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CHARACTERISTIC FUNCTIONS AND s-ORTHOGONALITY PROPERTIES OF CHEBYSHEV POLYNOMIALS OF THIRD AND FOURTH KIND

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Abstract. The properties of two families of s-orthogonal polynomials, which are connected with Chebyshev polynomials of third and fourth kind, are studied. Evaluations of the remainders are given and asymptotic formulae are calculated for the corresponding hyper-Gaussian formulae used for an approximate estimation of integrals.

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Introduction

In their various works A. Ossicini and F. Rosati dealt with the problem of construction of families of orthogonal polynomials which are at the same time s-orthogonal with respect to predefined weights.

Recently, in collaboration with the above-mentioned authors we have carried out a study of two families of s-orthogonal polynomials, connected with Chebyshev polynomials of first and second kind [1].

This paper is concerned with the properties of two families of s-orthogonal polynomials "connected" with Chebyshev polynomials of third and fourth kind.[1] Using proper formulae of an upper bound, *hyper-Gaussian functionals* are studied and used for an approximate estimation of "integrals with weight"; evaluations of the *remainders* are given and *asymptotic formulae* are derived. The above results allow one to go beyond those obtained by A. Ossicini and F. Rosati in [3] and [4].

1. s-Orthogonal Polynomials And Fundamental Formulae

Let $[a, b]$, $a < b$, be a finite interval on the x-axis, and $p(x)$ a fixed measurable function which is almost everywhere positive and summable in $[a, b]$ ($p(x) \in L[a, b]$).

Under such a hypothesis, having fixed an integer $s \geq 0$, it was proved ([5], [6]) that it is possible to determine a sequence $\{P_{s,m}(x)\}$ of polynomials of degree m (each polynomial being determined up to a multiplicative constant factor $c_{s,m}$)

[1]For the definition of such polynomials, which are of course Jacobi polynomials, see [2].

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s-orthogonal in $[a, b]$ with respect to the weight $p(x)$, i.e., such that for each integer $m \geq 1$

$$\int_a^b p(x)\Pi_{m-1}(x)[P_{s,m}(x)]^{2s+1}dx = 0\,,$$

where $\Pi_{m-1}(x)$ denotes *an arbitrary polynomial of degree* $\leq m-1$. Moreover, for $m \geq 1$ it follows that the m zeros of the polynomial $P_{s,m}(x)$ are *real* and *distinct* and located in the *interior* of $[a, b]$.

For $s = 0$, $P_{s,m}(x)$ are the classical orthogonal polynomials.

Such systems of s-orthogonal polynomials are of particular importance in studying of *hyper-Gaussian* quadrature formulae (see [7]) of the following type:

$$\int_a^b p(x)f(x)dx = \sum_{j=1}^{m}\sum_{h=0}^{2s} A_{hj}f^{(h)}(x_{m,j}) + R_{s,m}[f] \quad \text{for each} \quad f \in AC^{2s}[a, b]\,,$$

where the coefficients A_{hj} (dependent on s and m) are independent of f and are uniquely determined by means of the condition: $R_{s,m}[f] = 0$ if f is *an arbitrary* polynomial of degree $\leq 2m(s+1)-1$. Moreover, it follows that the *nodes* $x_{m,1}$, $x_{m,2}, \ldots, x_{m,m}$ (dependent in general on s) are necessarily m *zeros* of $P_{s,m}(x)$.

With these preliminary remarks, let us consider two families of s-orthogonal polynomials connected to *Chebyshev polynomials of* 3*rd and* 4*th kind* of degree $m = 0, 1, 2, \ldots$. We write such families as

$$\{c_m^* V_m(x)\}, \quad \{c_m^* W_m(x)\}, \quad x \in [-1, 1]\,, \tag{1}$$

where c_m^* is an appropriate normalization factor to be discussed later (see (18)). Let us specify the property of s-orthogonality of the above-mentioned systems of polynomials.

Theorem 1.1. *Polynomials* (1) *orthogonal with respect to the weights* $(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$ *and* $(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ *over the interval* $[-1, 1]$ *are s-orthogonal over the interval* $[-1, 1]$ *with respect to the weights*

$$p^{[1]}(x) = (1-x)^{\frac{1}{2}+s}(1+x)^{-\frac{1}{2}} \quad \text{and} \quad p^{[2]}(x) = (1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}+s} \tag{2}$$

for each integer $s \geq 0$ *(see, e.g.,* [3]*).*

Proof. For Chebyshev polynomials of 3rd and 4th kind given in (1) the formulae

$$(1-x)^s[V_m(x)]^{2s+1} = 2^{-s}\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s-k}V_{m(2k+1)+k}(x) \tag{3}$$

and

$$(1+x)^s[W_m(x)]^{2s+1} = 2^{-s}\sum_{k=0}^{s}\binom{2s+1}{s-k}W_{m(2k+1)+k}(x) \tag{4}$$

hold, if we set $x = \cos\theta$ and take into consideration the relations

$$(\sin\theta)^{2s+1} = 2^{-2s} \sum_{k=0}^{s} (-1)^k \binom{2s+1}{s-k} \sin(2k+1)\theta$$

and

$$(\cos\theta)^{2s+1} = 2^{-2s} \sum_{k=0}^{s} \binom{2s+1}{s-k} \cos(2k+1)\theta,$$

as well as the relations

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}, \qquad \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\,. \tag{5}$$

The proof follows from (3), (4) and the well known relations of orthogonality of the systems $V_m(x)$ and $W_m(x)$ (see also [3]). □

Let us recall now some general formulae which we will use for the so-called "characteristic functions" [8]. Such functions prove to be particularly useful to determine the convergence of quadrature formulae; it is convenient to operate in the complex plane (see [3], [4]; for an easier reference see the formulae in [1]).

Let us begin now to discuss a question of calculating the integral $I(f) = \int_{-1}^{1} p(x)f(x)dx$, where $f(x)$ is a trace in the interval $[-1,1]$ of a function $f(z)$, holomorphic in an open set $A \supset [-1,1]$. Having defined a regular domain $D \subset A$ such that $D \setminus \partial D \supset [-1,1]$, both the *hyper-Gaussian quadrature formula* and the *remainder* of the same formula can be formulated by means of integrals on $+\partial D$; the following relations hold:

$$I(f) = J_{s,m}[f] + R_{s,m}[f]\,,$$

with

$$J_{s,m}[f] = \frac{1}{2\pi i} \int_{+\partial D} f(z)\psi_A(z)dz,$$

having set

$$\psi_A(z) = \sum_{j=1}^{m} \sum_{h=0}^{2s} A_{hj} \frac{h!}{(z-x_{m,j})^{h+1}} \quad \forall\, z \notin \bigcup_{j=1}^{m} \{x_{m,j}\},$$

and with

$$R_{s,m}[f] = \frac{1}{2\pi i} \int_{+\partial D} f(z)\Phi_{s,m}(z)dz\,, \tag{6}$$

where $\Phi_{s,m}(z)$ is the *characteristic function* which is defined by

$$\Phi_{s,m}(z) = \frac{Q_{s,m}(z)}{[P_{s,m}(z)]^{2s+1}} \quad \forall\, z \notin [-1,1], \tag{7}$$

where

$$Q_{s,m} = \int_{-1}^{1} \frac{p(x)[P_{s,m}(x)]^{2s+1}}{z-x}\, dx, \quad m = 1,2,\dots\,, \tag{8}$$

and $P_{s,m}(z)$ denotes the complex expression of $P_{s,m}(x)$.[2]

We assume that ∂D to be one of the confocal ellipse, E_ρ ($\rho > 1$) which have focuses at the ends of the segment $[-1, 1]$, and are identified by the equations

$$z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}) = \frac{1}{2}(\rho + \rho^{-1})\cos\theta + \frac{i}{2}(\rho - \rho^{-1})\sin\theta\,, \tag{9}$$

and have the semiaxes

$$a_\rho = \frac{1}{2}(\rho + \rho^{-1}), \quad b_\rho = \frac{1}{2}(\rho - \rho^{-1})\,, \tag{10}$$

an eccentric angle θ, a focal semidistance $c = \sqrt{a_\rho^2 - b_\rho^2} = 1$.

The equations of E_ρ can also be put in the complex form

$$|z \pm \sqrt{z^2 - 1}| = \rho^{\pm 1}, \tag{11}$$

where the principal value is taken as a root.

For $z \in E_\rho$, we have some basic inequalities

$$|\sqrt{z^2-1}| = \frac{1}{2}|(z + \sqrt{z^2-1}) - (z - \sqrt{z^2-1})| \leq \frac{1}{2}(\rho + \rho^{-1}), \tag{12}$$

$$|z| = \frac{1}{2}|(z + \sqrt{z^2-1}) + (z - \sqrt{z^2-1})| \leq \frac{1}{2}(\rho + \rho^{-1}). \tag{13}$$

We also have:

$$\left|\sqrt{\frac{z \pm 1}{2}}\right| \leq \frac{1}{2}(\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}}) \tag{14}$$

which can be immediately obtained by raising to square and considering (13).

Finally let us add that $|z - x|$, $z \in E_\rho$, $x \in [-1, 1]$ with $E_\rho \cap [-1, 1] = \varnothing$, has an absolute minimum which is obtained when z coincides with a vertex of E_ρ on the major axis and x with the focus near this vertex. Hence it follows that

$$|z - x| \geq a_\rho - 1\,. \tag{15}$$

2. Case in which $P_{s,m}(x) = c_m^* V_m(x)$

Now, putting $x = \cos\theta$, we can calculate the Chebyshev polynomials of third and fourth kind (1):

$$V_m(\cos\theta) = \frac{\sin(2m+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}} = U_{2m}\left(\cos\frac{\theta}{2}\right), \tag{16}$$

$$W_m(\cos\theta) = \frac{\cos(2m+1)\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{1}{\cos\frac{\theta}{2}} T_{2m+1}\left(\cos\frac{\theta}{2}\right), \tag{17}$$

where $U_{2m}(\cdot)$ and $T_{2m+1}(\cdot)$ denote respectively *Chebyshev polynomials of first and second kind.*

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[2](6), (7), (8) correspond to (2.5), (2.8), (2.9) of [1].

(16) and (17) allow us to construct the requested Chebyshev polynomials, s-orthogonal with respect to the weights $p(x)$ (see (2)), having used a "particular normalization"

$$c_m^* = 2^{-m} \tag{18}$$

(it should be noted that $c_m^* V_m = x^m + \dots$ and, likewise, $c_m^* W_m = x^m + \cdots$).

Taking into account the first equality (5), we obtain the *s-orthogonal Chebyshev polynomials of third and fourth kind*:

$$P_{s,m}^{[1]}(x) = 2^{-m} V_{2m}\left(\sqrt{\frac{1+x}{2}}\right), \tag{19}$$

$$P_{s,m}^{[2]}(x) = 2^{-m}\sqrt{\frac{2}{1+x}}\, T_{2m+1}\left(\sqrt{\frac{1+x}{2}}\right). \tag{20}$$

Complex expressions of polynomials which appear in (19)and (20) can be given at once if we replace x by z, whenever necessary.

In the case of polynomials $P_{s,m}^{[1]}(x)$ we will give (Theorem 2.1) an estimate of the remainder $R_{s,m}[f]$ of (6). First we will establish the upper bounds of $|Q_{s,m}(z)|$ of (8), and of $|\Phi_{s,m}(z)|$ of (7).

Lemma 2.1. *For the function $\Phi_{s,m}(z)$ given by (7) the following inequality holds on the family of ellipses E_ρ ($\rho > 1$) given by (9) and (10):*

$$|\Phi_{s,m}^{[1]}(z)| \le 2^{s+2}\frac{1}{a_\rho - 1}\,\frac{(2s)!!}{(2s+1)!!}\left(\frac{\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}}}{\rho^{m+\frac{1}{2}} - \rho^{-m-\frac{1}{2}}}\right)^{2s+1}. \text{ [3]} \tag{21}$$

Proof. Having taken care of (15), from (8) it follows that

$$|Q_{s,m}^{[1]}(z)| \le \frac{1}{a_\rho - 1}\int_{-1}^{1}(1-x)^{\frac{1}{2}+s}(1+x)^{-\frac{1}{2}}|c_m^* V_m(x)|^{2s+1}dx.$$

Putting $x = \cos\theta$ and keeping in mind (16) and (18), we obtain

$$|Q_{s,m}^{[1]}(z)| \le \frac{1}{a_\rho - 1}\int_0^\pi (1-\cos\theta)^{\frac{1}{2}+s}(1+\cos\theta)^{-\frac{1}{2}}\sin\theta\left|2^{-m}\frac{\sin\frac{2m+1}{2}\theta}{\sin\frac{\theta}{2}}\right|^{2s+1} d\theta\,.$$

The above expression is reduced by the use of (5) to

$$|Q_{s,m}^{[1]}(z)| \le \frac{2^{s+1}}{2^{m(2s+1)}}\,\frac{1}{a_\rho - 1}\int_0^\pi \sin\frac{\theta}{2}\left|\sin\frac{2m+1}{2}\,\theta\right|^{2s+1} d\theta\,,$$

and further to

$$|Q_{s,m}^{[1]}(z)| \le \frac{2^{s+1}}{2^{m(2s+1)}}\,\frac{1}{a_\rho - 1}\int_0^\pi \left|\sin\frac{2m+1}{2}\,\theta\right|^{2s+1} d\theta\,. \tag{22}$$

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[3] Analogously to what it has been done in (2), we supply with superscripts [1] and [2] the entities relevant respectively to $P_{s,m}^{[1]}$ and to $P_{s,m}^{[2]}$.

In order to evaluate the integral in (22) let us carry out the substitution $\varphi = \frac{2m+1}{2}\theta$. Then the considered integral becomes

$$\frac{2}{2m+1} \int_0^{(2m+1)\pi/2} |\sin\varphi|^{2s+1} d\varphi\,. \tag{23}$$

Having divided the integration interval $[0,(2m+1)\pi/2]$ into $2m+1$ intervals of length $\pi/2$ and taken into account periodicity of the integrand function, we obtain $\int_0^{(2m+1)\pi/2} |\sin\varphi|^{2s+1} d\varphi = (2m+1)\int_0^{\pi/2} |\sin\varphi|^{2s+1} d\varphi$. Hence after substituting (23) into (22), we have the following upper bound of $|Q^{[1]}_{s,m}(z)|$:

$$|Q^{[1]}_{s,m}(z)| \le \frac{2^{s+2}}{2^{m(2s+1)}}\frac{1}{a_\rho - 1}\int_0^{\pi/2} (\sin\varphi)^{2s+1} d\varphi = \frac{2^{s+2}}{2^{m(2s+1)}\,(a_\rho - 1)}\frac{(2s)!!}{(2s+1)!!}\,. \tag{24}$$

Let us now proceed to proving (21) using (7) and (24). We obtain

$$|\Phi^{[1]}_{s,m}(z)|_{z\in E_\rho} \le \frac{2^{s+2}}{2^{m(2s+1)}}\frac{1}{a_\rho - 1}\frac{(2s)!!}{(2s+1)!!}\frac{1}{\min|P_{s,m}(z)|^{2s+1}_{z\in E_\rho}}\,. \tag{25}$$

Consider now the polynomial $P^{[1]}_{s,m}$ from (19), after using its complex expression and its modulus, i.e., $2^{-m}|V_{2m}(\sqrt{\frac{1+z}{2}})|$.

Recalling the well-known formula

$$V_{2m}(\zeta) = \frac{1}{2\sqrt{\zeta^2-1}}\left[(\zeta+\sqrt{\zeta^2-1})^{2m+1} - (\zeta-\sqrt{\zeta^2-1})^{2m+1}\right] \tag{26}$$

with the complex variable ζ and putting $\zeta^2 = \frac{z+1}{2}$ and $\zeta^2 - 1 = \frac{z-1}{2}$, we have

$$\begin{aligned} |P^{[1]}_{s,m}(z)|_{z\in E_\rho} = 2^{-m-1}\left|\sqrt{\frac{2}{z-1}}\right| \Bigg| &\left(\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right)^{2m+1} \\ &- \left(\sqrt{\frac{z+1}{2}}-\sqrt{\frac{z-1}{2}}\right)^{2m+1}\Bigg|\,. \end{aligned} \tag{27}$$

Now, due to (14), from (27), on account of the following relation for $z \in E_\rho$

$$\left|\sqrt{\frac{z+1}{2}} \pm \sqrt{\frac{z-1}{2}}\right| = \rho^{\pm\frac{1}{2}}, \quad ^{4} \tag{28}$$

we have:

$$|P^{[1]}_{s,m}(z)|_{z\in E_\rho} \ge 2^{-m}\frac{\rho^{m+\frac{1}{2}} - \rho^{-m-\frac{1}{2}}}{\rho^{\frac{1}{2}} + \rho^{-\frac{1}{2}}}\,. \tag{29}$$

Having stated that, from (25) the assertion follows. □

[4]See [4]. On the other hand, (28) can be immediately verified by raising to square and taking into account (8).

Theorem 2.1. *Having fixed an integer s, the following asymptotic property holds with respect to* (6) :

$$\lim_{m\to\infty} R_{s,m}[f] = 0$$

and, more precisely, $\quad R_{s,m}[f] = O(\rho^{-m(2s+1)}), \quad m \to \infty\,.$

Proof. Let us denote by L_ρ the length of the ellipse $E_\rho = \partial D$ and by M_ρ the maximum of $|f(z)|$ on E_ρ. Then (6) and Lemma 2.1 imply

$$|R_{s,m}[f]| \le \frac{1}{\pi}\, L_\rho M_\rho \frac{2^{s+1}}{a_\rho - 1}\,\frac{(2s)!!}{(2s+1)!!}\left(\frac{\rho^{\frac12}+\rho^{-\frac12}}{\rho^{m+\frac12}-\rho^{-m-\frac12}}\right)^{2s+1}. \qquad \square$$

3. Case in which $P_{s,m}(x) = c_m^* P_m W_m(x)$

In this case Lemma 2.1 and Theorem 2.1 are formulated in the same way, with proofs similar to those in Section 2, related however to (20); we will provide the details only concerning those steps which are different in the two cases.

From (2) it follows that

$$|Q_{s,m}^{[2]}(z)| \le \frac{1}{a_\rho - 1}\int_{-1}^{1}(1-x)^{-\frac12}(1+x)^{\frac12+s}|c_m^* W_m(x)|^{2s+1}dx\,.$$

Putting $x = \cos\theta$ in the integral, (5) gives

$$|Q_{s,m}^{[2]}(z)| \le \frac{2^{s+1}}{2^{m(2s+1)}}\,\frac{1}{a_\rho - 1}\int_0^\pi \cos\frac{\theta}{2}\,\left|\cos\frac{2m+1}{2}\,\theta\right|^{2s+1} d\theta\,.$$

Then, repeating the procedure used to find (23), (24), it follows that

$$|Q_{s,m}^{[2]}(z)| \le \frac{2^{s+2}}{2^{m(2s+1)}}\,\frac{1}{a_\rho - 1}\int_0^{\pi/2}(\cos\varphi)^{2s+1}d\varphi = \frac{2^{s+2}}{2^{m(2s+1)}}\,\frac{1}{a_\rho - 1}\,\frac{(2s)!!}{(2s+1)!!}\,.$$

In the case which we are now handling, $|P_{s,m}(z)|$ has to be considered as given by (20), subject to the complex expression and modulus, i.e.,

$$2^{-m}\left|\sqrt{\frac{2}{1+z}}T_{2m+1}\left(\sqrt{\frac{1+z}{2}}\,\right)\right|.$$

We apply the analogous formula of (26), i.e., the well-known formula:

$$T_{2m+1}(\zeta) = \frac12\,[(\zeta+\sqrt{\zeta^2-1})^{2m+1} + (\zeta-\sqrt{\zeta^2-1})^{2m+1}]\,,$$

which for $\zeta = \sqrt{\frac{z+1}{2}}$, due to (20) and with transformations analogous to those of Section 2, gives

$$|P_{s,m}^{[2]}(z)|_{z\in E_\rho} = 2^{-m-1}\left|\frac{2}{\sqrt{z+1}}\right|\,\left|\left[\left(\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right)^{2m+1}\right.\right.$$

$$+\left(\sqrt{\frac{z+1}{2}}-\sqrt{\frac{z-1}{2}}\right)^{2m+1}\Bigg]\Bigg|.$$

Then we obtain a formula analogous to (29)

$$|P^{[2]}_{s,m}(z)|_{z\in E_\rho} \geq 2^{-m}\frac{\rho^{m+\frac{1}{2}}-\rho^{-m-\frac{1}{2}}}{\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}}}\,.$$

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