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Connection Properties in FN-Spaces

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Abstract

In this paper we have studied the connection properties in fuzzy nearness spaces. We have defined the fuzzy uniform local uniform connectedness and fuzzy uniform local connectedness of fuzzy nearness spaces. We have shown that a fuzzy nearness space is fuzzy uniformly locally uniformly connected iff its completion is. Also every topological fuzzy locally connected fuzzy nearness space is fuzzy uniformly locally uniformly connected. We also establish the result for a fuzzy topological space X that X has a fuzzy locally connected regular T_1- extension iff X is the underlying fuzzy topological space of a fuzzy nearness space Y which is concrete, regular and fuzzy uniformly locally uniformly connected. Lastly we achieve the equivalence of the concepts of fuzzy uniform local uniform connectedness, fuzzy uniform local connectedness and fuzzy local connectedness.

Keywords: *Fuzzy uniformly locally connected space, Fuzzy uniformly locally uniformly connected space.*

1 Preliminaries

In [12], Samanta S.K. has introduced the concept of fuzzy nearness spaces (FNS) using near collections as well as using fuzzy uniform covers. He has shown the equivalence of both the concepts. Using fuzzy uniform covers, a FNS is axiomatized as below.

Definition 1.1. [12] *Let X be any set. A collection* $\mu\ X\ \mathcal{P}\,(I^X)$ *is called a fuzzy covering structure of X if*

(FU1) ($\mathcal{A}\quad \mathcal{B}$ and $\mathcal{A} \in \mu$) $\Psi\ \mathcal{B} \in \mu$,

(FU2) $\mathcal{A} \in \mu\ \Psi$ for each $x \in X$, there is a pair $A_1, A_2 \in \mathcal{A}$ such that $A_1(x) + A_2(x) J 1$,

(FU3) $\phi\ f\ \mu\ f\ \mathcal{P}(I^X)$,

(FU4) $\mathcal{A}, \mathcal{B} \in \mu\ \ \Psi\ \mathcal{A}\, \varpi\, \mathcal{B} \in \mu$,

(FU5) $\mathcal{A} \in \mu\ \ \Psi\ \{\mathrm{int}_\mu\ A \mid A \in \mathcal{A}\} \in \mu$, where

$$\mathrm{int}_\mu\ A = \underline{1} - \inf\{\sigma \in \mathrm{Pt}\,(I^X) \mid \{\sigma, A\} \in \mu\}.$$

The space (X, μ) is called a **fuzzy nearness space(FNS).**
The members of μ are called **fuzzy uniform covers**.

Definition 1.2. [12] *If* (X, μ) *and* (Y, η) *are fuzzy nearness spaces then the fuzzy map* f^{ϖ} $: (X, \mu)\ \varpi\ (Y, \eta)$ *is called* ***fuzzy uniformly continuous*** *if for each fuzzy uniform cover* $\mathcal{A}$ *of Y,* $f^{\varpi}(\mathcal{A}) = \{\, f^{\varpi}(A) \mid A \in \mathcal{A}\}$ *is a fuzzy uniform cover of X.*

Definition 1.3. [12] *A FNS* (X, μ) *is called a* ***topological FNS*** *if it satisfies* $\omega\{\mathrm{int}_\mu\ A \mid A \in \mathcal{A}\} = \underline{1} \Leftrightarrow \mathcal{A} \in \mu$.

2 Fuzzy Uniformly Connected FNS

In [7], Kulkarni S. D. and Nimse S. B. have defined fuzzy uniformly connected FNS.

Definition 2.1. [7] *A FNS* (X, μ) *is said to be* ***Fuzzy uniformly connected*** *if every uniformly continuous fuzzy map from* I^X *to a discrete FNS is constant.*

Definition 2.2. [7] *The fuzzy subspace A of X is called* ***Fuzzy uniformly connected*** *provided* (A, μ_A) *is fuzzy uniformly connected where* $\mu_A = \{\mathcal{U}\ \varpi \{A\} \mid \mathcal{U} \in \mu\}$.

Some of the important results regarding uniform connectedness are quoted here.

Remark 2.3. *A connected FNS is always fuzzy uniformly connected but not vice versa.*

Result 2.4. A FNS is fuzzy uniformly connected iff it is fuzzy well chained.

Result 2.5. The fuzzy uniformly continuous image of a fuzzy uniformly connected FN-space is fuzzy uniformly connected.

Result 2.6. A topological FNS (X, μ) is fuzzy uniformly connected iff X is a connected fuzzy topological space.

Result 2.7. If (X, μ) is a FNS such that its underlying fuzzy topological space is connected then (X, μ) is fuzzy uniformly connected but the converse does not hold.

Result 2.8. Let (X, μ) be a FNS. Let A, G ∈ I^X such that cl A = $\underline{1}$ and G is open. If G is a fuzzy uniformly connected subspace of (X, μ) then G ϖ A is fuzzy uniformly connected.

3 Fuzzy Uniformly Locally Uniformly Connected FNS

In this section we study the properties of fuzzy uniformly locally uniformly connected FNS.

Definition 3.1. *A FNS (X, m) is said to be* ***Fuzzy uniformly locally uniformly connected*** *iff each fuzzy uniform cover $\mathcal{U}$ of (X, m) is refined by a fuzzy uniform cover $\mathcal{V}$ of (X, m) each member of which is fuzzy uniformly connected as a fuzzy subspace.*

Proposition 3.2. *Let (X, m) be a fuzzy uniformly locally uniformly connected FNS and let G ∈ I^X be a fuzzy subspace which is an open fuzzy subset of the underlying topological space. Then each fuzzy uniformly connected component of G is open in I^X.*

Proof. Let B ∈ I^X be a fuzzy uniformly connected component of G. Let ρ i B such that ρ ∈ Pt (I^X). Then {G, ρ^c}∈ m. Then there exists $\mathcal{U}$ ∈ m which refines {G, ρ^c} and each member of $\mathcal{U}$ is fuzzy uniformly connected. Then there is a V ∈ $\mathcal{U}$ with ρ i int_X V.
L V i G . But V is fuzzy uniformly connected. Hence V i B . Thus ρ ∈ int_X B.
L B is open in I^X. ◊

Proposition 3.3. *A FNS (X, m) is fuzzy uniformly locally uniformly connected iff each fuzzy uniform cover $\mathcal{U}$ of (X, m) has a refinement $\mathcal{V}$ which is an open fuzzy uniform cover with each member of $\mathcal{V}$ being fuzzy uniformly connected.*

Proof. Assume that (X, m) is fuzzy uniformly locally uniformly connected. If $\mathcal{U}$ ∈ m then int_X $\mathcal{U}$ ∈ m . Then by hypothesis, int_X $\mathcal{U}$ is refined by $\mathcal{V}$ ∈ m, each member of $\mathcal{V}$ is fuzzy uniformly connected. Now for each V ∈ $\mathcal{V}$, choose U ∈ $\mathcal{U}$ with V i int_X U_V . Let C_V denote the fuzzy uniformly connected component of int_X U_V containing V then {C_V | V ∈ $\mathcal{V}$} is a fuzzy uniform cover of (X, m). The converse follows. ◊

Proposition 3.4. *Let AXX. Let (A, m_A) be a dense fuzzy subspace of a fuzzy uniformly locally uniformly connected FNS (X, m). Then (A, m_A) is fuzzy uniformly locally uniformly connected.*

Proof. Assume that (X, m) is fuzzy uniformly locally uniformly connected FNS.
Let $\mathcal{U} \in m_A$. Then there exists a fuzzy uniform cover $\mathcal{C}$ of (X, m) such that $\mathcal{U} = \{\underline{1}_A\} \varpi\ \mathcal{C}$. Since (X, m) is fuzzy uniformly locally uniformly connected, there exists a fuzzy open uniform cover $\mathcal{G} \in$ m which refines $\mathcal{C}$ such that each member of $\mathcal{G}$ is fuzzy uniformly connected. Then $\mathcal{V} = \{\underline{1}_A\} \varpi\ \mathcal{G} \in$ m refines $\mathcal{U}$ where $\mathcal{V}$ contains fuzzy uniformly connected members. Hence (A, m_A) is fuzzy uniformly locally uniformly connected. ◊

Lemma 3.5. *Let AXX. Let (X, m) be a FNS with (X*, m*) as its completion. For an open fuzzy subspace (A, m_A), the FN-space (A, m_A) is fuzzy uniformly connected iff (opA, m_{opA}) is fuzzy uniformly connected.*

[Here op A = $\underline{1}_{X*}$ - cl_{X*} (A^c)]

Proof. Since op A = $\underline{1}_{X*}$ - cl_{X*} (A^c), clearly A=$\underline{1}_X\ \varpi$ opA. If opA is fuzzy uniformly connected then by result 2.8 , A is fuzzy uniformly connected. Conversely, Suppose (A, m_A) is fuzzy uniformly connected. We know that A i op A i cl_{X*} A. So A is dense in opA. Hence opA is fuzzy uniformly connected. ◊

Proposition 3.6. *A FNS is fuzzy uniformly locally uniformly connected iff its completion is fuzzy uniformly locally uniformly connected.*

Proof. Assume that a FNS (X, m) is fuzzy uniformly locally uniformly connected. Consider the completion (X^*, m^*) of (X, m). Let $\mathcal{U} \in m^*$. Then there exists a fuzzy open uniform cover $\mathcal{G} \in$ m such that op $\mathcal{G}$ refines $\mathcal{U}$. For some fuzzy open uniform cover $\mathcal{V} \in$ m, $\mathcal{V}$ refines $\mathcal{G}$ and each member of $\mathcal{V}$ is fuzzy uniformly connected. Therefore op$\mathcal{V}$ refines $\mathcal{U}$ and op $\mathcal{V} \in m^*$ and each member of op $\mathcal{V}$ is fuzzy uniformly connected. Thus (X^*, m^*) is fuzzy uniformly locally uniformly connected. Conversely, suppose the completion (X^*, m^*) is fuzzy uniformly locally uniformly connected FNS. Then (X, m) being fuzzy dense subspace of (X^*, m^*), by proposition 3.4, (X, m) is fuzzy uniformly locally uniformly connected FNS. ◊

Proposition 3.7. *If a topological FNS is fuzzy locally connected then it is fuzzy uniformly locally uniformly connected.*

Proof. Let (X, m) be a fuzzy locally connected topological FNS. Let $\mathcal{U} \in$ m be open.
If ρ i U for U ∈ $\mathcal{U}$, then there exists a quasi- neighbourhood V of ρ with V i U. Then (V, m_V) is a connected fuzzy subspace of (X, m) in topological sense. So (V, m_V) is a

connected FN-subspace and hence fuzzy uniformly connected. Clearly there exists a fuzzy uniform cover $\mathcal{V}$ of (X, m) which refines $\mathcal{U}$ and every V $\in \mathcal{V}$ is fuzzy uniformly connected. Hence (X, m) is fuzzy uniformly locally uniformly connected. ◊

Proposition 3.8. *For a fuzzy topological space X, the following conditions are equivalent:*

1) X has a fuzzy locally connected, regular T_1 extension.

2) X is the underlying fuzzy topological space of a fuzzy nearness space Y which is concrete, regular and fuzzy uniformly locally uniformly connected.

Proof. Let (X, δ) be a fuzzy topological space. Then (X, m_δ) is a topological FNS.

To Prove **(1) ⇒ (2)**:

Suppose X has a fuzzy locally connected regular T_1 extension Y. Then (X, m_δ) is a fuzzy uniformly locally uniformly connected by proposition 3.7. Then X is dense in Y. Hence Y is fuzzy uniformly locally uniformly connected.

To Prove **(2) ⇒ (1)**:

Suppose X is the underlying fuzzy topological space of a FNS Y which is fuzzy concrete, regular and fuzzy uniformly locally uniformly connected. Consider the completion Y^* of Y. Then Y^* is fuzzy topological space. We will show that Y^* is the required fuzzy locally connected regular T_1 extension. Let $\mathcal{G}$ be a fuzzy open cover of Y^*. But Y^* is fuzzy regular. Hence there exists a fuzzy open cover $\mathcal{D}$ of Y^* such that $cl_{Y*}\ \mathcal{D} = \{cl_{Y*} D \mid D \in \mathcal{D}\}$ refines $\mathcal{G}$. As $\mathcal{D}$ is a fuzzy uniform cover of Y^*, there exists a fuzzy uniform cover $\mathcal{U}$ of Y such that op $\mathcal{U}$ refines $\mathcal{D}$. But Y is fuzzy uniformly locally uniformly connected. So there exists a fuzzy uniform cover $\mathcal{U}$ of Y which refines $\mathcal{U}$ such that each V $\in \mathcal{V}$ is fuzzy uniformly connected. Then $cl_{Y*}\mathcal{V}$ is a fuzzy uniform cover of Y^* which refines $\mathcal{G}$. For each V $\in \mathcal{V}$, $cl_{Y*}V$ is fuzzy uniformly connected since V is fuzzy uniformly connected and $cl_{Y*}V$ is topological (Since Y^* is topological). Hence $cl_{Y*}V$ is connected. Hence Y^* is fuzzy locally connected regular T_1 extension of X. ◊

4 Fuzzy Uniformly Locally Connected FN-Spaces

We define fuzzy uniform local connectedness of a FN-space as below.

Definition 4.1. *A FNS (X, m) is called* ***Fuzzy uniformly locally connected*** *if each* $\mathcal{U} \in m$ *is refined by* $\mathcal{V} \in m$ *such that for each* V $\in \mathcal{V}$*, the underlying fuzzy topological space of V is connected.*

Proposition 4.2. *Every fuzzy uniformly locally connected FNS is fuzzy uniformly locally uniformly connected.*

Proof. Let (X, m) is a fuzzy uniformly locally connected FNS. Then each $\mathcal{U} \in$ m is refined by $\mathcal{V} \in$ m such that the underlying topological space of each $V \in \mathcal{V}$ is connected. Then each (V, m_V) is F-uniformly connected as a subspace of (X, m). Thus $\mathcal{U}$ is refined by $\mathcal{V}$, each member of which is a F-uniformly connected subspace of (X, m). Hence (X, m) is fuzzy uniformly locally uniformly connected. ◊

Proposition 4.3. *If a FNS (X, m) is fuzzy uniformly locally connected then its underlying fuzzy topological space is locally connected.*

Proof. Let (X, m) be a fuzzy uniformly locally connected FNS. Since $\mathrm{int}_m \underline{1}_X = \underline{1}_X$, then $\underline{1}_X$ is a fuzzy open cover and is refined by it self. Hence $\underline{1}_X$ is locally connected. Hence the underlying fuzzy topological space is locally connected. ◊

Remark 4.4. *Consider the usual fuzzy uniformity on* $I^{Q\,[I]}$. *Then it is fuzzy uniformly locally uniformly connected but not uniformly locally connected.*

Proposition 4.5. *If a regular topological FNS is fuzzy uniformly locally uniformly connected then it is fuzzy uniformly locally connected.*

Proof. Suppose (X, m) is fuzzy regular, topological FNS which is fuzzy uniformly locally uniformly connected. Let $\mathcal{U} \in$ m. Then by the regularity, there exists $\mathcal{H} \in$ m such that $\mathrm{cl}_X \mathcal{H}$ refines $\mathcal{U}$. There exists $\mathcal{V} \in$ m such that $\mathcal{V}$ refines $\mathcal{H}$ and each member of $\mathcal{V}$ is fuzzy uniformly connected. Then $\mathrm{cl}_X \mathcal{V} \in$ m which refines $\mathcal{U}$. Further for each $V \in \mathcal{V}$, $\mathrm{cl}_X V$ is fuzzy uniformly connected. Since X is topological FNS, $\mathrm{cl}_X V$ is also topological and hence is fuzzy connected. So (X, m) is fuzzy uniformly locally connected. ◊

Corollary 4.6. *For a regular topological FNS (X, m), the following are equivalent:*

1) (X, m) is fuzzy uniformly locally uniformly connected.
2) (X, m) is fuzzy uniformly locally connected.
3) (X, m) is fuzzy locally connected.

Proof. From Proposition 4.5, (1) $\Rightarrow$ (2). From Proposition 4.3, (2) $\Rightarrow$ (3). From Proposition 3.7, (3) $\Rightarrow$ (1). ◊

Proposition 4.7. *A FNS (X, m) is fuzzy uniformly locally connected iff each* $\mathcal{U} \in m$ *is refined by an open* $\mathcal{V} \in m$, *each member of which is connected as a fuzzy topological subspace of the underlying topological space of (X, m).*

Proof. Suppose (X, m) is fuzzy uniformly locally connected FNS. Let $\mathcal{U} \in$ m. Then

$\text{int}_X\, \mathcal{U} = \{\text{int}_X\, U \mid U \in \mathcal{U}\} \in m$. Then $\text{int}_X\, \mathcal{U}$ is refined by $\mathcal{A} \in m$ such that for each $A \in \mathcal{A}$, the underlying fuzzy topological space of A is fuzzy connected. For each $A \in \mathcal{A}$, choose $U_A \in \mathcal{U}$ such that A i $\text{int}_X\, U_A$ and let V_A denote the fuzzy component of $\text{int}_X\, U_A$ such that A i V_A. As the underlying fuzzy topological space of X is fuzzy locally connected and $\text{int}_X\, U_A$ is open. Hence V_A is open in the underlying fuzzy topological space of X. L $\mathcal{V} = \{V_A \mid A \in \mathcal{A}\}$ is the required open fuzzy uniform cover of X as it is refined by $\mathcal{A}$. Converse follows. ◊ ◊

5 Conclusion

We conclude that a FN-space is fuzzy uniformly locally uniformly connected iff its completion is. Also every topological fuzzy locally connected FN-space is fuzzy uniformly locally uniformly connected.
We also established the result for a fuzzy topological space X that X has a fuzzy locally connected regular T_1- extension iff X is the underlying fuzzy topological space of a FN-space Y which is concrete, regular and fuzzy uniformly locally uniformly connected.
Lastly we achieved the equivalence of the concepts of fuzzy uniform local uniform connectedness, fuzzy uniform local connectedness and fuzzy local connectedness.

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