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PROPER MOVING AVERAGE REPRESENTATIONS AND OUTER FUNCTIONS IN TWO VARIABLES

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Abstract. In this work, we consider the problem of moving average representations for random fields. As in the Kolmogorov–Wiener case, such representations lead to interesting questions in harmonic analysis in the polydisc. In particular, we study outer functions with respect to half-space, semigroup and quarterplane and their interrelations.

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1. Introduction

Professor N. Vakhania has been interested in the area of Prediction Theory and Analysis for infinite dimensional stationary processes, [1]. In this work we study the moving average (**MA**) representations for weakly stationary random fields under half–spaces, semigroups and quarter–planes. We relate the properness of the **MA** representation to the analytic properties of the factor of the spectral density. We interpret the results of Helson and Lowdenslager, [3], in this context, and relate them to the H-outer property. Using a result in [3], one can show the equivalence of the outer property of the factor, and the properness of the semigroup induced **MA** representation. We use this result to relate λ-outer functions (in one variable, with the other variable acting as a parameter), and half-space **MA** representations.

Finally, we take up the quarter-plane **MA** representation and relate its properness to H-outer property of the factor. This can be exploited to obtain a probabilistic result of Soltani, [9]. Connection between outer functions in the sense of [3] and outer functions in $H^2(T^2)$ is used to obtain the analytic results of Izuchi and Matsugu, [5], with very simple proofs. Throughout the paper, we use the standard terminology of the book [2].

2. Half-Space MA Representations

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space, and $L^2(\Omega, \mathcal{F}, P)$ be the complex Hilbert space of (equivalence classes of) P-square integrable complex-valued functions. A family $\left\{X_t, t \in \mathbf{Z}^d\right\} \subset L^2(\Omega, \mathcal{F}, P)$ is called a *weakly stationary random field*

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if $E(X_t) = c$ (from now on $c = 0$), and

$$E(X_t X_{t'}) = \mathcal{R}(t - t').$$

In the case of $d = 1$, $\{X_n, n \in \mathbf{Z}\}$ is called a *weakly stationary process*. With $\{X_n\}$, one associates a spectral measure F through the Bochner Theorem,

$$\mathcal{R}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\lambda}\, dF(\lambda).$$

A stationary process has a **MA** representation if

$$X_n = \sum_{k=0}^{\infty} a_k \xi_{n-k}, \tag{2.1}$$

where $\{\xi_k, k \in \mathbf{Z}\}$ are orthonormal elements of $L^2(\Omega, \mathcal{F}, P)$. It is well known by a result of Kolmogorov and Wiener, that X_n has a **MA** representation if and only if $F \ll$ Leb, and the density is given by $f\left(e^{i\lambda}\right) = \left|\varphi\left(e^{i\lambda}\right)\right|^2$, where $\varphi\left(e^{i\lambda}\right) = \sum_{k=0}^{\infty} \bar{a}_k e^{ik\lambda}$.

From the point of view of prediction, one needs to know when the **MA** representation (2.1) is proper, i.e., when

$$H(X : n) = H(\xi : n), \quad \text{for each } n. \tag{2.2}$$

Here, $H(X : n) = \overline{\text{sp}}\{X_k, k \leq n\}$.

It is obvious that the above condition holds true if and only if $\varphi\left(e^{i\lambda}\right)$ is H-outer, i.e., $\overline{\text{sp}}\left\{e^{in\lambda}\varphi\left(e^{i\lambda}\right) : n \geq 0\right\} = H^2(T)$, where $H^2(T) = \overline{\text{sp}}\left\{e^{in\lambda}, n \geq 0\right\}$, and the closure refers to the space $L^2([-\pi, \pi], \text{Leb})$. An analytic condition for φ to be H-outer is given by

$$\log\left|\int_{-\pi}^{\pi} \varphi\left(e^{i\lambda}\right)\, d\sigma\right| = \int_{-\pi}^{\pi} \log\left|\varphi\left(e^{i\lambda}\right)\right|\, d\sigma,$$

where σ is the normalized Lebesgue measure on $[-\pi, \pi]$.

In case $G = \mathbf{Z}^2$, Helson and Lowdenslager ([3]) considered the analogue of this problem by putting an ordering on $\mathbf{Z}^2$ induced by a semigroup S so that $S \cup -S = \mathbf{Z}^2$, and $S \cap -S = \{(0,0)\}$. A particular example of S is

$$S = \{(j,k) : j \in \mathbf{Z}_+ \text{ for } k = 0 \text{ and } j \in \mathbf{Z} \text{ for } k \geq 1\}. \tag{2.3}$$

They showed that a stationary random field has a **MA** representation, i.e.,

$$X_{m,n} = \sum_{(j,k)\in S} a_{j,k}\xi_{m-j,n-k},$$

with $\{\xi_{j,k}\}_{(j,k)\in\mathbf{Z}^2}$, orthonormal elements in $L^2(\Omega, \mathcal{F}, P)$ if and only if the spectral measure of $\{X_{m,n}\}$ on T^2 is absolutely continuous with respect to σ_2, the normalized Lebesgue measure on T^2, and $f\left(e^{i\lambda}, e^{i\mu}\right) = \left|\varphi\left(e^{i\lambda}, e^{i\mu}\right)\right|^2$. Here $\varphi\left(e^{i\lambda}, e^{i\mu}\right) = \sum_{(j,k)\in S} \bar{a}_{j,k} e^{ij\lambda + ik\mu}$ is of analytic type (in the sense of S).

Using the ordering on $\mathbf{Z}^2$ induced by S, i.e., $(j,k) < (j',k')$ if $(j'-j, k'-k) \in S$, we define

$$H\left(X:(m,n)\right) = \overline{\mathrm{sp}}\left\{X_{j,k} : (j,k) < (m,n) \in \mathbf{Z}^2\right\}.$$

We say that a **MA** S-representation is proper if

$$H\left(X:(m,n)\right) = H\left(\xi:(m,n)\right), \text{ for all } (m,n) \in \mathbf{Z}^2.$$

From the result in [3] we get that a **MA** representation is proper if and only if

$$\log\left|\int\limits_{T^2} \varphi\left(e^{i\lambda}, e^{i\mu}\right) d\sigma_2\right| = \int\limits_{T^2} \log\left|\varphi\left(e^{i\lambda}, e^{i\mu}\right)\right| d\sigma_2, \tag{2.4}$$

where φ is the function of S-analytic type associated with the **MA** S-representation. A function φ of S-analytic type is called *outer* if it satisfies equation (2.4).

Given a set $A \subset \mathbf{Z}^2$, and $g \in L^2\left(T^2, \sigma_2\right)$, denote $[A] := \overline{\mathrm{sp}}\left\{e^{ij\lambda+ik\mu} : (j,k) \in A\right\}$, and $[g]_A := \overline{\mathrm{sp}}\left\{e^{ij\lambda+ik\mu} g : (j,k) \in A\right\}$, closed linear subspaces of $L^2\left(T^2, \sigma_2\right)$. The following result is a consequence of Theorem 6 in [3].

Theorem 2.1. *Let φ be of S-analytic type. Then the following statements are equivalent:*

(i) *φ is outer,*
(ii) *$[\varphi]_S = [S]$.*

Let us denote by $A_\lambda := \mathbf{Z} \times \mathbf{Z}_+$ the half-space in $\mathbf{Z}^2$, and $H^2_\lambda := [A_\lambda]$. The condition (ii) of Theorem 2.1 implies that an outer function φ of S-analytic type satisfies

$$H^2_\lambda = [A_\lambda] = [[S]]_{A_\lambda} = [[\varphi]_S]_{A_\lambda} = [\varphi]_{A_\lambda},$$

giving $[\varphi]_{A_\lambda} = H^2_\lambda$.

Following [5], we define a function $g \in L^2(T^2, \sigma_2)$ as λ-outer if the cut function $g_\lambda\left(e^{i\mu}\right) = g\left(e^{i\lambda}, e^{i\mu}\right)$ is outer in the variable $e^{i\mu}$, σ-a.e. (in $e^{i\lambda}$).

The following lemma is a consequence of a result in [4].

Lemma 2.2. *If $g \in H^2_\lambda$, then g is λ-outer if and only if $[g]_{A_\lambda} = H^2_\lambda$.*

Proof. By Theorem 2 in [4], $[g]_{A_\lambda} = qH^2_\lambda$, where q is a unimodular function, such that the cut function q_λ of q is inner in $e^{i\mu}$. Since $g = qh$, and the cut function of g is outer, we get that q is constant in $e^{i\mu}$, σ-a.e. Thus $qH^2_\lambda = H^2_\lambda$, giving the necessity.

To get the sufficiency, we assume without lost of generality that $g\left(e^{i\lambda}, e^{i\mu}\right)$ is in $H^2\left(T_\mu\right)$ for all λ, and as in [2], for each λ, we define

$$G_r\left(e^{i\lambda}, e^{i\mu}\right) = \exp\left(\int\limits_{T} \frac{e^{i\lambda'} + re^{i\mu}}{e^{i\lambda'} - re^{i\mu}} \log\left|g\left(e^{i\lambda}, e^{i\mu}\right)\right| d\sigma(\lambda')\right).$$

Then for all $e^{i\lambda}$, $\lim_{r\to 1^-} G_r\left(e^{i\lambda}, e^{i\mu}\right) = G\left(e^{i\lambda}, e^{i\mu}\right)$ is an outer factor (in $e^{i\mu}$) of g as a function of $e^{i\mu}$. Since

$$\int_{T^2} |G|^2 d\sigma_2 = \int_{T^2} |g|^2 d\sigma_2 < \infty,$$

the function $G \in L^2\left(T^2, \sigma_2\right)$, we obtain that G is λ-outer. Then $q = g/G$ is inner in $e^{i\mu}$ for all λ. Hence, $[g]_{A_\lambda} = q[G]_{A_\lambda}$. From the proof of the necessity, $[G]_{A_\lambda} = H^2_\lambda$. Thus $H^2_\lambda = [g]_{A_\lambda} = qH^2_\lambda$ and consequently, for each λ, q is constant in μ. Thus $g = qG$ is λ-outer. $\square$

As a consequence, we obtain the following theorem.

Theorem 2.3. *Let g be of S-analytic type. If g is outer, then g is λ-outer.*

Given a function g of S-analytic type, we have

$$g\left(e^{i\lambda}, e^{i\mu}\right) = \sum_{j=0}^{\infty} a_{j,0} e^{ij\lambda} + \sum_{k\geq 1} \sum_{j\in\mathbf{Z}} a_{j,k} e^{ij\lambda + ik\mu}.$$

Let

$$h(e^{i\lambda}) = \sum_{j=0}^{\infty} a_{j,0} e^{ij\lambda}, \tag{2.5}$$

and observe that

$$\int_T g\left(e^{i\lambda}, e^{i\mu}\right) \, d\sigma(\mu) = h\left(e^{i\lambda}\right).$$

Thus if h is outer, then

$$\begin{aligned} \log\left|\int_{T^2} g\left(e^{i\lambda}, e^{i\mu}\right) \, d\sigma_2\right| &= \log\left|\int_T \int_T g\left(e^{i\lambda}, e^{i\mu}\right) \, d\sigma(\mu) d\sigma(\lambda)\right| \\ &= \int_T \log\left|\int_T g\left(e^{i\lambda}, e^{i\mu}\right)\right| \, d\sigma(\lambda). \end{aligned}$$

If further g is λ-outer, then $\log\left|\int_T g\left(e^{i\lambda}, e^{i\mu}\right) \, d\sigma(\mu)\right| = \log \int_T \left|g\left(e^{i\lambda}, e^{i\mu}\right)\right| \, d\sigma(\mu)$. Hence we get

Lemma 2.4. *Let g be of S-analytic type, and h be as in* (2.5). *If h is outer and g is λ-outer, then g is outer.*

3. Quarterplane MA Representations

In [6], a quarterplane **MA** (**QMA**) representation was studied for $G = \mathbf{Z}^2$. Here, one does not have an ordering on $\mathbf{Z}^2$ (except lexicographic). We say that $\{X_{m,n}, (m,n) \in \mathbf{Z}^2\}$ has a **QMA** representation if

$$X_{m,n} = \sum_{k\geq 0}\sum_{j\geq 0} a_{j,k}\xi_{m-j,n-k}, \tag{3.1}$$

where $\{\xi_{j,k}, (j,k) \in \mathbf{Z}^2\}$ is an orthonormal family. We say that a function $\varphi \in H^2(T^2)$ if

$$\varphi\left(e^{i\lambda}, e^{i\mu}\right) = \sum_{j\geq 0}\sum_{k\geq 0} b_{j,k}e^{ij\lambda+ik\mu}. \tag{3.2}$$

We note that $\varphi \in H^2(T^2)$ implies that φ is of S-analytic type. It is easy to verify that $\{X_{m,n}\}$ has the **QMA** representation (3.1) if and only if its spectral measure $F \ll \sigma_2$, and the density $f\left(e^{i\lambda}, e^{i\mu}\right) = \left|\varphi\left(e^{i\lambda}, e^{i\mu}\right)\right|^2$, where $\varphi \in H^2(T^2)$ and

$$\varphi\left(e^{i\lambda}, e^{i\mu}\right) = \sum_{j,k\geq 0} \bar{a}_{j,k}e^{ij\lambda+ik\mu}.$$

A **QMA** representation is proper if and only if

$$H\left(X : (m,n)\right) = \overline{\text{sp}}\left\{X_{j,k} : j \leq m, k \leq n\right\} = H\left(\xi : (m,n)\right).$$

For any second order random field $\{y_{m,n} : (m,n) \in \mathbf{Z}^2\}$, we denote

$$\begin{aligned} L^1(y:m) &= \overline{\text{sp}}\left\{y_{j,k} : j \leq m, k \in \mathbf{Z}\right\} \\ \text{and}\quad L^2(y:n) &= \overline{\text{sp}}\left\{y_{j,k} : j \in \mathbf{Z}, k \leq n\right\}. \end{aligned}$$

Also, let $p^i(y:m)$ be the projection onto $L^i(y:m)$, $i = 1,2$. We shall drop the dependence on y when it is clear from the context. We observe that for $\{\xi_{j,k}, (j,k) \in \mathbf{Z}^2\}$,

$$p^1(m)p^2(n) = p(m,n), \tag{3.3}$$

where $p(m,n)$ is the projection on $H\left(\xi : (m,n)\right)$. Thus we obtain that if the **QMA** representation (3.1) is proper, i.e., $H\left(X : (m,n)\right) = H\left(\xi : (m,n)\right)$, then for the process $\{X_{m,n} : (m,n) \in \mathbf{Z}^2\}$, equality (3.3) holds true. This condition was introduced in [6]. Denote, as in [6], $L^i(X : -\infty) = \bigcap_m L^i(X : m)$, $i = 1,2$. It was further proved in [6] that under condition (3.3), and the condition:

$$\overline{\text{sp}}\left\{L^1(X:-\infty) \cup L^2(X:-\infty)\right\} = \{0\}, \tag{3.4}$$

a weakly stationary random field has a proper **QMA** representation.

We observe

Theorem 3.1. *The **QMA** representation* (3.1) *for* $\{X_{m,n}\}$ *is proper if and only if the following three conditions are satisfied:*

(i) *the spectral measure,* F*, of* $\{X_{m,n}\}$*, satisfies* $F \ll \sigma_2$*,*

(ii) *the density* $f\left(e^{i\lambda}, e^{i\mu}\right) = \left|\varphi\left(e^{i\lambda}, e^{i\mu}\right)\right|^2$*, with* $\varphi \in H^2(T^2)$*,*

(iii) $[\varphi]_{\mathbf{Z}_+^2} = H^2(T^2)$.

From the above theorem, we obtain the following result of Soltani [9].

Theorem 3.2. *A second order stationary random field has a proper **QMA** representation if and only if it satisfies conditions* (3.3), (3.4), *and* (i)–(iii) *of Theorem* 3.1.

Since $\varphi \in H^2(T^2)$, with $[\varphi]_{\mathbf{Z}_+^2} = H^2(T^2)$ (H-outer for two variables), implies that $[\varphi]_S = \left[[\varphi]_{\mathbf{Z}_+^2}\right]_S = [H^2(T^2)]_S = [S]$, we get that φ is outer. This was originally proved in [2]. A counter example to the converse of the above statement was provided in [8]. In [7], a necessary and sufficient condition was given for the equivalence of the properties for a function to be H-outer and outer. Thus, in Theorem 3.1, we cannot replace the condition (iii) by the requirement that φ be outer.

We note that if $g \in H^2(T^2)$, then g is of S-analytic type. One can ask whether one can improve Lemma 2.4 under the assumption that $g \in H^2(T^2)$. Let us observe that, with h as in (2.5), we have

$$g\left(e^{i\lambda}, e^{i\mu}\right) = h\left(e^{i\lambda}\right) + e^{i\mu}\tilde{h}\left(e^{i\lambda}, e^{i\mu}\right),$$

where $\tilde{h} \in H^2(T^2)$. If g is outer, then, with $H^2(T_\lambda) = \overline{\mathrm{sp}}\left\{e^{ij\lambda} : j \in \mathbf{Z}_+\right\}$ in $L^2(T^2, \sigma_2)$, we get

$$H^2(T_\lambda) \subset [S] = [g]_S = \overline{\mathrm{sp}}\left\{e^{ij\lambda}h\left(e^{i\lambda}\right)\right\} \oplus M,$$

with $M \perp H^2(T_\lambda)$. Since $h \in H^2(T_\lambda)$, we obtain

$$H^2(T_\lambda) = \overline{\mathrm{sp}}\left\{e^{ij\lambda}h\left(e^{i\lambda}\right), j \geq 0\right\}.$$

This shows that h is H-outer in T_λ or, equivalently, that h is outer in T. In combination with Lemma 2.4 we obtain the following result from [5], with a simple proof.

Theorem 3.3. *Let $g \in H^2(T^2)$; then g is outer if and only if h, defined in* (2.5) *is outer in T, and g is λ-outer.*

Remark 3.4. One can derive other results in [5] with simple variations of the above arguments.

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