\lambda \in [0, 1]$. *By considering these facts, one can see that the distance* $D^b(uv,\overline{0})$

$$\begin{aligned}
&= \sup_{\lambda\in[0,1]} \max\{|(uv)^-(\lambda)|^p, |(uv)^+(\lambda)|^p\} \\
&\leq \sup_{\lambda\in[0,1]} \max\{|u_\lambda^-|^p|v_\lambda^-|^p, |u_\lambda^-|^p|v_\lambda^+|^p, |u_\lambda^+|^p|v_\lambda^-|^p, |u_\lambda^+|^p|v_\lambda^+|^p\} \\
&\leq \sup_{\lambda\in[0,1]} \max\{D^b(u,\overline{0})|v_\lambda^-|^p, D^b(u,\overline{0})|v_\lambda^+|^p, D^b(u,\overline{0})|v_\lambda^-|^p, D^b(u,\overline{0})|v_\lambda^+|^p\} \\
&= D^b(u,\overline{0}) \sup_{\lambda\in[0,1]} \max\{|v_\lambda^-|^p, |v_\lambda^+|^p\} \\
&= D^b(u,\overline{0})D^b(v,\overline{0}),
\end{aligned}$$

which completes the proof of part (i).

(ii) This is trivial by using the fact given by (iv) of Proposition 3.5 and Definition 2.4(a).

Representation Theorem 1 *Let* $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ *for* $u \in E^1$ *and for each* $\lambda \in [0, 1]$. *Then the following statements hold:*

(i) *$u^-(\lambda)$ is a b-bounded and non-decreasing left b-continuous and $u^+(\lambda)$ is a b-bounded and non-increasing left b-continuous function on $(0,1]$.*

(ii) *The functions $u^-(\lambda)$ and $u^+(\lambda)$ are right b-continuous at the point $\lambda = 0$.*

Proof 3.8 *The proof can be obtained in a similar way from [38] which consists as a generalization of this theroem. We omit the detail.*

Remark 3.9 *If the number u is the uniform b-limit in Definition 2.8 of the sequence $\{u_n\}$ of fuzzy numbers on E^1, write $u_n \xrightarrow{\rho} u$, as $n \to \infty$ then the real sequences of functions $u_n^-(\lambda) \xrightarrow{u\rho} u_\lambda^-$ and $u_n^+(\lambda) \xrightarrow{u\rho} u_\lambda^+$ in $[0,1]$, respectively.*

Theorem 3.10 *Let $(u_k), (v_k) \in \omega(F)$ with ${}^b\lim_k u_k = a$, ${}^b\lim_k v_k = b$. Then,*

(i) ${}^b\lim_k(u_k + v_k) = a + b$ *as* $k \to \infty$,

(ii) ${}^b\lim_k(u_k - v_k) = a - b$ *as* $k \to \infty$,

(iii) ${}^b\lim_k u_k v_k = ab$ *as* $k \to \infty$,

(iv) ${}^b\lim_k u_k/v_k = a/b$ *as* $k \to \infty$ *where* $0 \in [u_k]_0$ *for all* $k \in \mathbb{N}$ *and* $0 \in [u]_0$.

Definition 3.11 *A sequence $(u_k) \in \omega(F)$ is called fuzzy b-bounded if and only if the set of fuzzy numbers consisting of the terms of the sequence (u_k) is a b-bounded set. That is to say that a sequence $(u_k) \in \omega(F)$ is said to be fuzzy b-bounded if and only if there exist two fuzzy numbers m and M such that $m \preceq u_k \preceq M$ for all $k \in \mathbb{N}$. This means that $m_\lambda^- \leq u_k^-(\lambda) \leq M_\lambda^-$ and $m_\lambda^+ \leq u_k^+(\lambda) \leq M_\lambda^+$ for all $\lambda \in [0,1]$.*

The fact that the fuzzy b-boundedness of the sequence $(u_k) \in \omega(F)$ is equivalent to the uniform b-boundedness of the functions $u_k^-(\lambda)$ and $u_k^+(\lambda)$ on $[0,1]$. Therefore, one can see by using the relation (2) that the fuzzy b-boundedness of the sequence $(u_k) \in \omega(F)$ is equivalent to the fact that

$$\sup_{k\in\mathbb{N}} D^b(u_k, \overline{0}) = \sup_{k\in\mathbb{N}} \sup_{\lambda\in[0,1]} \max\{|u_k^-(\lambda)|^p, |u_k^+(\lambda)|^p\} < \infty \quad (p \geq 1)$$

Now, prior to stating and proving the lemma concerning the sum of a b-convergent series of fuzzy numbers we give the following definition (cf. Kim and Ghil [50]):

Definition 3.12 *Let $(u_k) \in \omega(F)$ and D^b be a fuzzy b-metric on E^1. Then the expression $\sum u_k$ is called a series of fuzzy numbers. Denote $s_k = \sum_{k=0}^n u_k$ for all $n \in \mathbb{N}$, if the sequence (s_k) b-converges to a fuzzy number u then we say* *that the series* $\sum u_k$ *of fuzzy numbers b-converges to* u *and write* $\sum_{k=0}^{\infty} u_k = u$ *which implies as* $n \to \infty$ *that*

$$\sum_{k=0}^{n} u_k^-(\lambda) \xrightarrow{u\rho} u^-(\lambda) \quad \textit{and} \quad \sum_{k=0}^{n} u_k^+(\lambda) \xrightarrow{u\rho} u^+(\lambda),$$

uniformly in $\lambda \in [0,1]$. *On the other hand by taking into account* ρ *is a b-metric on* $\mathbb{R}$, *there exists an integer* $N(\epsilon)$ *such that*

$$\rho\left(\sum_k u_k^{\pm}(\lambda), u^{\pm}(\lambda)\right) < \varepsilon \quad \textit{whenever} \quad n > N(\epsilon) \quad \textit{and for all} \quad \lambda \in [0,1].$$

Conversely, if the fuzzy numbers $u_k = \{(u^-(\lambda), u^+(\lambda)) : \lambda \in [0,1]\}$, $\sum_k u_k^-(\lambda) = u^-(\lambda)$ *and* $\sum_k u_k^+(\lambda) = u^+(\lambda)$ *u*ρ*-converge in* λ, *then* $u = \{[u_\lambda^-, u_\lambda^+] : \lambda \in [0,1]\}$ *defines a fuzzy number such that* $u = \sum u_k$.

Lemma 3.13 *(cf. [32]) If the fuzzy number* $u_k = \{[u_k^-(\lambda), u_k^+(\lambda)] : \lambda \in [0,1]\}$, $\sum_k u_k^-(\lambda) = u_\lambda^-$ *and* $\sum_k u_k^+(\lambda) = u_\lambda^+$ *u*ρ*-converge in* λ, *then* $u = \{[u_\lambda^-, u_\lambda^+] : \lambda \in [0,1]\}$ *defines a fuzzy number such that* $u = \sum u_k$.

Proof 3.14 *To prove the lemma, we must show that the pair of functions* u^- *and* u^+ *satisfies the conditions of Representation Theorem. For this, we prove that* u^- *is a b-bounded, non-decreasing, left b-continuous function on* $(0,1]$ *and right b-continuous at the point* $\lambda = 0$. u_k^-*'s are the b-bounded, non-decreasing, left b-continuous functions on* $(0,1]$ *and right b-continuous at the point* $\lambda = 0$ *for each* $k \in \mathbb{N}$.

- **(i)** *Let* $\lambda_1 < \lambda_2$. *Then,* $u_k^-(\lambda_1) \le u_k^-(\lambda_2)$ *for each* $k \in \mathbb{N}$. *Therefore, we have* $\sum_k u_k^-(\lambda_1) \le \sum_k u_k^-(\lambda_2)$ *which yields the fact that* $u^-(\lambda_1) \le u^-(\lambda_2)$. *Hence,* u^- *is non-decreasing.*

- **(ii)** *By taking into account the uniform b-convergence in* λ *of* $\lim_{\lambda \to \lambda_0^-} u_k^-(\lambda) = u_k^-(\lambda_0)$, $\sum_k u_k^-(\lambda) = u^-(\lambda)$ *for each* $k \in \mathbb{N}$ *we obtain for* $\lambda_0 \in (0,1]$ *that*

$$\lim_{\lambda \to \lambda_0^-} u^-(\lambda) = \lim_{\lambda \to \lambda_0^-} \sum_k u_k^-(\lambda) = \sum_k \lim_{\lambda \to \lambda_0^-} u_k^-(\lambda) = \sum_k u_k^-(\lambda_0) = u^-(\lambda_0)$$

  which shows that u^- *is a left b-continuous function on* $(0,1]$.

- **(iii)** *By using the uniform b-convergence in* λ *in the expressions* $\lim\limits_{\lambda \to 0^+} u_k^-(\lambda) = u_k^-(0)$ *for each* $k \in \mathbb{N}$ *and* $\sum_k u_k^-(\lambda) = u^-(\lambda)$, *we see that*

$$\lim_{\lambda \to 0^+} u^-(\lambda) = \lim_{\lambda \to 0^+} \sum_k u_k^-(\lambda) = \sum_k \lim_{\lambda \to 0^+} u_k^-(\lambda) = \sum_k u_k^-(0) = u^-(0). \quad (3)$$

  This means that u^- *is a right b-continuous function at the point* $\lambda = 0$.

(iv) *There exists $M_k > 0$ such that $\rho(u_k^-(\lambda), 0) \leq M_k$ for all $\lambda \in [0,1]$ and for all $k \in \mathbb{N}$. Since the series $\sum_k u_k^-(\lambda) = u^-(\lambda)$ $u\rho$-converges in λ there exists $n_0 \in \mathbb{N}$ for all $\varepsilon > 0$ such that $\rho\left(\sum_{k=n+1}^{\infty} u_k^-(\lambda), 0\right) < \varepsilon$ for all $n \geq n_0$ and for all $\lambda \in [0,1]$. Therefore, we have*

$$\begin{aligned}
\rho(u_\lambda^-, 0) = \rho\left(\sum_{k=0}^{\infty} u_k^-(\lambda), 0\right) &= \rho\left(\sum_{k=0}^{n} u_k^-(\lambda) + \sum_{k=n+1}^{\infty} u_k^-(\lambda), 0\right) \\
&\leq \sum_{k=0}^{n} \rho(u_k^-(\lambda), 0) + \rho\left(\sum_{k=n+1}^{\infty} u_k^-(\lambda), 0\right) \\
&\leq \sum_{k=0}^{n} M_k + \varepsilon \leq K_\varepsilon
\end{aligned}$$

This leads us to the fact that u^- is a b-bounded function.

Since one can establish in the similar way that u^+ is a b-bounded, non-increasing, left b-continuous function on $(0,1]$, and right b-continuous at the point $\lambda = 0$, we omit the detail.

Therefore, it is deduced that $[u]_\lambda = [u_k^-(\lambda), u_k^+(\lambda)]$ defines a fuzzy number. Finally, we show that $\sum u_k = u$. Since the series of functions $\sum_k u_k^-(\lambda)$ and $\sum_k u_k^+(\lambda)$ $u\rho$-converge in λ to $u^-(\lambda)$ and $u^+(\lambda)$, respectively, for all $\varepsilon > 0$ and $s = 2^{p-1}$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
D^b\left(\sum_{k=0}^{n} u_k, u\right) &= \sup_{\lambda\in[0,1]} \max\left\{\left|\sum_{k=0}^{n} u_k^-(\lambda) - u^-(\lambda)\right|^p, \left|\sum_{k=0}^{n} u_k^+(\lambda) - u^+(\lambda)\right|^p\right\} \\
&\leq \max\left\{\sup_{\lambda}\left|\sum_{k=0}^{n} u_k^-(\lambda) - u_\lambda^-\right|^p, \sup_{\lambda}\left|\sum_{k=0}^{n} u_k^+(\lambda) - u_\lambda^+\right|^p\right\} < \varepsilon
\end{aligned}$$

for all $n \geq n_0$, the sequence $(\sum_{k=0}^{n} u_k)$ b-converges to the fuzzy number u, i.e. $\sum u_k = u$.

Example 3.15 *As an example for b-convergent series corresponding fuzzy b-metric D^b with $s = 2^{p-1}$, consider the series $\sum u_k$ with*

$$u_k(t) = \begin{cases} 2^k t - 1 & , \quad \frac{1}{2^k} \leq t \leq \frac{2}{2^k}, \\ 1, & , \quad \frac{2}{2^k} < t \leq \frac{4}{2^k}, \\ 2 - 2^{k-2} t & , \quad \frac{4}{2^k} < t \leq \frac{8}{2^k}, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for all $k \in \mathbb{N}$. It is obvious that $u_k^-(\lambda) = \frac{\lambda+1}{2^k}$ and $u_k^+(\lambda) = \frac{4(2-\lambda)}{2^k}$ for all $\lambda \in [0,1]$. Therefore by using the uniform b-convergence of b-metric we see *that* $\sum_k (u_k)^-_\lambda = 2^p(\lambda+1)$ *and* $\sum_k (u_k)^+_\lambda = 8^p(2-\lambda)$. *Then, it is conclude that* $\sum u_k = u$, *where*

$$u(t) := \begin{cases} \frac{t}{2^p} - 1 & , \quad (2 \le t \le 4), \\ 1 & , \quad (4 \le t \le 8), \\ 2 - \frac{t}{8^p} & , \quad (8 \le t \le 16), \\ 0 & , \quad (otherwise). \end{cases}$$

3.1 Completeness of Some Fuzzy Sequence Spaces

Lemma 3.16 *Define the relation* $D^{b,\lambda}$ *on the space* γ *by*

$$\begin{array}{rcll} D_\infty^{b,\lambda} & : & \gamma \times \gamma & \longrightarrow \mathbb{R}^+ \\ & & (u,v) & \longmapsto D_\infty^{b,\lambda}(u,v) = \sup_{k\in\mathbb{N}} D^b(u_k, v_k) = \sup_{k\in\mathbb{N}} \sup_{\lambda\in[0,1]} \rho(u_k, v_k) \end{array}$$

where γ *denotes any of the spaces* $\ell_\infty(b,\lambda)$, $c(b,\lambda)$, $c_0(b,\lambda)$ *and* $u = (u_k), v = (v_k) \in \gamma$. *Then,* $(\gamma, D_\infty^{b,\lambda})$ *is a b-complete metric space.*

Proof 3.17 *Since the proof is similar for the spaces* $c(b,\lambda)$ *and* $c_0(b,\lambda)$, *we prove the theorem only for the space* $\ell_\infty(b,\lambda)$.

One can easily show by a routine verification that $D_\infty^{b,\lambda}$ *satisfies* (b_1) *and* (b_2). *So, we prove only* (b_3). *Let* $u = (u_k)$, $v = (v_k)$ *and* $w = (w_k) \in \ell_\infty(b,\lambda)$. *Then,*

(b_3) *By using the axiom* (b_3) *in Definition 2.2 we get*

$$\begin{aligned} D_\infty^{b,\lambda}(u_k, w_k) &= \sup_{k\in\mathbb{N}} \big\{ D^b(u_k, w_k) \big\} \le \sup_{k\in\mathbb{N}} \big\{ s[D^b(u_k, v_k) + D^b(v_k, w_k)] \big\} \\ &\le s \sup_{k\in\mathbb{N}} \big\{ D^b(u_k, v_k) \big\} + s \sup_{k\in\mathbb{N}} \big\{ D^b(v_k, w_k) \big\} \\ &= s \left[\sup_{k\in\mathbb{N}} \big\{ D^b(u_k, v_k) \big\} + \sup_{k\in\mathbb{N}} \big\{ D^b(v_k, w_k) \big\} \right] \\ &= s \left[D_\infty^{b,\lambda}(u_k, v_k) + D_\infty^{b,\lambda}(v_k, w_k) \right] \end{aligned}$$

for all $s \ge 1$. *Therefore* $(\ell_\infty(b,\lambda), D_\infty^{b,\lambda})$ *is b-metric space on* $\ell_\infty(b,\lambda)$. *It remains to prove the b-completeness of the space* $\ell_\infty(b,\lambda)$.

Let $x_m = \{x_1^{(m)}, x_2^{(m)}, \ldots\}$ *be a b-Cauchy sequence in* $\ell_\infty(b,\lambda)$. *Then, for any* $\epsilon > 0$ *there exists a positive integer* m_0 *such that* $D_\infty^{b,\lambda}(x_m, x_r) = \sup_{k\in\mathbb{N}} D^b(x_k^{(m)}, x_k^{(r)}) < \epsilon$ *for all* $m, r > m_0$. *A fortiori, for every fixed* $k \in \mathbb{N}$ *and for* $m, r > m_0$ *then*

$$\left\{ D^b \left(x_k^{(m)}, x_k^{(r)} \right) : k \in \mathbb{N} \right\} < \epsilon. \tag{4}$$

In this case for any fixed $k \in \mathbb{N}$, *by using completeness of* (E^1, D^b) *in Corollary 3.3 , we say that* $x_k^{(m)}$ *is a b-Cauchy sequence and is b-convergent. That is to* *say that* $(x_\lambda^-)_k^{(m)}$ *and* $(x_\lambda^+)_k^{(m)}$ *are up-convergent for all* $\lambda \in [0,1]$. *Now, we suppose that* ${}^b\lim_{m\to\infty} x_k^{(m)} = x_k$ *and* $x = (x_1, x_2, \ldots)$. *We must show that* ${}^b\lim_{m\to\infty} D_\infty^{b,\lambda}(x_m, x) = 0$ *and* $x \in \ell_\infty(b,\lambda)$.

The constant $m_0 \in \mathbb{N}$ *for all* $m > m_0$, *taking the b-limit as* $r \to \infty$ *in (4), we obtain* $D^b(x_k^{(m)}, x_k) < \epsilon$ *for all* $k \in \mathbb{N}$. *Since* $\left(x_k^{(m)}\right) \in \ell_\infty(b,\lambda)$, *there exists a positive number* $\delta > 0$ *such that* $D^b(x_k^{(m)}, 0) \leq \delta$. *By taking into account b-metric axiom* (b_3) *we get*

$$D^b(x_k, 0) \leq s[D^b(x_k, x_k^{(m)}) + D^b(x_k^{(m)}, 0)] < s(\epsilon + \delta) \tag{5}$$

for all $s \geq 1$. *It is clear that (5) holds for every* $k \in \mathbb{N}$ *whose right-hand side does not involve* k. *This leads us to the consequence that* $x = (x_k) \in \ell_\infty(b,\lambda)$. *Also, we immediately deduce that the inequality*

$$D_\infty^{b,\lambda}(x_m, x) = \sup_{k\in\mathbb{N}} D^b\left(x_k^{(m)}, x_k\right) < \epsilon$$

holds for $m > m_0$. *This shows that* $D_\infty^{b,\lambda}(x_m, x) \to 0$ *as* $m \to \infty$. *Since* (x_m) *is an arbitrary b-Cauchy sequence,* $\ell_\infty(b,\lambda)$ *is b-complete.*

Lemma 3.18 *Define the distance function* $D_q^{b,\lambda}$ *by*

$$\begin{array}{rcccl} D_q^{b,\lambda} & : & \ell_q(b,\lambda) \times \ell_q(b,\lambda) & \longrightarrow & \mathbb{R}^+ \\ & & (u,v) & \longmapsto & D_q^{b,\lambda}(u,v) := \left\{\sum_k D^b(u_k, v_k)^q\right\}^{1/q} \end{array}$$

where $x = (x_k)$, $y = (y_k) \in \ell_q(b,\lambda)$ *and* $1 \leq q < \infty$. *Then,* $(\ell_q(b,\lambda), D_q^{b,\lambda})$ *is a b-complete metric space.*

Proof 3.19 *It is obvious that* $D_q^{b,\lambda}$ *satisfies* (b_1) *and* (b_2). *Let* $u = (u_k)$, $v = (v_k)$ *and* $w = (w_k) \in \ell_q(b,\lambda)$. *Then, we derive by applying the Minkowski's inequality* $D_q^{b,\lambda}(u,w)$ *can be evaluated as*

$$\begin{aligned} &= \left\{\sum_k \left[D^b(u_k, w_k)\right]^q\right\}^{1/q} \leq \left\{\sum_k \left(s[D^b(u_k, v_k) + D^b(v_k, w_k)]\right)^q\right\}^{1/q} \\ &\leq (s^q)^{1/q}\left\{\left(\sum_k \left[D^b(u_k, v_k)\right]^q\right)^{1/q} + \left(\sum_k \left[D^b(v_k, w_k)\right]^q\right)^{1/q}\right\} \\ &= s\left\{\left(\sum_k \left[D^b(u_k, v_k)\right]^q\right)^{1/q} + \left(\sum_k \left[D^b(v_k, w_k)\right]^q\right)^{1/q}\right\} \\ &= s\ [D_q^{b,\lambda}(u,v) + D_q^{b,\lambda}(v,w)]. \end{aligned}$$

This shows that (b_3) *also holds. Therefore* $(\ell_q(b,\lambda), D_q^{b,\lambda})$ *is b-metric space with* $s = 2^{q-1}$. *Since the remaining part of this proof is analogous Lemma 3.16 we omit the detail. Hence* $(\ell_q(b,\lambda), D_q^{b,\lambda})$ *is a b-complete metric space.*

Concluding Remarks

In this research some new sequence spaces of fuzzy numbers are introduced by using the notion of b-metric with respect to the usual metric, and shown that the given spaces are b-complete. In addition, this work presents a new tool for the description of fixed point theorems for fuzzy contractive mappings. The potential applications of the obtained results include the establishment of new sequence spaces which are interesting topics for the future works. Of course, it will be meaningful to determine the alpha-, beta- and gamma-duals of b-metric sequence spaces. We should record that one can study on the domain of some triangle matrices in b-metric sequence spaces $\ell_\infty(b, \lambda)$, $c(b, \lambda)$ and $\ell_q(b, \lambda)$ which is a new development on the theory of sequence spaces and matrix transformations. Finally, we should note from now on that our next papers will be devoted to Kothe-Toeplitzs duals and matrix transformations between some classical sets of sequences of fuzzy numbers with respect to b-metric.

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References

[1] A. Aghajani, S. Radenovic and J.R. Roshan, Common fixed point results for four mappings satisfying almost generalized (S-T)-contractive condition in partially ordered metric spaces, *Appl. Math. Comput.*, 218(2012), 5665-5670.

[2] A. Aghajani, M. Abbas and J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Math. Slovaca*, (in Press).

[3] A.C.M. Ran and M.C.B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, *Proc. Amer. Math. Soc.*, 132(2004), 1435-1443.

[4] H. Aydi, M.F. Bota, E. Karapinar and S. Moradi, A common fixed point for weak ψ-contractions on b-metric spaces, *Fixed Point Theory*, 13(2) (2012), 337-346.

[5] H. Aydi, M.F. Bota, E. Karapinar and S. Mitrovic, A fixed point theorem for set-valued quasi-contractions in b-metric spaces, *Fixed Point Theory Appl.*, 88(1) (2012).

[6] L. Ciric, B. Samet, H. Aydi and C. Vetro, Common fixed points of generalized contractions on partial metric spaces and an application, *Appl. Math. Comput.*, 218(6) (2011), 2398-2406.

[7] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Semin. Mat. Fis. Univ. Modena*, 46(2) (1998), 263-276.

[8] F. Başar and B. Altay, On the space of sequences of p-bounded variation and related matrix mappings, *Ukrainian Math. J.*, 55(1) (2003), 136-147.

[9] H.K. Nashine and B. Samet, Fixed point results for mappings satisfying weakly contractive condition in partially ordered metric spaces, *Nonlinear Anal.*, 74(2011), 2201-2209.

[10] J. Esmaily, S.M. Vaezpour and B.E. Rhoades, Coincidence point theorem for generalized weakly contractions in ordered metric spaces, *Appl. Math. Comput.*, 219(2012), 1536-1548.

[11] J. Harjani and K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, *Nonlinear Anal.*, 72(3-4) (2010), 1188-1197.

[12] J.J. Nieto, R.L. Pouso and R. Rodriguez-Lopez, Fixed point theorems in ordered abstract sets, *Proc. Amer. Math. Soc.*, 135(2007), 2505-2517.

[13] J.J. Nieto and R. RodrÄśguez-Lopez, Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sin., Engl. Ser.,* 23(2007), 2205-2212.

[14] J.R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, Common fixed points of almost generalized $(\psi, \phi)_s$ -contractive mappings in ordered b-metric spaces, *Fixed Point Theory Appl.*, 159(1) (2013).

[15] J.R. Roshan, V. Parvaneh and I. Altun, Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations, *Appl. Math. Comput.*, 226(2014), 725-737.

[16] M.A. Khamsi and N. Hussain, KKM mappings in metric type spaces, *Nonlinear Anal.*, 73(9) (2010), 3123-3129.

[17] M.A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, *Fixed Point Theory Appl.*, Article ID 315398 (2010).

[18] K.H. Lee, *First Course on Fuzzy Theory and Applications*, Springer, Berlin - Heidelberg - NewYork, (2005).

[19] L.A. Zadeh, Fuzzy Sets, *Inform. and Control*, 8(1965), 338-353.

[20] M. Abbas, T. Nazir and S. Radenovic, Common fixed points of four maps in partially ordered metric spaces, *Appl. Math. Lett.*, 24(2011), 1520-1526.

[21] M. Abbas, V. Parvaneh and A. Razani, Periodic points of T-Ciric generalized contraction mappings in ordered metric spaces, *Georgian Math. J.*, 19(4) (2012), 597-610.

[22] M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractals in b-metric spaces, *Cent. Eur. J. Math.*, 8(2) (2010), 367-377.

[23] M. Boriceanu, Fixed point theory for multivalued generalized contractions on a set with two b-metrics, *Stud. Univ. Babe-Bolyai, Math. LIV*, 3(2009), 1-14.

[24] M. Boriceanu, Strict fixed point theorems for multivalued operators in b-metric spaces, *Int. J. Mod. Math.*, 4(3) (2009), 285-301.

[25] M. Rojes-Medar and H. Roman-Flores, On the equivalence of convergences of fuzzy sets, *Fuzzy Sets Syst.*, 80(1996), 217-224.

[26] M. Stojaković and Z. Stojaković, Addition and series of fuzzy sets, *Fuzzy Sets Syst.*, 83(1996), 341-346.

[27] M. Matloka, Sequences of fuzzy numbers, *Busefal*, 28(1986), 28-37.

[28] M. Mursaleen and M. Başarır, On some new sequence spaces of fuzzy numbers, *Indian J. Pure Appl. Math.*, 34(9) (2003), 1351-1357.

[29] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.*, 7(2) (2006), 289-297.

[30] N. Hussain, D. Doric, Z. Kadelburg and S. Radenovic, Suzuki-type fixed point results in metric type spaces, *Fixed Point Theory Appl.*, 126(1) (2012).

[31] N. Hussain and M.H. Shah, KKM mappings in cone b-metric spaces, *Comput. Math. Appl.*, 62(2011), 1677-1684.

[32] Ö. Talo and F. Başar, Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations, *Comput. Math. Appl.*, 58(2009), 717-733.

[33] Ö. Talo and F. Başar, On the space $bv_p(b,\lambda)$ of sequences of p-bounded variation of fuzzy numbers, *Acta Math. Sin. Eng. Ser.*, 24(6) (2008), 965-972.

[34] Ö. Talo and F. Başar, Certain spaces of sequences of fuzzy numbers defined by a modulus function, *Demonstratio Mathematica*, XLIII(1) (2010).

[35] Ö. Talo and F. Başar, Quasilinearity of the classical sets of sequences of the fuzzy numbers and some applications, *Taiwanese J. Math.*, 14(5) (2010), 1799-1819.

[36] M. Pacurar, Sequences of almost contractions and fixed points in b-metric spaces, *An. Univ. Vest. Timis., Ser. Theor. Math. Comput. Phys.*, XLVIII(3) (2010), 125-137.

[37] P. Diamond and P. Kloeden, Metric spaces of fuzzy sets, *Fuzzy Sets Syst.*, 35(1990), 241-249.

[38] R. Goetschel and W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets Syst.*, 18(1986), 31-43.

[39] J.R. Roshan, N. Shobkolaei, S. Sedghi and M. Abbas, Common fixed point of four maps in b-metric spaces, *Hacet. J. Math.Stat.*, (in Press).

[40] R.P. Agarwal, M.A. El-Gebeily and D.O. Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, 87(1) (2008), 109-116.

[41] S. Nanda, On sequences of fuzzy numbers, *Fuzzy Sets Syst.*, 33(1989), 123-126.

[42] S. Radenovic and Z. Kadelburg, Generalized weak contractions in partially ordered metric spaces, *Comput. Math. Appl.*, 60(2010), 1776-1783.

[43] S.L. Singh and B. Prasad, Some coincidence theorems and stability of iterative procedures, *Comput. Math. Appl.*, 55(2008), 2512-2520.

[44] U. Kadak and F. Başar, Power series of fuzzy numbers, *AIP Conference Proceedings*, 1309(2010), 538-550.

[45] U. Kadak and F. Başar, Power series of fuzzy numbers with reel or fuzzy coefficients, *Filomat*, 25(3) (2012), 519-528.

[46] U. Kadak and F. Başar, On some sets of fuzzy-valued sequences with the level sets, *Contemp. Anal. Appl. Math.*, 1(2) (2013), 70-90.

[47] U. Kadak and F. Başar, On Fourier series of fuzzy valued functions, *The Scientific World Journal*, Article ID 782652, (2014).

[48] U. Kadak and M. Ozluk, On partial metric spaces of fuzzy numbers with the level sets, *Adv. Fuzzy Sets Syst.*, 16(1) (2013), 31-64.

[49] W. Shatanawi and B. Samet, On weakly contractive condition in partially ordered metric spaces, *Comput. Math. Appl.*, 62(2011), 3204-3214.

[50] Y.K. Kim and B.M. Ghil, Integrals of fuzzy-number-valued functions, *Fuzzy Sets Syst.*, 86(1997), 213-222.

[51] Y. Altın, M. Et and R. Çolak, Lacunary statistical and lacunary strongly convergence of generalized difference sequences of fuzzy numbers, *Comput. Math. Appl.*, 52(2006), 1011-1020.

[52] Z. Gong and C. Wu, Bounded variation, absolute continuity and absolute integrability for fuzzy-number-valued functions, *Fuzzy Sets Syst.*, 129(2002), 83-94.