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Modified Sequential Covering Algorithm for Finding a Global Minimizer of Nondifferentiable Functions and Applications

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Abstract

In this paper, the one-dimensional unconstrained global optimization problem of continuous functions satifying a Hölder condition is considered. We first extend the algorithm of sequential covering SCA for Lipschitz functions to a large class of Hölder functions, and we propose a modification MSCA of this algorithm. We show that the implementation of the algorithm MSCA has the advantage of avoiding the use of auxiliary calculations. The algorithm MSCA is then some what easily which permits to considerably reduce the calculation time. The convergence of the method is also studied. The algorithm MSCA can be applied to systems of nonlinear equations. Finally, some numerical examples are presented and illustrate the efficiency of the present approach.

Keywords: *Global optimization, Hölder functions, Sequential covering method, Systems of nonlinear equations.*

1 Introduction

This paper investigates the global optimization problem of minimizing a one-dimensional multiextremal function $f(x)$ having a finite number of local minima over an interval $[a, b]$ of $\mathbb{R}$, i.e., the finding of a point $x^* \in [a, b]$ and the value f^* such that

$$f^* = f(x^*) = \min_{x\in[a,b]} f(x). \qquad (1)$$

Throughout this work, we assume that the objective function f satisfies a Hölder condition, i.e.,

$$|f(x) - f(y)| \leq h\,|x-y|^{\alpha}\,, \quad \text{for all } x, y \in [a,b]\,, \qquad (2)$$

where $0 < h < \infty$ is called the Hölder constant and $0 < \alpha < 1$ the Hölder exponent.

Let $\varepsilon > 0$ be the desired accuracy with which the global minimum of $f(x)$ is to be searched.

Now, we review the concept of Hölder functions and give several properties. Obviously, a Hölder function $f(x)$ is a Lipschitz one on $[a,b]$ when $\alpha = 1$, [11]. If $f(x)$ is a Hölder function with constant $h > 0$ and exponent $0 < \alpha < 1$ on $[a,b]$, then it is also Hölder function with constant $h^{'} > h$ and exponent $\alpha^{'} < \alpha$. Although a Hölder function is always continuous, it need not to be differentiable. A Hölder function with a low value of α is much more irregular than a Hölder function with a high value of α (in fact, this statement only makes sense if we consider the highest value of α for which (2) holds).

Global optimization problems arise in many disperse fields of science and technology. The existence of multiple local minima of a general nonconvex objective function makes global optimization a great challenge.

This problem is of interest for at least two reasons. Firstly, because such an interest is motivated by a number of real life applications where such problems arise, for instance, the simple plant location problem under a uniform delivered price policy, see Hanjoul et al. [12], infinte horizon optimization problems see [15] and secondly, because such objective functions are more irregular than the Lipschitz ones, [17].

Many approaches using an auxiliary function have been proposed to deal with the continuous global optimization problems [9-11]. In [8], the authors have proposed an extension of the method of Piyavskii [22], [26] to case of a univariate Hölder function where the parameters h and α are a priori known. They used a technique which is based on the construction of piecewise parabolic lower bounding functions of f on $[a,b]$. At each iteration, the method necessitates the determination of the unique global minimizer of a convex nondifferentiable functions. This amounts to solving a nonlinear equation of order α, which is not always easy to treat explicitly. In [18], the authors have suggested at each iteration, an approximation of the global minimizer of lower bounding functions by the intersection point of two secantes linked to the piecewise parabols. In [23], we have extended the Piyavskii's algorithm by constructing a lower bounding functions which are piecewise linear. The purpose of the present work is to avoid the use of the construction of piecewise lower bounding functions of f on $[a,b]$. We propose a sequential covering algorithm SCA which is best known most discused in literature [2], [4-7] is constructed for Hölder functions without differentiability not convex property of f, but the parameters h and α must be known. We construct an iterative sequence $(x_k)_{k\geq 1}$of points of $[a,b]$, such that with each point x_i we associate a neighbouring interval $I(x_i, r_{ik})$ with center x_i and radius r_{ik}. When the interval $[a,b]$ is completely covered then we prove the convergence of the sequence $(x_k)_{k\geq 1}$ to a global minimizer x^* of f on $[a,b]$.

This paper modifies the sequential covering algorithm for Hölder continuous functions. An advantage of our modification is that it reduces the number of evaluation points necessary to the convergence. Moreover, it is a new optimization method to solve a system of nonlinear equations without differentiability assumption on the objective function.

The rest of this paper is organized as follows: In section 2, the method of sequential covering is briefly described. The corresponding algorithm for univariate Hölder functions is presented in section 3. In section 4, we propose a modifications of the sequential covering algorithm. In section 5, test functions and numerical experiments are reported. In sections 6, we apply the modified algorithm for solving systems of nonlinear nonsmooth equations. Finally, some conclusions are drawn in section 7.

2 General Idea of the Method of Coverings

Covering algorithms, [5], [11], [24-25], constitute an important class of methods for solving problems of type (1) for very large classes of functions f. Let S^* be the set of all global minimizers of the objective function f (the solutions set) and the set of ε-optimal (approximate) solutions of problem (1) is defined as follows

$$S_{\varepsilon}^* = \{x \in [a,b] : \ f(x) \leq f^* + \varepsilon\}. \tag{3}$$

The sets S^*, S_{ε}^* are nonempty because of the compactness of $[a,b]$ and the continuity of f on $[a,b]$.

It is clear that $S^* \subset S_{\varepsilon}^* \subset [a,b]$. Our goal is to find at least one point $x_{\varepsilon} \in S_{\varepsilon}^*$. Any value $f(x_{\varepsilon})$ where $x_{\varepsilon} \in S_{\varepsilon}^*$ is called an ε-optimal value of f on $[a,b]$.

Let $X_k = \{x_1, x_2, ..., x_k\}$ be a finite set of k points in $[a,b]$. After evaluating f at these points, we define the record value

$$m_i^* = \min\left(f(x_1), f(x_2), ..., f(x_i)\right) = f(x_r), \text{ for } 1 \leq i \leq k. \tag{4}$$

Any such point x_r , $i \leq r \leq k$ is called a record point. We say that a numerical algorithm solves the problem (1) after k evaluations if a set X_k is such that $m_k^* \leq f^* + \varepsilon$ or, equivalently, $(x_r \in X_k)$ belongs to the set S_{ε}^*. The algorithm is defined by a rule for constructing such a set X_k.

It is follows from (4) that the sequence $(m_i^*)_{1\leq i\leq k}$ is an nonincreasing one. With each point x_i, we associate its neighbor interval $[a,b]_i$.
Consider the following sets S_i and their union U_k,

$$S_i \;=\; \{x : x_i \in [a,b]_i\,,\;\; m_i^* - \varepsilon \leq f(x)\}, 1 \leq i \leq k,\; 5$$

(2)

$$U_k \;=\; \overset{k}{\underset{i=1}{\cup}} S_i.$$

We say that the sets S_i cover the interval $[a,b]$, if

$$[a,b] \subset U_k. \tag{6}$$

If the condition (6) is not fulfilled, then the search for minimum is continued on the set $[a,b]\setminus U_k$. The process is finished when the whole interval $[a,b]$ is completely covered. Instead of finding the global minimum on $[a,b]$, one finds the global minimum on subintervals whose union contains $[a,b]$.

3 Sequential Covering Algorithm for Hölder Functions

In this section, we show how to construct a covering of $[a,b]$ for functions satisfying the Hölder condition (2). To construct such a covering, we must know how to construct a minorant of the objective function f on $[a,b]$. Assume that the function f satisfies the condition (2). Hence, we obtain the minorant of f on the interval $[a,b]$:

$$f(y) - h\,|x-y|^{\alpha} \leq\; f(x), \quad \forall x,y \in [a,b] \tag{7}$$

Using this minorant we prove the following theorem:

Theorem 1.1 *Let f satisfy* (2). *We assume that f is evaluated at the points $x_1, x_2, ..., x_k$. We put $m_k^* = \min\limits_{1\leq i\leq k} f(x_i)$ and designate by $(I(x_i, r_{ik}))_{1\leq i\leq k}$ the set of intervals centered at $\{x_i\}_{1\leq i\leq k}$ and with radius r_{ik}such that $r_{ik} = \left(\frac{f(x_i)-m_k^*+\varepsilon}{h}\right)^{1/\alpha}$.*

If the union $\overset{k}{\underset{i=1}{\cup}} I(x_i, r_{ik})$ covers $[a,b]$, *then m_k^* is a global minimum of f on $[a,b]$.*

Proof. First, the inequality (7) yields that the inequality

$$m_k^* - \varepsilon \leq f(x), \tag{8}$$

will be satisfied for all x satisfying

$$m_k^* - \varepsilon \leq \; f(y) - h\,|x - y|^{\alpha}\,. \tag{9}$$

Let x_i belongs to X_k, $i \leq k$. Introduce the intervals $I(x_i, r_{ik})$ such that

$$I(x_i, r_{ik}) = \{x \in \mathbb{R}, \;\; |x - x_i| \leq r_{ik}\}\,. \tag{10}$$

From the inequality (9) and for $y = x_i$, we have

$$|x - x_i| \leq \left(\frac{f(x_i) - m_k^* + \varepsilon}{h}\right)^{1/\alpha}.$$

The center of the interval $I(x_i, r_{ik})$ is the point x_i and the radius is

$$r_{ik} = \left(\frac{f(x_i) - m_k^* + \varepsilon}{h}\right)^{1/\alpha}. \tag{11}$$

From $(7), (10)$ and (11) it follows that (8) holds for all $x \in I(x_i, r_{ik})$.
Now, let $m = \min\limits_{x\in[a,b]} f(x) = f(x_0)$, where $x_0 \in [a, b]$, since $\overset{k}{\underset{i=1}{\cup}} I(x_i, r_{ik})$ contains the interval $[a, b]$, then $x_0 \in \overset{k}{\underset{i=1}{\cup}} I(x_i, r_{ik})$, hence there exists a $1 \leq i_0 \leq k$ such that $x_0 \in I(x_{i_0}, \left(\frac{f(x_{i_0}) - m_k^* + \varepsilon}{h}\right)^{1/\alpha})$. Consequently

$$|x_{i_0} - x_0| \leq \left(\frac{f(x_{i_0}) - m_k^* + \varepsilon}{h}\right)^{1/\alpha},$$

therefore

$$h\,|x_{i_0} - x_0|^{\alpha} \leq f(x_{i_0}) - m_k^* + \varepsilon. \tag{12}$$

From the inequality (12) and, taking into account the Hölder condition of f for all x_{i_0} and x_0 in $[a, b]$, we obtain

$$m_k^* - m \leq \varepsilon.$$

We conclude that m_k^* is a global minimum of f on $[a, b]$.

3.1 SCA Algorithm

At realizing the method, the intervals $(I(x_i, r_{ik}))_{1\leq i\leq k}$ and X_k are sequentially constructed. We suppose that the last computation of f was fulfilled at the point x_k, the record point was x_r and the record m_k^* is determined from (4). If at the new point x_{k+1} we have $f(x_{k+1}) < m_k^*$, we set $m_{k+1}^* = f(x_{k+1})$ and replace the radius r_{ik} by $r_{i(k+1)}$. If the intervals $(I(x_i, r_{i(k+1)}))_{1\leq i\leq k}$ cover the interval $[a,b]$, the computations end; otherwise, we take a new point x_{k+2} and the minimization procedure is continued on $[a,b]\setminus \bigcup_{i=1}^{k} I(x_i, r_{i(k+1)})$. The procedure terminates when the interval $[a,b]$ is completely covered. Since $[a,b]$ is bounded, such a covering is accomplished in a finite number of steps. The interval $[a,b]$ is covered by intervals of different radii (nonuniformity). Consider a point x_i at which $m_i^* = f(x_i)$, $1 \le i \le k$, obviously we have $f(x_i) = m_i^* \ge m_k^*$, hence,

$$r_{ik} = \left(\frac{f(x_i) - m_k^* + \varepsilon}{h}\right)^{1/\alpha} \ge \left(\frac{\varepsilon}{h}\right)^{1/\alpha}.$$

Then the shortest radius is at points x_i at which $m_i^* = f(x_i)$ i.e., $\left(\frac{\varepsilon}{h}\right)^{1/\alpha}$. We then take $x_1 = a + \left(\frac{\varepsilon}{h}\right)^{1/\alpha}$, which is the center of the interval $I(x_1, r_{11})$ and radius $r_{11} = \left(\frac{\varepsilon}{h}\right)^{1/\alpha}$, because at the initialization $f(x_1) = m_1^*$. With this value of x_1, we save time and we are sure not to have ignored the global minimum in a neighbourhood of this point. Indeed, if

$$x^* = \underset{x\in[a,b]}{\arg\min} f(x) \in I(x_1, r_{11}),$$

and, taking into account the Hölder condition of f for x_1, x^* in $[a,b]$, we obtain

$$f(x_1) - f(x^*) \le \varepsilon.$$

For the point x_2 which is the center of the interval $I(x_2, r_{22})$ and radius $r_{22} = \left(\frac{f(x_2)-m_2^*+\varepsilon}{h}\right)^{1/\alpha}$, we shoud take $x_2 = x_1 + r_{11} + r_{22}$. But r_{22} is unknown, and since the shortest radius is $\left(\frac{\varepsilon}{h}\right)^{1/\alpha}$, then we take

$$x_2 = x_1 + r_{11} + \left(\frac{\varepsilon}{h}\right)^{1/\alpha} = x_1 + 2\left(\frac{\varepsilon}{h}\right)^{1/\alpha}.$$

For $k \ge 2$, the sequence $(x_k)_{k\ge1}$ has the form:

$$\begin{cases} x_1 = a + \left(\frac{\varepsilon}{h}\right)^{1/\alpha} \\ x_{k+1} = x_k + \left(\frac{\varepsilon}{h}\right)^{1/\alpha} + \left(\frac{f(x_k)-m_k^*+\varepsilon}{h}\right)^{1/\alpha} \text{ for } k \ge 1. \end{cases}$$

However, the evaluation points are not equidistant and form an iterative sequence. This choice allows us not to miss the global minimum of f on each subintervals of $[a,b]$.

Algorithm SCA
1. Initialization.
Put $k = 1,\ x_1 = a + \left(\frac{\varepsilon}{h}\right)^{1/\alpha},\ x_\varepsilon = x_1,\ m_\varepsilon^* = f(x_\varepsilon)$

2. Steps $k = 2, 3, ...$

Put $x_{k+1} = x_k + \left(\frac{\varepsilon}{h}\right)^{1/\alpha} + \left(\frac{f(x_k) - m_\varepsilon^* + \varepsilon}{h}\right)^{1/\alpha}$.

If $x_{k+1} > b - \left(\frac{\varepsilon}{h}\right)^{1/\alpha}$, then stop.

Otherwise, determine $f(x_{k+1})$.

If $f(x_{k+1}) < m_\varepsilon^*$, then put $x_\varepsilon = x_{k+1}, m_\varepsilon^* = f(x_{k+1})$.

Put $k = k + 1$ go to 2.

4 A Modified Sequential Covering Algorithm

Some modifications have been proposed for the case of Lipschitz functions [5]. In this paper we propose a new modification for the case of Hölder functions. For this, we give the following result.

Theorem 1.2 *Let f be a real Hölder function with constant $h > 0$ and exponent α $(0 < \alpha < 1)$, defined on the interval $[a, b]$ of $\mathbb{R}$. We assume that f is evaluated at the points $x_1, x_2, ..., x_k$. We put $m_k^* = \min\limits_{1 \le i \le k} f(x_i)$ and designate by $(I(x_i, r_{ik}))_{1 \le i \le k}$ the set of intervals centered at $\{x_i\}_{1 \le i \le k}$ and with radius r_{ik}such that* $\mathrm{r}_{ik} = \frac{f(x_i) - m_k^* + \frac{\varepsilon}{2}}{C_{\varepsilon/2}}$, where

$$C_\varepsilon = \begin{cases} \frac{\alpha h^{\frac{1}{\alpha}}}{\varepsilon^{\frac{1-\alpha}{\alpha}}} & \textit{for} \quad \alpha \in \left\{\frac{1}{m}, m \in \mathbb{N}^* \setminus \{1\}\right\} \\ \\ \frac{\alpha h^{\frac{\alpha+1}{\alpha}}}{\varepsilon^{\frac{1}{\alpha}}} & \textit{for} \quad \alpha \in]0, 1[\setminus \left\{\frac{1}{m}, m \in \mathbb{N}^* \setminus \{1\}\right\}. \end{cases}$$

If the union $\bigcup\limits_{i=1}^{k} I(x_i, r_{ik})$ contains $[a, b]$, *then m_k^* is a global minimum of f on* $[a, b]$.

Let us first give the following result:

Theorem 1.3 *Let f be a real Hölder function with constant $h > 0$ and exponent $0 < \alpha < 1$, defined on the interval $[a, b]$ of $\mathbb{R}$. Then there exists a constant $C_\varepsilon > 0$* such that

$$|f(x) - f(y)| \le C_\varepsilon |x - y| + \varepsilon \quad , \quad \forall \varepsilon > 0 \textit{ and } \forall\, x, y \in [a, b]. \tag{13}$$

We give the following lemma:

Lemma 1. *Let $\delta > 0$ and $0 < \alpha < 1$, then there exists a constant $C > 0$ such that:*

$$z^\alpha - Cz - \delta \le 0, \qquad \forall\, z > 0. \tag{14}$$

Proof. Indeed:

1) If $\alpha = \frac{1}{m},\ m \in \mathbb{N}^* \setminus \{1\},$ from the inequality (14), we have

$$z \leq (Cz + \delta)^m .$$

By the Binomial formula, we obtain, for any $C > 0$:

$$(Cz+\delta)^m = {}_{i=0}^{m} \binom{m}{i} C^i z^i \delta^{m-i} = \delta^m + mCz\delta^{m-1} + ... + C^m z^p = mCz\delta^{m-1} + R(z, C, \delta).$$

Where $\binom{m}{i} := \frac{m!}{i!(m-i)!}$, and $R(z, C, \delta) > 0$ is the rest of Binomial formula. Hence,

$$(Cz + \delta)^m \geq mCz\delta^{m-1},$$

and therefore

$$Cz + \delta \geq (mC\delta^{m-1})^{\frac{1}{m}} z^{\frac{1}{m}},$$

then

$$z^{\frac{1}{m}} \leq \frac{C}{(mC\delta^{m-1})^{\frac{1}{m}}} z + \frac{\delta}{(mC\delta^{m-1})^{\frac{1}{m}}}. \tag{15}$$

Taking $C = \frac{\delta^{1-m}}{m}$ we obtain the inequality (14).

2) If $\alpha \neq \frac{1}{m},\ m \in \mathbb{N}^* \setminus \{1\}$, then there exists $p \in \mathbb{N}^*$ such that

$$\frac{1}{p+1} < \alpha < \frac{1}{p},$$

therefore, from the inequality (15),

$$z^\alpha \leq z^{\frac{1}{p+1}} \leq \frac{\delta^{-p}}{p+1} z + \delta \leq \alpha\delta^{-\frac{1}{\alpha}} z + \delta.$$

Hence (14) is obtained with $C = \alpha\delta^{-\frac{1}{\alpha}}$.

Proof of the theorem 1.3

If in the condition (2) and by lemma 1, we put $z = |x - y|$ in the inequality (14), then there exists a constant $C > 0$ such that

$$|x - y|^\alpha \leq C\,|x - y| + \delta, \tag{16}$$

with

$$C = \begin{cases} \alpha\delta^{\frac{\alpha-1}{\alpha}} & \text{for } \alpha \in \left\{\frac{1}{m}, m \in \mathbb{N}^* \setminus \{1\}\right\} \\ \alpha\delta^{-\frac{1}{\alpha}} & \text{for } \alpha \in \left]0, 1\right[\setminus \left\{\frac{1}{m}, m \in \mathbb{N}^* \setminus \{1\}\right\}. \end{cases}$$

Setting $h\delta = \varepsilon$, and since f is hölderian, we deduce the theorem 1.3 from (2) and (16).

Proof of the theorem 1.2

By the inequality (13) we have $\forall \varepsilon > 0$ and $\forall\ x, y \in [a, b]$, there exists a constant $C_{\varepsilon/2} > 0$, such that

$$f(y) - C_{\varepsilon/2}\,|x-y| - \frac{\varepsilon}{2} \leq f(x).$$

With the modification which we have proposed above, and for a fixed value $y \in [a, b]$, if a certain x satisfy

$$m_k^* - \varepsilon \leq\ f(y) - C_{\varepsilon/2}\,|x-y| - \frac{\varepsilon}{2}, \tag{17}$$

then $m_k^* - \varepsilon \leq f(x)$. If we take $y = x_i$, and by the inequality (17) we have

$$|x - x_i| \leq \frac{f(x_i) - m_k^* + \frac{\varepsilon}{2}}{C_{\varepsilon/2}}.$$

Hence, we can cover the interval $[a, b]$ by the family of the intervals

$$I(x_i, r_{ik}) = \left\{ x \in \mathbb{R},\ \ |x - x_i| \leq \frac{f(x_i) - m_k^* + \frac{\varepsilon}{2}}{C_{\varepsilon/2}} \right\};$$

and the constant C_ε is given explicitly by theorem 1.2.

Algorithm MSCA$_1$
1. Initialization.
Put $k = 1,\ x_1 = a + \frac{\varepsilon}{2C_{\varepsilon/2}},\ x_\varepsilon = x_1,\ m_\varepsilon^* = f(x_\varepsilon)$
2. Step $k = 2, 3, ...$
Put $x_{k+1} = x_k + \frac{f(x_i) - m_\varepsilon^* + \varepsilon}{C_{\varepsilon/2}}$
If $x_{k+1} > b$ then stop.
Otherwise, determine $f(x_{k+1})$.
If $f(x_{k+1}) < m_\varepsilon^*$, then put $x_\varepsilon = x_{k+1}, m_\varepsilon^* = f(x_{k+1})$
Put $k = k + 1$ go to step 2.

4.1 Remark

If a radius r'_{ik} is such that $r'_{ik} > r_{ik}$, then the modified nonuniform covering algorithm MSCA$_2$ that we suggest will be quickened a little. However, it is interesting to look for other constants C'_ε smaller than C_ε to show that $r'_{ik} > r_{ik}$ and in this case, the covering by the intervals $I(x_i, r'_{ik})$ becomes quicher than that of the intervals $I(x_i, r_{ik})$. We shall numerically establish this result. The following lemma is needed for our purpose in proving.

Lemma 2. *Let* $h, \varepsilon > 0$ *and* $0 < \alpha < 1$. *We consider the function* $g_\alpha :]0, \infty[\to \mathbb{R}$ *defined by*

$$g_\alpha(z) = \frac{hz^\alpha - \varepsilon}{z}.$$

Then

$g_{\alpha\max} = g_\alpha(\left(\frac{\varepsilon}{h(1-\alpha)}\right)^{1/\alpha}) = \frac{\alpha\varepsilon}{1-\alpha}\left(\frac{h}{\varepsilon}(1-\alpha)\right)^{1/\alpha}$.

Proof. We observe that the unique solution of the equation

$$g_\alpha'(z) = 0,$$

is $z_0 = \left(\frac{\varepsilon}{h(1-\alpha)}\right)^{1/\alpha} \in \left]0, \infty\right[$. The function $g_\alpha(z)$ is increasing on $]0, z_0[$ and decreasing on $]z_0, \infty[$. Thus, the global maximum for $g_\alpha(z)$ is attained at z_0. From the inequality (13) we deduce:

$$C_\varepsilon' \geq \frac{|f(x) - f(y)| - \varepsilon}{|x-y|}\ , \quad \forall \varepsilon > 0 \text{ and } \forall\ x, y \in [a, b]\,.$$

And since f is hölderian on $[a, b]$ then we have

$$\frac{|f(x) - f(y)| - \varepsilon}{|x-y|} \leq g_\alpha(|x-y|),\ \forall \varepsilon > 0 \text{ and } \forall\ x, y \in [a, b]\,.$$

Therefore, we can take

$$C_\varepsilon' = \sup_{x,y\in[a,b]} g_\alpha(|x-y|) \geq \sup_{x,y\in[a,b]} \frac{|f(x) - f(y)| - \varepsilon}{|x-y|}\ .$$

And with the lemma 2 we have

$$C_\varepsilon' = \sup_{x,y\in[a,b]} g_\alpha(|x-y|),$$

hense

$$C_\varepsilon' = \frac{\alpha\varepsilon}{1-\alpha}\left(\frac{h}{\varepsilon}(1-\alpha)\right)^{1/\alpha}.$$

We can now show that $C_\varepsilon' < C_\varepsilon$ for all $h, \varepsilon > 0$ and $0 < \alpha < 1$.

5 Numerical Experiments

This section reports numerical results of the algorithm SCA and the proposed algorithms MSCA_1 and MSCA_2 with the Hölder condition (2) in solving some test problems. The following problems are found in literature. In order to find the global minimum of the following examples, we use all of the above algorithms which have been implemented in MATLAB and the experiments have been executed at a PC with Intel Core Duo 1.6G and 1G RAM.

Problem 1. Consider the following global minimization problem [20] :

$$\min_{x\in[0,1]} f_1(x) = -\cos x \exp\left(1 - \frac{\sqrt{|\sin \pi x - 0.5|}}{\pi}\right).$$

$h = 4.3, \alpha = 1/2$

Problem 2. Consider the following global minimization problem [18] :

$$\min_{x\in[0,10]} f_2(x) = \sum_{k=1}^{5} k\left|\sin\left((3k+1)\,x + k\right)\right| |x - k|^{\frac{1}{5}}.$$

$h = 77, \alpha = 1/5$

Problem 3. Consider the following global minimization problem [8] :

$$\min_{x\in[-0.5,0.5]} f_3(x) = |x + 0.25|^{\frac{2}{3}} - 3\cos\tfrac{x}{2}.$$

$h = 4.26, \alpha = 2/3$

All these test functions are neither differentiable nor Lipschitz on their feasible domains but only hölderian.

Table. Numerical tests

	Tol(ε)	SCA	MSCA_1	MSCA_2
f_1	0.1	x^* =0.167117	0.167117	0.167117
		$f(x^*)$ =-2.650706	-2.650706	-2.650706
		$\boldsymbol{n}_{ep}$ =191	259	185
f_2	0.1	x^* =7.388878	2.8186e-009	7.385764
		$f(x^*)$ =14.067589	14.0676	14.067532
		$\boldsymbol{n}_{ep}$ =100000	100000	100000
f_3	0.1	x^* =-0.251838	-0.252227	-0.251957
		$f(x^*)$ =-2.961241	-2.959120	-0.952687
		$\boldsymbol{n}_{ep}$ =56	64	51

$\boldsymbol{n}_{ep}$:Number of evaluation points.

6 Solving Systems of Nonlinear Equations

Many applied problems are reduced to solving systems of nonlinear equations [3], [13], [21], which is one of the most difficult numerical computation problems in mathematics. This task has applications in scientific fields such as physics [16], chemistry [14], economics [1] etc. There are several methods proposed in the literature to tackle this problem, however a complete solution has not yet been achieved. Newton and quasi-Newton type methods and their modifications [19] are traditional optimization-based methods for solving systems of nonlinear equations. But these methods need the differentiability assumption for the functions of the system. In this work, we propose a modified algorithm MSCA to solve systems of nonlinear equations formulated as an optimization problem.

Let us consider the problem of computing at least one solution of nonlinear and nondifferentiable systems with simple bound constraints. We can express this problems as

$$\begin{cases} f_1(x) = 0 \\ f_2(x) = 0 \\ \quad\vdots \\ f_n(x) = 0 \end{cases} \tag{S}$$

Let $[a, b] \subseteq \mathbb{R}$ be the interval where there exists one or more roots of the nonlinear system (S). Let us suppose that the function $f_i : \mathbb{R} \to \mathbb{R}$, for all $i = 1, ..., n$, can be nondifferentiable, but it must be bounded in $[a, b]$. We shall suppose the Hölder continuity of all functions f_i on $[a, b]$, i.e.,

$$|f_i(x) - f_i(y)| \leq h_i |x - y|^{\alpha_i}, \quad \forall x, y \in [a, b],$$

where $h_i > 0$ and $0 < \alpha_i < 1$, for each $i = 1, ..., n$, are a priori known .
We first convert the nonlinear system (S) into an equivalent global optimization problem, and then apply the modified algorithm MSCA to the problem. Let us introduce the notation

$$H(x) = (f_1(x), f_2(x), ..., f_n(x))^T,$$

(^{T} denotes transpose of the row vector $(f_1(x), f_2(x), ..., f_n(x))$. Then (S) can be written as

$$H(x) = 0, \ x \in [a, b].$$

To solve this problem approximately it suffices to find at least one point x^* from the set

$$[a, b]_\varepsilon = \{x \in [a, b] : \|H(x)\|_2 \leq \varepsilon\},$$

where $\|.\|_2$ is the Euclidean norm in $\mathbb{R}^n$.
The typical methods for solving (S) are global optimization methods in which the system (S) can be reformulated as the following global optimization problem (OP) :

$$\min_{x \in [a,b]} f(x) := \|H(x)\|_2 . \tag{OP}$$

In (OP), $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is a nonnegative, nonsmooth and possibly multimodal function. Since the system represented by (S) has solution(s) in $[a, b]$, then, in terms of results, to solve this system is equivalent to find a global minimum(a) of the optimization problem given by (OP).
The following proposition, needed in order to justify the approach taken, shows the Hölder property of f.

Proposition 1. *If the functions $f_i, i = 1, ..., n$, are Hölder continuous with the parameters $h_i > 0$ and $0 < \alpha_i < 1$ on $[a, b]$, then f is also Hölderian on $[a, b]$, with the parameters $h = (\sum_{i=1}^{n} h_i^2)^{1/2}$ and $\alpha = \min_{1 \leq i \leq n} \alpha_i$.*

Proof If we use the well known triangle inequality

$$\left|\|a\|_2 - \|b\|_2\right| \leq \|a-b\|_2 ,$$

and from the Hölder property of the functions f_i, we obtain for all pairs $x_1, x_2 \in [a,b]$, the chain of relations

$$|f(x_1) - f(x_2)| = \left|\|H(x_1)\|_2 - \|H(x_2)\|_2\right| \leq \|H(x_1) - H(x_2)\|_2$$

$$\leq \left(\sum_{i=1}^{n} h_i^2 \left|x_1 - x_2\right|^{2\alpha_i}\right)^{1/2} \leq (\sum_{i=1}^{n} h_i^2)^{1/2}. \left|x_1 - x_2\right|^{\min_i \alpha_i} .$$

This shows that f defines a Hölder function on $[a,b]$.

Proposition 2. *x^* is a solution of system (S) if and only if*

$$0 = f_* = \|H(x^*)\|_2 = \min\left\{\|H(x)\|_2 , x \in [a,b]\right\}.$$

Proof By the definition, $f(x) \geq 0$. Thus, for the global minimizer x^* of $f(x)$ it holds $f(x^*) \geq 0$. If there exists $x^* \in [a,b]$ such that $f(x^*) = 0$, then it implies that x^* is a global minimizer and subsequently $f_1(x^*) = f_2(x^*) = ... = f_n(x^*) = 0$ and then x^* is a root for the corresponding system of nonlinear equations. The proof is an immediate consequence of the norm property, i.e., $\|z\|_2 \geq 0$, $\forall z$ and $\|z\|_2 = 0 \Leftrightarrow z = 0$.

By virtue of the proposition 2, the optimization problem contains all of the information on (S). We see that $\min\limits_{x\in[a,b]} f(x) > 0$ holds if and only if the system (S) has no solution, and in the case $\min\limits_{x\in[a,b]} f(x) = 0$ the set of solutions of the problem (OP) coincides with the set of solutions of (S).

Example The following example is proposed by the author. Consider the system of nonlinear equations:

$$\begin{cases} f_1(x) = \sqrt{\frac{9}{4} - x^2} - \frac{\sqrt{5}}{2} = 0 \\ f_2(x) = \left|\sin(\frac{\pi}{2}x)\right| \left|\frac{\sqrt{2}-x}{\sqrt{2}-1}\right|^{\frac{1}{3}} - x = 0 \\ f_3(x) = -\cos(x + \frac{\pi}{2} - 1)e^{1-\frac{\sqrt{\left|\sin\pi(x+\frac{\pi}{2}-1)-\frac{1}{2}\right|}}{\pi}} = 0. \end{cases} \qquad (S')$$

The functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ of the system (S') are hölderian on the interval $[-1.5, 1.5]$ with respectively the constants
$(h_1 = \sqrt{3},\ \alpha_1 = 1/2)$, $(h_2 = 5.4,\ \alpha_2 = 1/3)$ et $(h_3 = 7.3,\ \alpha_3 = 1/2)$.

The system (S') is equivalent to the following global minimization problem:

$$\min_{x\in[-1.5,1.5]} f(x),$$

where $f(x) = \|H(x)\|_2 = \sqrt{(f_1(x))^2 + (f_2(x))^2 + (f_3(x))^2}$, is a Hölder function with the constant $h = \sqrt{h_1^2 + h_2^2 + h_3^2} = 9.2439$ and $\alpha = \min(\alpha_1, \alpha_2, \alpha_3) = 1/3$. Applying the modified nonuniform covering algorithm we obtain for $\epsilon = 0.1$, the solution of (S'), by $x = 1.0000062$.

7 Conclusion

The global optimization of Hölder functions has already been studied. In our work we bring an ordered sequential covering algorithm for univariate Hölder functions without using any assumptions of differentiability or convexity of the objective function, but only the knowledge of the parameters h and α. The method thus obtained is simple, efficient and its convergence is proved. We have applied the modified algorithm for solving nonsmooth system of nonlinear equations as an equivalent converted global optimization problem. Numerical examples are given and show the effectiveness of our proposed algorithm. In the future, the author will focus on the applicability of the algorithm MSCA for the case of Hölder functions with several variables defined on the hyper-rectangle $[a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]$ of $\mathbb{R}^n$.

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