Georgian Mathematical Journal
Volume 9 (2002), Number 4, 709–721

THE WHITEHEAD CATEGORICAL GROUP OF DERIVATIONS

A. R. GARZÓN AND A. DEL RÍO

Dedicated to H. Inassaridze on his 70th birthday

Abstract. Given a categorical crossed module $\mathbb{H} \to G$, where G is a group, we show that the category of derivations, $Der(G, \mathbb{H})$, from G into $\mathbb{H}$ has a natural monoidal structure. We introduce the Whitehead categorical group of derivations as the Picard category of $Der(G, \mathbb{H})$ and then we characterize the invertible derivations, with respect to the tensor product, in this monoidal category.

2000 Mathematics Subject Classification: 18D10, 20J05.
Key words and phrases: Derivation, monoidal category, categorical group, crossed module.

1. Introduction

If G is a group and $G \to \mathrm{Aut}(H)$ is a group homomorphism, that is, H is a G-group, there is no natural structure on the set $\mathrm{Der}(G, H)$ of derivations from G into H; however, in the course of investigations of the properties of relative homotopy groups, J. H. C. Whitehead [20] showed that if H is a G-crossed module, then $\mathrm{Der}(G, H)$ has a natural monoid structure, and he further characterized the units in $\mathrm{Der}(G, H)$. The resulting group $D(G, H)$ is called the Whitehead group of regular derivations. These results have then allowed the application of derivation groups to the variety of situations concerned with the notion of crossed module. Such an application, to holomorphs of groups, was given by A. S. T. Lue [12]; moreover, derivation groups appear, in the automorphism structures for crossed modules studied by K. Norrie [14] and R. Brown and N. D. Gilbert [2], and also in the development by N. D. Gilbert [9] of Whitehead's ideas by using the equivalence between crossed modules and internal groups in the category of groupoids.

The object of this paper is to introduce the Whitehead "categorical group" of regular derivations in order to obtain analogous applications at the level of categorical groups.

Categorical groups are monoidal groupoids in which each object is invertible, up to isomorphism, with respect to the tensor product [1, 10, 16, 17] and they have been widely used in various fields such as ring theory, group cohomology and algebraic topology [3, 4, 6, 8, 11, 15, 17, 19].

In this setting of categorical groups, the notion of crossed module was introduced by L. Breen [1] and it has been used recently by P. Carrasco and J. Martínez [5] in order to study 2-fold extensions of a group G by a symmetric

ISSN 1072-947X / $8.00 /

G-categorical group $\mathbb{A}$. Of course, this notion of crossed module recovers the classic one for groups when groups are seen as discrete categorical groups.

Derivations of $\mathbb{G}$ into $\mathbb{H}$, when $\mathbb{G}$ is a categorical group acting on a braided categorical group $\mathbb{H}$, have been studied in [7], where it is shown how the braiding in $\mathbb{H}$ allows the definition of a categorical group structure in the category of derivations $\mathrm{Der}(\mathbb{G},\mathbb{H})$. However, if $\mathbb{H}$ is not braided, there is no natural structure on the category $\mathrm{Der}(\mathbb{G},\mathbb{H})$.

Now, for any group G, we consider categorical crossed modules of the form $\mathbb{H} \to G[0]$, where $G[0]$ denotes the discrete categorical group associated to G, which we call categorical G-crossed modules. Then we analyze the natural monoidal structure, in the category of derivations $\mathrm{Der}(G,\mathbb{H})$, owing to the G-crossed module structure in the categorical group $\mathbb{H}$. For any monoidal category $\mathscr{C}$, there is an associated categorical group $\mathscr{P}(\mathscr{C})$ [19], called the Picard categorical group of $\mathscr{C}$, consisting of invertible objects in $\mathscr{C}$ and isomorphisms between them. Then we introduce the Whitehead categorical group of derivations $D(G,\mathbb{H})$ exactly as the Picard categorical group of the monoidal category $\mathrm{Der}(G,\mathbb{H})$ and, finally, we characterize the objects of $D(G,\mathbb{H})$ in terms of the endomorphisms, of G and $\mathbb{H}$, associated to any derivation from $G[0]$ into $\mathbb{H}$.

2. Preliminaries

Let us recall [13] that a *monoidal category* $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$ consists of a category $\mathbb{G}$, a functor (tensor product) $\otimes : \mathbb{G}\times\mathbb{G} \to \mathbb{G}$, an object I of $\mathbb{G}$, called the unit object, and natural isomorphisms called, respectively, the associativity, left-unit and right-unit constraints

$$a = a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$
$$l = l_X : I \otimes X \to X,\ r = r_X : X \otimes I \to X,$$

such that the usual coherence conditions are satisfied. In a monoidal category, an object X is said to be 2-*regular*, or *invertible*, if the functors $Y \mapsto X \otimes Y$ and $Y \mapsto Y \otimes X$ are equivalences. A *categorical group* $\mathbb{G}$ is a monoidal small category where every arrow is invertible and every object is 2-regular. In this case, it is possible to define a contravariant functor $(-)^* : \mathbb{G} \to \mathbb{G}$, $X \mapsto X^*$, $f \mapsto f^*$, and natural isomorphisms

$$\gamma_X : X \otimes X^* \to I,\ \vartheta_X : X^* \otimes X \to I,$$

such that $l_X \cdot (\gamma_X \otimes 1) = r_X \cdot (1 \otimes \vartheta_X) \cdot a_{X,X^*,X}$, for all objects $X \in \mathbb{G}$. The triple $(X^*, \gamma_X, \vartheta_X)$ is termed an *inverse* for X.

Let us remark that, once an inverse for any object has been chosen, there is a natural isomorphism

$$\upsilon_{X,Y} : (X \otimes Y)^* \to Y^* \otimes X^*\ ,$$

satisfying $a_{Z^*,Y^*,X^*} \cdot (\upsilon_{Y,Z} \otimes 1) \cdot \upsilon_{X,Y\otimes Z} = (1 \otimes \upsilon_{X,Y}) \cdot \upsilon_{X\otimes Y,Z} \cdot a^*_{X,Y,Z}$.

A functor $T : \mathbb{G} \to \mathbb{H}$ between categorical groups is termed a *homomorphism* if it is supplied with natural isomorphisms

$$\mu = \mu_{X,Y} : T(X \otimes Y) \to T(X) \otimes T(Y),$$

compatible with a in the sense that:

$$(1 \otimes \mu_{Y,Z}) \cdot \mu_{X,Y\otimes Z} \cdot T(a_{X,Y,Z}) = a_{T(X),T(Y),T(Z)} \cdot (\mu_{X,Y} \otimes 1) \cdot \mu_{X\otimes Y,Z} \ . \qquad (1)$$

Below we will denote by $\mathscr{CG}$ the category of categorical groups and homomorphisms between them.

Note that if (T,μ) is a homomorphism, then there exists a (unique) isomorphism [6], $\mu_0 : T(I) \to I$, such that $T(r_X) = r_{T(X)} \cdot (1 \otimes \mu_0) \cdot \mu_{X,I}$ and $T(l_X) = l_{T(X)} \cdot (\mu_0 \otimes 1) \cdot \mu_{I,X}$. With respect to the inverses, there exist (unique) isomorphisms

$$\lambda_X : T(X^*) \to T(X)^*,$$

such that the equalities $\mu_0 \cdot T(\gamma_X) = \gamma_{T(X)} \cdot (1 \otimes \lambda_X) \cdot \mu_{X,X^*}$ and $\mu_0 \cdot T(\vartheta_X) = \vartheta_{T(X)} \cdot (\lambda_X \otimes 1) \cdot \mu_{X^*,X}$ hold.

Given homomorphisms of categorical groups $(T,\mu), (T'\mu') : \mathbb{G} \to \mathbb{H}$, a *morphism* from (T,μ) to (T',μ') consists of a natural transformation $\epsilon : T \to T'$ such that, for any objects $X, Y \in \mathbb{G}$, the following condition holds:

$$(\epsilon_X \otimes \epsilon_Y) \cdot \mu_{X,Y} = \mu'_{X,Y} \cdot \epsilon_{X\otimes Y} \ . \qquad (2)$$

If G is a group, a *G-categorical group* is defined [5] as a categorical group $\mathbb{H}$ together with a homomorphism of categorical groups (a *G-action*)

$$(T,\mu) : G[0] \to \mathscr{E}q(\mathbb{H})$$

from the discrete categorical group defined by G (denoted by $G[0]$) to the categorical group, $\mathscr{E}q(\mathbb{H})$, of equivalences of $\mathbb{H}$ [1]. It is easy to see that giving a G-action on $\mathbb{H}$ is equivalent to giving equivalences

$${}^x(-) : \mathbb{H} \to \mathbb{H},\ A \mapsto {}^xA\ ,\ f \mapsto {}^xf$$

for each $x \in G$, together with natural isomorphisms

$$\psi = \psi_{x,A,A'} : {}^x(A \otimes A') \to {}^xA \otimes {}^xA',$$

$$\phi = \phi_{x,x',A} : {}^{xx'}A \to {}^x({}^{x'}A),$$

$$\phi_0 = \phi_{0,A} : {}^1A \to A,$$

satisfying suitable coherence conditions (see [6, 18]).

Moreover, for any $x \in G$, there exists a unique isomorphism

$$\psi_0 = \psi_{0,x} : {}^xI \to I,$$

such that $l_{{}^xA} \cdot (\psi_{0,x} \otimes 1) \cdot \psi_{x,I,A} = {}^xl_A$ and $r_{{}^xA} \cdot (1 \otimes \psi_{0,x}) \cdot \psi_{x,A,I} = {}^xr_A$, for any element $x \in G$ and any object $A \in \mathbb{H}$.

A G-categorical group is termed *strict* when all the isomorphisms $\psi_{x,A,A'}$, $\phi_{x,x',A}$, $\phi_{0,A}$ and $\psi_{0,x}$ are equalities.

Let $\mathbb{H}$ and $\mathbb{E}$ be G-categorical groups. A *G-equivariant* homomorphism $(\mathbf{T},\nu) : \mathbb{H} \to \mathbb{E}$ consists of a categorical group homomorphism $\mathbf{T} = (T,\mu) : \mathbb{H} \to \mathbb{E}$ and a family of natural isomorphisms

$$\nu = (\nu_{x,A} : T({}^xA) \to {}^xT(A))_{(x,A)\in G\times Obj(\mathbb{H})}$$

that are compatible with ψ, ϕ and ϕ_0 in the appropriate sense (see [18]).

3. The Category of Derivations $\mathrm{Der}(G,\mathbb{H})$

Let G be a group and let $\mathbb{H}$ be a G-categorical group. A *derivation* from G into $\mathbb{H}$ is a pair (D,β) where $D: G[0]\to\mathbb{H}$ is a functor and

$$\beta = (\beta_{x,y}: D(xy)\to D(x)\otimes {}^{x}D(y))_{x,y\in G}$$

is a family of isomorphisms such that, for any $x,y,z\in G$, the following coherence condition holds

$$(1\otimes\psi_{x,D(y),{}^{y}D(z)})\cdot(1\otimes {}^{x}\beta_{y,z})\cdot\beta_{x,yz}$$
$$=(1\otimes(1\otimes\phi_{x,y,D(z)}))\cdot a_{D(x),{}^{x}D(y),{}^{xy}D(z)}\cdot(\beta_{x,y}\otimes 1)\cdot\beta_{xy,z}\ . \tag{3}$$

If (D,β) is a derivation, there exists an isomorphism $\bar\beta_0: D(1)\to I$ (where 1 is the neutral element of G) that is determined by the equalities

$$r_{D(x)}\cdot(1\otimes\psi_{0,x})\cdot(1\otimes {}^{x}\bar\beta_0)\cdot\beta_{x,1}=id_{D(x)}=l_{D(x)}\cdot(1\otimes\phi_{0,D(x)})\cdot(\bar\beta_0\otimes 1)\cdot\beta_{1,x}\ .$$

Let $D_0: G[0]\to\mathbb{H}$ be the constant functor with value the unit object $I\in\mathbb{H}$. Then, if for any $x\in G$ we consider the arrow in $\mathbb{H}$ $(\beta_0)_x=(1\otimes\psi_{0,x}^{-1})\cdot l_I^{-1}: I\to I\otimes I\to I\otimes {}^{x}I$, the pair (D_0,β_0) is clearly a derivation, called *the trivial derivation* from G into $\mathbb{H}$.

Given two derivations $(D,\beta),(D',\beta'): G[0]\to\mathbb{H}$, an arrow from (D,β) to (D',β') consists of a natural transformation $\epsilon: D\to D'$ such that, for any $x,y,z\in G$, the following condition holds:

$$(\epsilon_x\otimes {}^{x}\epsilon_y)\cdot\beta_{x,y}=\beta'_{x,y}\cdot\epsilon_{xy}\ . \tag{4}$$

The vertical composition of natural transformations determines a composition for arrows between derivations, and therefore we can consider the category, $\mathrm{Der}(G,\mathbb{H})$, of derivations from G into $\mathbb{H}$, which is actually a groupoid.

Note that, if H is a G-group, then $H[0]$ is a G-categorical group and thus the category $\mathrm{Der}(G,H[0])$ is exactly the discrete category associated to the set $\mathrm{Der}(G,H)$. Recall that, in this case, there is a bijection between the set $\mathrm{Der}(G,H)$ and the set of group homomorphisms f, from G to the semidirect product $H\rtimes G$, such that $pr\cdot f=id_G$, where $pr: H\rtimes G\to G$ is the projection.

Now, if $\mathbb{H}$ is a G-categorical group, we can consider the semidirect product $\mathbb{H}\rtimes G[0]$ (see [6]), which is the categorical group where the underlying groupoid is $\mathbb{H}\times G[0]$ and with tensor product given by $(A,x)\otimes(B,y)=(A\otimes {}^{x}B,xy)$. Since $\mathscr{CG}$ is a 2-category, where the 2-cells are the morphisms between homomorphisms, we can consider the category $\mathrm{Hom}_{\mathscr{CG}}(G[0],\mathbb{H}\rtimes G[0])$ and then we have the following:

Proposition 3.1. *For any group G and any G-categorical group $\mathbb{H}$ there is an isomorphism of categories between $\mathrm{Der}(G,\mathbb{H})$ and the full subcategory $\mathscr{A}$ of $\mathrm{Hom}_{\mathscr{CG}}(G[0],\mathbb{H}\rtimes G[0])$ whose objects are those homomorphisms $\mathbf{T}=(T,\mu): G[0]\to\mathbb{H}\rtimes G[0]$ such that $\mathbf{pT}=id_{G[0]}$, where $\mathbf{p}:\mathbb{H}\rtimes G[0]\to G[0]$ is the projection.*

Proof. We shall define functors

$$\kappa:\mathrm{Der}(G,\mathbb{H})\longrightarrow\mathscr{A}\ ;\ \Sigma:\mathscr{A}\longrightarrow\mathrm{Der}(G,\mathbb{H})$$

such that $\kappa \cdot \Sigma = id$ and $\Sigma \cdot \kappa = id$.

For any object $(D, \beta) \in \mathrm{Der}(G, \mathbb{H})$, we let $\kappa((D, \beta)) = (T, \mu)$, where $T(x) = (D(x), x) \in \mathbb{H} \rtimes G[0]$, and $\mu_{x,y} : T(xy) \to T(x) \otimes T(y)$ is given by $\mu_{x,y} = (\beta_{x,y}, id_{xy})$. For any arrow $\epsilon : (D, \beta) \to (D', \beta')$, $\kappa(\epsilon) : (T, \mu) \to (T', \mu')$ is the morphism determined by the natural transformation given by $\kappa(\epsilon)_x = (\epsilon_x, id_x)$.

To define the functor Σ, let us consider the projection $\partial : \mathbb{H} \rtimes G[0] \to \mathbb{H}$, $(A, x) \mapsto A$. Then, for any object $(T, \mu) \in \mathscr{A}$, $\Sigma((T, \mu)) = (D, \beta)$, where $D(x) = \partial T(x)$, and $\beta_{x,y} : D(xy) \to D(x) \otimes {}^{x}D(y)$ is given by $\beta_{x,y} = \partial(\mu_{x,y})$. For any morphism $\tau : (T, \mu) \to (T', \mu')$, $\Sigma(\tau) : (D, \beta) \to (D', \beta')$ is the arrow determined by the natural transformation whose component at $X \in G$ is $\Sigma(\tau)_x = \partial(\tau_x)$.

Now it is straightforward to check that κ and Σ are inverse to each other. □

4. The Monoidal Structure on $\mathrm{Der}(G, \mathbb{H})$

Recall that, if G is a group, a G-crossed module consists of a group homomorphism $\rho : H \to G$ together with an action $(x, h) \mapsto {}^{x}h$ of G on H satisfying $\rho({}^{x}h) = x\rho(h)x^{-1}$ and ${}^{\rho(h)}h' = hh'h^{-1}$, for all $x \in G$, $h, h' \in H$.

This notion has the following generalization in the setting of categorical groups.

Definition 4.1 ([5])**.** Let G be a group. A *categorical G-crossed module* consists of a triad $(\mathbb{H}, \rho, \xi)$, where $\mathbb{H}$ is a G-categorical group, $\rho : \mathbb{H} \to G[0]$ is a G-equivariant homomorphism (necessarily strict), considering in $G[0]$ the action given by conjugation (i.e. $\rho({}^{x}A) = x\rho(A)x^{-1}$ for any $x \in G$ and $A \in \mathbb{H}$), and

$$\xi = \left(\xi_{A,B} : {}^{\rho(A)}B \otimes A \to A \otimes B\right)_{(A,B) \in \mathbb{H}}$$

is a family of natural isomorphisms in $\mathbb{H}$, such that for all objects $A, B, C \in \mathbb{H}$ the following diagrams are commutative:

$$\begin{array}{ccc} {}^{\rho(A)}(B \otimes C) \otimes A & \xrightarrow{\xi_{A,B\otimes C}} & A \otimes B \otimes C \\ \downarrow{\scriptstyle \psi\otimes 1} & & \uparrow{\scriptstyle \xi_{A,B}\otimes 1} \\ {}^{\rho(A)}B \otimes {}^{\rho(A)}C \otimes A & \xrightarrow[1\otimes\xi_{A,C}]{} & {}^{\rho(A)}B \otimes A \otimes C \end{array} , \qquad (5)$$

$$\begin{array}{ccc} {}^{\rho(A\otimes B)}C \otimes A \otimes B & \xrightarrow{\xi_{A\otimes B,C}} & A \otimes B \otimes C \\ \downarrow{\scriptstyle \phi\otimes 1} & & \uparrow{\scriptstyle 1\otimes\xi_{B,C}} \\ {}^{\rho(A)}({}^{\rho(B)}C) \otimes A \otimes B & \xrightarrow[\xi_{A,{}^{\rho(B)}C}\otimes 1]{} & A \otimes {}^{\rho(B)}C \otimes B \end{array} , \qquad (6)$$

$$\begin{array}{ccccc}
 & {}^x({}^{\rho(A)}B \otimes A) & \xrightarrow{\psi} & {}^x({}^{\rho(A)}B) \otimes {}^xA & \xleftarrow{\phi\otimes 1} {}^{(x\rho(A))}B \otimes {}^xA \\
{}^x\xi_{A,B} \swarrow & & & & \Vert \\
{}^x(A \otimes B) & & & & \\
\searrow \psi & & & & {}^{(\rho({}^xA)x)}B \otimes {}^xA \, . \\
 & {}^xA \otimes {}^xB & \xleftarrow[\xi_{{}^xA,{}^xB}]{} & {}^{\rho({}^xA)}({}^xB) \otimes {}^xA & \xleftarrow{\phi\otimes 1}
\end{array} \qquad (7)$$

Note that if H is a group and $(H[0], \rho : H[0] \to G[0], \xi)$ is a categorical G-crossed module, then ξ is necessarily an identity and therefore we obtain a G-equivariant group homomorphism $\rho : H \to G$, which is a crossed module in the usual sense. In addition, Conduché's 2-crossed modules are particular cases of categorical G-crossed modules (see [5]). Categorical G-modules (see [4]) are braided categorical groups [10] provided with a coherent left action of $G[0]$. If $(\mathbb{A}, c)$ is a G-module then the trivial homomorphism $\mathbb{A} \to G[0]$ is clearly a categorical G-crossed module where ξ is given by the braiding c.

If H is a G-crossed module, then the set $\mathrm{Der}(G, H)$ of derivations from G into H has a natural monoid structure [20]. Now, if $(\mathbb{H}, \rho, \xi)$ is a categorical G-crossed module, then $\mathbb{H}$ is a G-categorical group and our aim below is to show that the category $\mathrm{Der}(G, \mathbb{H})$ has a natural monoidal structure, that is inherited from the G-crossed module structure in $\mathbb{H}$. To do so, we first show the following:

Lemma 4.2. *Let $(\mathbb{H}, \rho, \xi)$ be a categorical G-crossed module. For any derivation $(D, \beta) : G[0] \to \mathbb{H}$, there are endomorphisms of $G[0]$ and of $\mathbb{H}$,*

$$\sigma_{_D} : G[0] \to G[0] \ , \ \theta_{_D} = (\theta_{_D}, \mu_{_\theta}) : \mathbb{H} \longrightarrow \mathbb{H},$$

where $\sigma_{_D}(x) = \rho(D(x))x$, $x \in G$, and $\theta_{_D}$ is defined, for any object $A \in \mathbb{H}$, by $\theta_{_D}(A) = D(\rho(A)) \otimes A$, for any arrow $f : A \to B$, by $\theta_{_D}(f) = D(\rho(f)) \otimes f = 1 \otimes f$ and where $(\mu_{_\theta})_{A,B} : \theta_{_D}(A \otimes B) \to \theta_{_D}(A) \otimes \theta_{_D}(B)$, $A, B \in \mathbb{H}$, is given by the following composition:

$$\begin{array}{ccc}
D(\rho(A \otimes B)) \otimes A \otimes B & \xrightarrow{(\mu_\theta)_{A,B}} & D(\rho(A)) \otimes A \otimes D(\rho(B)) \otimes B \\
\Vert & & \uparrow {\scriptstyle 1\otimes\xi_{A,D(\rho(B))}\otimes 1} \\
D(\rho(A)\rho(B)) \otimes A \otimes B & \xrightarrow{\beta_{\rho(A),\rho(B)}\otimes 1} & D(\rho(A)) \otimes {}^{\rho(A)}D(\rho(B)) \otimes A \otimes B \, .
\end{array}$$

Moreover, if $\epsilon : (D, \beta) \to (D', \beta')$ is an arrow in $\mathrm{Der}(G, \mathbb{H})$, then $\sigma_{_D} = \sigma_{_{D'}}$ and there is a morphism of categorical group homomorphisms $\tau : \theta_{_D} \to \theta_{_{D'}}$ given, for any $A \in \mathbb{H}$, by $\tau_{_A} = \epsilon_{_{\rho(A)}} \otimes id_{_A}$.

Moreover, if $D_0 : G[0] \to \mathbb{H}$ is the trivial derivation, then $\sigma_{_{D_0}} = id_G$ and there is a morphism $\theta_{_{D_0}} \to id_{\mathbb{H}}$ given, for any $A \in \mathbb{H}$, by $l_{_A} : I \otimes A \to A$.

Proof. σ_D is a group homomorphism since, for any $x, y \in G$,

$$\begin{aligned} \sigma_D(xy) &= \rho(D(xy))\, xy = \rho[D(x) \otimes {}^{x}D(y)]\, xy \\ &= \rho(D(x))\rho({}^{x}D(y))\, xy = \rho(D(x))\, x\, \rho(D(y))\, x^{-1}xy \\ &= \rho(D(x))\, x\, \rho(D(y))\, y = \sigma_D(x)\sigma_D(y)\ . \end{aligned}$$

As for θ_D, we obtain the coherence condition (1) for μ_θ as follows:

$$\begin{aligned} &(1 \otimes (\mu_\theta)_{B,C}) \cdot (\mu_\theta)_{A,B\otimes C} \cdot \theta_D(a_{A,B,C}) = \\ &\overset{def}{=} [(1 \otimes (1 \otimes \xi_{B,D(\rho(C))} \otimes 1)) \cdot (1 \otimes (\beta_{\rho(B),\rho(C)} \otimes 1))] \cdot \\ &\quad [(1 \otimes \xi_{A,D(\rho(B\otimes C))} \otimes 1) \cdot (\beta_{\rho(A),\rho(B\otimes C)} \otimes 1)] \cdot [1 \otimes a_{A,B,C}] \\ &\overset{nat}{=} (1 \otimes (1 \otimes \xi_{B,D(\rho(C))} \otimes 1)) \cdot (1 \otimes \xi_{A,D(\rho(B))\otimes {}^{\rho(B)}D(\rho(C))} \otimes 1) \cdot \\ &\quad ((1 \otimes {}^{\rho(A)}\beta_{\rho(B),\rho(C)}) \otimes 1 \otimes 1) \cdot (\beta_{\rho(A),\rho(B\otimes C)} \otimes 1) \cdot [1 \otimes a_{A,B,C}] \\ &\overset{(5)}{=} (1 \otimes (1 \otimes \xi_{B,D(\rho(C))} \otimes 1)) \cdot (1 \otimes \xi_{A,D(\rho(B))} \otimes a_{\theta_D(A),D(\rho(B))\otimes {}^{\rho(B)}D(\rho(C)),B\otimes C}) \cdot \\ &\quad (1 \otimes 1 \otimes \xi_{A,{}^{\rho(B)}D(\rho(C))} \otimes 1) \cdot (1 \otimes \psi_{\rho(A),D(\rho(B)),{}^{\rho(B)}D(\rho(C))} \otimes a^{-1}_{A,B,C}) \cdot \\ &\quad ((1 \otimes {}^{\rho(A)}\beta_{\rho(B),\rho(C)}) \otimes 1 \otimes 1) \cdot (\beta_{\rho(A),\rho(B\otimes C)} \otimes 1) \cdot [1 \otimes a_{A,B,C}] \\ &\overset{(3)}{=} (1 \otimes (1 \otimes \xi_{B,D(\rho(C))} \otimes 1)) \cdot (1 \otimes \xi_{A,D(\rho(B))} \otimes 1) \cdot a_{\theta_D(A),D(\rho(B))\otimes {}^{\rho(B)}D(\rho(C)),B\otimes C} \cdot \\ &\quad (1 \otimes 1 \otimes \xi_{A,{}^{\rho(B)}D(\rho(C))} \otimes 1) \cdot (1 \otimes (1 \otimes \phi_{\rho(A),\rho(B),D(\rho(C))}) \otimes 1) \cdot \\ &\quad (a_{D(\rho(A)),{}^{\rho(A)}D(\rho(B)),{}^{\rho(A\otimes B)}D(\rho(C))} \otimes 1) \cdot ((\beta_{\rho(A),\rho(B)} \otimes 1) \otimes 1) \cdot (\beta_{\rho(A\otimes B),\rho(C)} \otimes 1) \\ &\overset{(6)}{=} (1 \otimes (1 \otimes \xi_{B,D(\rho(C))} \otimes 1)) \cdot (1 \otimes \xi_{A,D(\rho(B))} \otimes 1) \cdot a_{\theta_D(A),D(\rho(B))\otimes {}^{\rho(B)}D(\rho(C)),B\otimes C} \cdot \\ &\quad (1 \otimes (1 \otimes \xi^{-1}_{B,D(\rho(C))} \otimes 1)) \cdot (1 \otimes (1 \otimes \xi_{A\otimes B,D(\rho(C))} \otimes 1)) \cdot \\ &\quad (a_{D(\rho(A)),{}^{\rho(A)}D(\rho(B)),{}^{\rho(A\otimes B)}D(\rho(C))} \otimes 1) \cdot ((\beta_{\rho(A),\rho(B)} \otimes 1) \otimes 1) \cdot (\beta_{\rho(A\otimes B),\rho(C)} \otimes 1) \\ &\overset{nat}{=} (1 \otimes \xi_{A,D(\rho(B))} \otimes a_{\theta_D(A),\theta_D(B),\theta_D(C)}) \cdot (a_{D(\rho(A)),{}^{\rho(A)}D(\rho(B)),A\otimes B} \otimes 1) \cdot \\ &\quad (1 \otimes (\xi_{A\otimes B,D(\rho(C))} \otimes 1)) \cdot ((\beta_{\rho(A),\rho(B)} \otimes 1) \otimes 1) \cdot (\beta_{\rho(A\otimes B),\rho(C)} \otimes 1) \\ &\overset{nat}{=} a_{\theta_D(A),\theta_D(B),\theta_D(C)} \cdot [(1 \otimes \xi_{A,D(\rho(B))} \otimes 1) \cdot ((\beta_{\rho(A),\rho(B)} \otimes 1) \otimes 1)] \cdot \\ &\quad [(1 \otimes \xi_{\rho(A\otimes B),D(\rho(C))} \otimes 1) \cdot (\beta_{\rho(A\otimes B),\rho(C)} \otimes 1)] \\ &\overset{def}{=} a_{\theta_D(A),\theta_D(B),\theta_D(C)} \cdot ((\mu_\theta)_{A,B} \otimes 1) \cdot (\mu_\theta)_{A\otimes B,C}\ . \end{aligned}$$

Finally, $\sigma_D = \sigma_{D'}$ since, for any $x \in G$, $\rho(\epsilon_x) : \rho(D(x)) \to \rho(D'(x))$ is necessarily an identity, because $G[0]$ is discrete, and therefore $\sigma_D(x) = \rho(D(x))x = \rho(D'(x))x = \sigma_{D'}(x)$. On the other hand, to prove that $\tau : \theta_D \to \theta_{D'}$, given by $\tau_A = \epsilon_{\rho(A)} \otimes id_A : D(\rho(A)) \otimes A \to D'(\rho(A)) \otimes A$, is a morphism is routine. □

Now, using the endomorphism σ_D of G associated to any derivation D, we have:

Proposition 4.3. *Let G be a group and $(\mathbb{H}, \rho, \xi)$ a categorical G-crossed module. If (D_1, β_1) and (D_2, β_2) are derivations from $G[0]$ into $\mathbb{H}$, then there is a derivation from $G[0]$ into $\mathbb{H}$, $(D_1 \otimes D_2, \beta_1 \otimes \beta_2)$, where $D_1 \otimes D_2 : G[0] \to \mathbb{H}$ is given by $(D_1 \otimes D_2)(x) = D_1(\sigma_{D_2}(x)) \otimes D_2(x)$, $x \in G$, and $(\beta_1 \otimes \beta_2) = ((\beta_1 \otimes \beta_2)_{x,y} : (D_1 \otimes D_2)(xy) \to (D_1 \otimes D_2)(x) \otimes {}^{x}(D_1 \otimes D_2)(y))$ is the family of* *natural isomorphisms given by the following composition:*

$$\begin{array}{c} D_1(\sigma_{D_2}(x)\sigma_{D_2}(y)) \otimes D_2(xy) \\ \Big\downarrow {\scriptstyle (\beta_1)_{\sigma_{D_2}(x),\sigma_{D_2}(y)} \otimes (\beta_2)_{x,y}} \\ D_1(\sigma_{D_2}(x)) \otimes^{\rho(D_2(x))x} D_1(\sigma_{D_2}(y)) \otimes D_2(x) \otimes {}^{x}D_2(y) \\ \Big\downarrow {\scriptstyle 1\otimes\phi\otimes 1} \\ D_1(\sigma_{D_2}(x)) \otimes^{\rho(D_2(x))} ({}^{x}D_1(\sigma_{D_2}(y))) \otimes D_2(x) \otimes {}^{x}D_2(y) \\ \Big\downarrow {\scriptstyle 1\otimes\xi_{D_2(x),{}^{x}D_1(\sigma_{D_2}(y))}\otimes 1} \\ D_1(\sigma_{D_2}(x)) \otimes D_2(x) \otimes {}^{x}D_1(\sigma_{D_2}(y)) \otimes {}^{x}D_2(y) \\ \Big\downarrow {\scriptstyle 1\otimes\psi^{-1}} \\ D_1(\sigma_{D_2}(x)) \otimes D_2(x) \otimes {}^{x}(D_1(\sigma_{D_2}(y)) \otimes D_2(y)). \end{array}$$

Moreover, in such conditions, $\sigma_{D_1\otimes D_2} = \sigma_{D_1} \cdot \sigma_{D_2}$.

Proof. It is straightforward to check that $\beta_1 \otimes \beta_2$ satisfies (3) so that $(D_1 \otimes D_2, \beta_1 \otimes \beta_2)$ is a derivation from $G[0]$ to $\mathbb{H}$.

As for the second statement we have:

$$\begin{aligned} \sigma_{D_1\otimes D_2}(x) &= \rho((D_1 \otimes D_2)(x))x = \rho(D_1(\sigma_{D_2}(x)) \otimes D_2(x))\, x \\ &= \rho(D_1(\rho(D_2(x))\, x))\rho(D_2(x))\, x = \rho(D_1(\rho(D_2(x))\, x))\sigma_{D_2}(x) \\ &= \rho(D_1(\sigma_{D_2}(x)))\sigma_{D_2}(x) = \sigma_{D_1}(\sigma_{D_2}(x)). \end{aligned}$$

□

The derivation $(D_1 \otimes D_2, \beta_1 \otimes \beta_2)$ built into the above proposition defines a functor

$$\mathrm{Der}(G, \mathbb{H}) \times \mathrm{Der}(G, \mathbb{H}) \xrightarrow{\otimes} \mathrm{Der}(G, \mathbb{H})$$
$$(D_1, \beta_1) \otimes (D_2, \beta_2) = (D_1 \otimes D_2, \beta_1 \otimes \beta_2),$$

that determines a monoidal structure in the category $\mathrm{Der}(G, \mathbb{H})$. The isomorphism of associativity, $\bar{a} : ((D_1, \beta_1) \otimes (D_2, \beta_2)) \otimes (D_3, \beta_3) \to (D_1, \beta_1) \otimes ((D_2, \beta_2) \otimes (D_3, \beta_3))$, is the arrow determined by the natural transformation $\bar{a} : (D_1 \otimes D_2) \otimes D_3 \to D_1 \otimes (D_2 \otimes D_3)$ given by $\bar{a}_x = a_{D_1(\sigma_{D_2\otimes D_3}(x)),D_2(\sigma_{D_3}(x)),D_3(x)}$, $x \in G$; the unit object $\bar{I}$, is the trivial derivation (D_0, β_0); the left unit constraint $\bar{l} = \bar{l}_{(D,\beta)} : (D_0, \beta_0) \otimes (D, \beta) \to (D, \beta)$, is the arrow determined by the natural transformation $\bar{l} : D_0 \otimes D \to D$ given by $\bar{l}_x = l_{D(x)}$; and the right unit constraint $\bar{r} = \bar{r}_{(D,\beta)} : (D, \beta) \otimes (D_0, \beta_0) \to (D, \beta)$, is the arrow determined by the natural transformation $\bar{r} : D \otimes D_0 \to D$ given by $\bar{r}_x = r_{D(x)}$.

Below we will see that another monoidal structure in $\mathrm{Der}(G, \mathbb{H})$ can be defined alternatively, by using the endomorphism $\theta_D : \mathbb{H} \to \mathbb{H}$, given in Lemma 4.2, instead of σ_D. To do so, we first show the following:

Proposition 4.4. *Let $(\mathbb{H},\rho,\xi)$ be a categorical G-crossed module and (D,β) a derivation from $G[0]$ to $\mathbb{H}$. Then we have:*

i) $\sigma_D \cdot \rho = \rho \cdot \theta_D : \mathbb{H} \to G[0]$.
ii) *Both $\theta_D \cdot D$ and $D \cdot \sigma_D$ are derivations from $G[0]$ into $\mathbb{H}$ and there is an arrow in* $\mathrm{Der}(G,\mathbb{H})$, $\epsilon : \theta_D \cdot D \to D \cdot \sigma_D$.
iii) *If $ac : G[0] \to Eq(\mathbb{H})$ denotes the G-action on $\mathbb{H}$ then, for any $x \in G$, there is a morphism of categorical group homomorphisms $\tau_x : \theta_D \cdot ac_x \to ac_{\sigma_D(x)} \cdot \theta_D$.*
iv) *Given two derivations $(D_1,\beta_1),(D_2,\beta_2) : G[0] \to \mathbb{H}$ there is, for any $x \in G$, a natural isomorphism $\chi_x : D_1(\sigma_{D_2}(x)) \otimes D_2(x) \to \theta_{D_1}(D_2(x)) \otimes D_1(x)$.*

Proof. i) $(\sigma_D \cdot \rho)(A) = \rho(D(\rho(A)))\rho(A) = \rho(D(\rho(A)) \otimes A) = (\rho \cdot \theta_D)(A)$.
ii) To prove that $\theta_D \cdot D$ and $D \cdot \sigma_D$ are derivations is straightforward. As for the arrow ϵ, it is given by the natural transformation whose component at $x \in G$, $\epsilon_x : (\theta_D \cdot D)(x) \to (D \cdot \sigma_D)(x)$, is determined by the following diagram:

$$\begin{array}{ccc}
\theta_D(D(x)) = D(\rho(D(x))) \otimes D(x) & \overset{\epsilon_x}{\dashrightarrow} & D(\sigma_D(x)) = D(\rho(D(x))x) \\
 & & \downarrow{\scriptstyle \beta_{\rho(D(x)),x}} \\
\downarrow{\scriptstyle 1\otimes r^{-1}} & & D(\rho(D(x))) \otimes {}^{\rho(D(x))}D(x) \\
 & & \downarrow{\scriptstyle 1\otimes r^{-1}} \\
D(\rho(D(x))) \otimes D(x) \otimes I & & D(\rho(D(x))) \otimes {}^{\rho(D(x))}D(x) \otimes I \\
\downarrow{\scriptstyle 1\otimes\gamma^{-1}} & & \downarrow{\scriptstyle 1\otimes\gamma^{-1}} \\
D(\rho(D(x))) \otimes D(x) \otimes D(x) \otimes D(x)^* & \xrightarrow{1\otimes\xi^{-1}_{D(x),D(x)}\otimes 1} & D(\rho(D(x))) \otimes {}^{\rho(D(x))}D(x) \otimes D(x) \otimes D(x)^*.
\end{array}$$

iii) For any object $A \in \mathbb{H}$, $\tau_{x,A} : \theta_D({}^xA) \to {}^{\sigma_D(x)}\theta_D(A)$ is the arrow in $\mathbb{H}$ given by the following diagram:

$$\begin{array}{ccc}
\theta_D({}^xA) = D(\rho({}^xA)) \otimes {}^xA = D(x\rho(A)x^{-1}) \otimes {}^xA & \overset{\tau_{x,A}}{\dashrightarrow} & {}^{\sigma_D(x)}\theta_D(A) = {}^{(\rho(D(x))\,x)}\theta_D(A) \\
\downarrow{\scriptstyle \beta_{x\rho(A),x^{-1}}\otimes 1} & & \downarrow{\scriptstyle \phi} \\
D(x\rho(A)) \otimes {}^{x\rho(A)}D(x^{-1}) \otimes {}^xA & & {}^{(\rho(D(x))}({}^x\theta_D(A)) \\
\downarrow{\scriptstyle 1\otimes\phi\otimes 1} & & \downarrow{\scriptstyle (1\otimes\gamma^{-1})\cdot r^{-1}} \\
D(x\rho(A)) \otimes {}^x({}^{\rho(A)}D(x^{-1})) \otimes {}^xA & & {}^{\rho(D(x))}({}^x\theta_D(A)) \otimes D(x) \otimes D(x)^* \\
\downarrow{\scriptstyle 1\otimes\psi^{-1}} & & \downarrow{\scriptstyle \xi_{D(x),{}^x\theta_D(A)}\otimes r^{-1}} \\
D(x\rho(A)) \otimes {}^x({}^{\rho(A)}D(x^{-1}) \otimes A) & & D(x) \otimes {}^x(D(\rho(A)) \otimes A) \otimes D(x)^* \otimes I \\
\downarrow{\scriptstyle \beta_{x,\rho(A)}\otimes {}^x\xi_{A,D(x^{-1})}} & & \downarrow{\scriptstyle 1\otimes\bar{\beta}_0^{-1}} \\
D(x) \otimes {}^xD(\rho(A)) \otimes {}^x(A \otimes D(x^{-1})) & & D(x) \otimes {}^x(D(\rho(A)) \otimes A) \otimes D(x)^* \otimes D(xx^{-1}) \\
\downarrow{\scriptstyle 1\otimes\psi} & & \downarrow{\scriptstyle 1\otimes\psi\otimes 1\otimes\beta_{x,x^{-1}}} \\
D(x) \otimes {}^xD(\rho(A)) \otimes {}^xA \otimes {}^xD(x^{-1}) & \xrightarrow[(1\otimes\vartheta^{-1}\otimes 1)(r^{-1}\otimes 1)]{} & D(x) \otimes {}^xD(\rho(A)) \otimes {}^xA \otimes D(x)^* \otimes D(x) \otimes {}^xD(x^{-1}).
\end{array}$$

iv) The arrow $\chi_x : D_1(\sigma_{D_2}(x)) \otimes D_2(x) \to \theta_{D_1}(D_2(x)) \otimes D_1(x)$ is defined as the composite $\chi_x = (1 \otimes \xi_{D_2(x),D_1(x)}) \cdot ((\beta_1)_{\rho(D_2(x)),x} \otimes 1)$. □

Now, consider the functor

$$\mathrm{Der}(G,\mathbb{H}) \times \mathrm{Der}(G,\mathbb{H}) \xrightarrow{\bar{\otimes}} \mathrm{Der}(G,\mathbb{H})$$
$$(D_1,\beta_1)\bar{\otimes}(D_2,\beta_2) = (D_1\bar{\otimes}D_2, \beta_1\bar{\otimes}\beta_2),$$

where $D_1\bar{\otimes}D_2 : G[0] \to \mathbb{H}$ is given by $(D_1\bar{\otimes}D_2)(x) = \theta_{D_1}(D_2(x)) \otimes D_1(x)$ and

$$(\beta_1\bar{\otimes}\beta_2)_{x,y} : (D_1\bar{\otimes}D_2)(xy) \to (D_1\bar{\otimes}D_2)(x) \otimes {}^x(D_1\bar{\otimes}D_2)(y)$$

is defined as the following composition:

$$\begin{array}{c}
\theta_{D_1}(D_2(xy)) \otimes D_1(xy) \\
\big\downarrow {\scriptstyle \theta_{D_1}((\beta_2)_{x,y}) \otimes (\beta_1)_{x,y}} \\
\theta_{D_1}(D_2(x) \otimes {}^xD_2(y)) \otimes D_1(x) \otimes {}^xD_1(y) \\
\big\downarrow {\scriptstyle \mu_{\theta_{D_1}} \otimes 1} \\
\theta_{D_1}(D_2(x)) \otimes \theta_{D_1}({}^xD_2(y)) \otimes D_1(x) \otimes {}^xD_1(y) \\
\big\downarrow {\scriptstyle 1 \otimes (\tau_1)_{x,D_2(y)} \otimes 1} \\
\theta_{D_1}(D_2(x)) \otimes {}^{\sigma_{D_1}(x)}\theta_{D_1}(D_2(y)) \otimes D_1(x) \otimes {}^xD_1(y) \\
\big\downarrow {\scriptstyle 1 \otimes \phi \otimes 1} \\
\theta_{D_1}(D_2(x)) \otimes {}^{\rho(D_1(x))}({}^x\theta_{D_1}(D_2(y))) \otimes D_1(x) \otimes {}^xD_1(y) \\
\big\downarrow {\scriptstyle 1 \otimes \xi_{D_1(x),{}^x\theta_{D_1}(D_2(y))} \otimes 1} \\
\theta_{D_1}(D_2(x)) \otimes D_1(x) \otimes {}^x\theta_{D_1}(D_2(y)) \otimes {}^xD_1(y) \\
\big\downarrow {\scriptstyle 1 \otimes \psi^{-1}} \\
\theta_{D_1}(D_2(x)) \otimes D_1(x) \otimes {}^x(\theta_{D_1}(D_2(y)) \otimes D_1(y)) \ .
\end{array}$$

To check that $(D_1\bar{\otimes}D_2, \beta_1\bar{\otimes}\beta_2)$ is a derivation is straightforward. It is also easy to see that there is a morphism of categorical group homomorphisms

$$\theta_{D_1} \cdot \theta_{D_2} \to \theta_{D_1\bar{\otimes}D_2} \tag{8}$$

whose component at $A \in \mathbb{H}$ is the arrow $(\mu_{\theta_{D_1}})_{D_2(\rho(A)),A}$.

The functor $\bar{\otimes}$ determines a monoidal structure in the category $\mathrm{Der}(G,\mathbb{H})$ and, for any derivations (D_1,β_1) and (D_2,β_2), there is a natural isomorphism $(D_1 \otimes D_2)(x) \to (D_1\bar{\otimes}D_2)(x)$ given by the isomorphism χ_x defined in Proposition 4.4 iv).

Then we summarize the above results in the following:

Theorem 4.5. *If G is a group and $(\mathbb{H},\rho,\xi)$ is a categorical G-crossed module, the functors*

$$\otimes : \mathrm{Der}(G,\mathbb{H}) \times \mathrm{Der}(G,\mathbb{H}) \longrightarrow \mathrm{Der}(G,\mathbb{H})$$

and

$$\bar{\otimes} : \mathrm{Der}(G,\mathbb{H}) \times \mathrm{Der}(G,\mathbb{H}) \longrightarrow \mathrm{Der}(G,\mathbb{H})$$

determine, in the category $\mathrm{Der}(G,\mathbb{H})$*, two monoidal structures connected by the natural isomorphism* χ_x *defined in Proposition* 4.4.

Let us recall now [19] that, for any monoidal category $\mathscr{C}$, the *Picard categorical group* $\mathscr{P}(\mathscr{C})$ of $\mathscr{C}$ is the subcategory of $\mathscr{C}$ given by invertible objects and isomorphisms between them. Clearly, $\mathscr{P}(\mathscr{C})$ is a categorical group and any monoidal functor $F:\mathscr{C}\to\mathscr{D}$ restricts to a homomorphism of categorical groups $\mathscr{P}(F):\mathscr{P}(\mathscr{C})\to\mathscr{P}(\mathscr{D})$. In this way, there is a functor $\mathscr{P}$ from the category of monoidal categories to $\mathscr{CG}$.

Definition 4.6. For any group G and any categorical G-crossed module $(\mathbb{H},\rho,\xi)$ we define the Whitehead categorical group of derivations $D(G,\mathbb{H})$ as the Picard categorical group, $\mathscr{P}(\mathrm{Der}(G,\mathbb{H}))$, of the monoidal category $(\mathrm{Der}(G,\mathbb{H}),\otimes)$.

Note that, when $\mathbb{H}=H[0]$ is the discrete categorial group associated to a group H, the categorial group $D(G,\mathbb{H})$ is exactly the discrete one associated to the Whitehead group $D(G,H)$ of the regular derivations from G into H. Also, note that if $(\mathbb{A},c)$ is a G-module then $D(G,\mathbb{A})=\mathrm{Der}(G,\mathbb{A})$, where the last is the categorical group of derivations studied in [7].

Finally, we characterize the objects of $D(G,\mathbb{H})$, that is, the invertible derivations, as follows:

Theorem 4.7. *Let G be a group and $(\mathbb{H},\rho,\xi)$ a categorical G-crossed module. Then, the following statements are equivalent:*

a) $(D,\beta)\in D(G,\mathbb{H})$,
b) $\sigma_{D}\in\mathrm{Aut}(G)$,
c) $\theta_{D}\in\mathscr{E}q(\mathbb{H})$.

Proof. a) $\Rightarrow$ b) If $(D,\beta)\in D(G,\mathbb{H})$, then there exists an inverse for (D,β), that is, a derivation $(D,\beta)^*=(\bar{D},\bar{\beta})\in D(G,\mathbb{H})$ and isomorphisms $(D,\beta)\otimes(\bar{D},\bar{\beta})\cong(\bar{D},\bar{\beta})\otimes(D,\beta)\cong(D_0,\beta_0)$. Thus, according to Lemma 4.2 and Proposition 4.3, $\sigma_{D}\cdot\sigma_{\bar{D}}=\sigma_{\bar{D}}\cdot\sigma_{D}=\sigma_{D_0}=id_G$ and therefore σ_{D} is an automorphism.
a) $\Rightarrow$ c) Using now (8) and Lemma 4.2, there exist morphisms of categorical group homomorphisms

$$\theta_{D}\cdot\theta_{\bar{D}}\longrightarrow\theta_{D\otimes\bar{D}}\longrightarrow\theta_{D_0}\longrightarrow id_{\mathbb{H}}$$

and

$$\theta_{\bar{D}}\cdot\theta_{D}\longrightarrow\theta_{\bar{D}\otimes D}\longrightarrow\theta_{D_0}\longrightarrow id_{\mathbb{H}}$$

and therefore θ_{D} is an equivalence of $\mathbb{H}$.
b) $\Rightarrow$ a) Let $(D,\beta)\in\mathrm{Der}(G,\mathbb{H})$ such that σ_{D} is an automorphism. An inverse $(D,\beta)^*=(\bar{D},\bar{\beta})$ for (D,β) is obtained as follows. Define $\bar{D}:G[0]\to\mathbb{H}$ by $\bar{D}(x)=(D(\sigma_{D}^{-1}(x)))^*$ and, by denoting $x'=\sigma_{D}^{-1}(x)$, $x\in G$, let $\bar{\beta}_{x,y}:\bar{D}(xy)\to$

$\bar{D}(x) \otimes {}^x\bar{D}(y)$ be the arrow in $\mathbb{H}$ determined by the diagram:

$$\begin{array}{ccc}
\bar{D}(xy) = (D(x'y'))^* & \overset{\bar{\beta}_{x,y}}{\dashrightarrow} & \bar{D}(x) \otimes {}^x\bar{D}(y) = (D(x'))^* \otimes {}^x(D(y')^*) \\
\downarrow (\beta^{-1}_{x',y'})^* & & \uparrow 1\otimes\lambda^{-1} \\
(D(x') \otimes {}^{x'}D(y'))^* & & D(x')^* \otimes ({}^xD(y'))^* \\
\downarrow (\xi_{D(x'),\,{}^{x'}D(y')})^* & & \uparrow \upsilon \\
({}^{\rho(D(x'))}({}^{x'}D(y')) \otimes D(x'))^* & \xrightarrow[(\phi\otimes 1)^*]{} & ({}^{\sigma_D(x')}D(y') \otimes D(x'))^*.
\end{array}$$

It is straightforward to see that $(\bar{D}, \bar{\beta}) \in \mathrm{Der}(G, \mathbb{H})$. In addition, for any $x \in G$, $(\bar{D} \otimes D)(x) = \bar{D}(\sigma_D(x)) \otimes D(x) = (D(\sigma_D^{-1}(\sigma_D(x))))^* \otimes D(x) = D(x)^* \otimes D(x) \simeq I = D_0(x)$ and therefore there is an arrow in $\mathrm{Der}(G, \mathbb{H})$, $(\bar{D}, \bar{\beta}) \otimes (D, \beta) \to (D_0, \beta_0)$. Thus, $(\bar{D}, \bar{\beta})$ is an inverse for (D, β) and so $(D, \beta) \in D(G, \mathbb{H})$.

c) $\Rightarrow$ a) The proof is similar to the above one by defining $\bar{D}(x) = \theta_D^{-1}(D(x)^*)$ where θ_D^{-1} is a quasi-inverse of θ_D. □

Acknowledgments

This work was financially supported by DGI:BFM2001-2886.

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(Received 29.04.2002)

Authors' address:
Departamento de Álgebra, Facultad de Ciencias
Universidad de Granada, 18071 Granada
Spain
E-mail: agarzon@ugr.es