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ON THE TAIL ESTIMATION OF THE NORM OF RADEMACHER SUMS *

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Abstract. The main aim of this paper is to prove a bilateral inequality for $P\big[\|\sum_1^n a_k r_k\| > t\big]$, where $t > 0$, (a_k) are elements of a normed space, while (r_k) are Rademacher functions. Then this inequality is applied for estimation of $E\|\sum_1^n a_k r_k\|$. As another corollary we give a maximal inequality for exchangeable random variables that recently has been published in [4].

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1. The Main Assertion

Theorem. *Let $(a_1, \dots, a_{2n})$ be a collection of elements of a normed space X (real or complex), $r_1, \dots, r_{2n}$ be Rademacher functions (i.i.d. random variables taking only two values -1 and $+1$ with equal probabilities) defined on a probability space $(\Omega, \mathcal{A}, P)$.*

(a) *If $\sum_1^{2n} a_k = 0$, then for any $t > 0$ the following bilateral inequality holds*

$$c\, m\Big[\|a_{k_1} + \cdots + a_{k_n}\| > \frac{t}{c}\,\Big] \;\le\; P\Big[\Big\|\sum_1^{2n} a_k r_k\Big\| > t\Big]$$
$$\le C\, m\Big[\|a_{k_1} + \cdots + a_{k_n}\| > \frac{t}{C}\,\Big], \qquad (1)$$

where c, C are absolute constants, while m is the uniform distribution on the set of all collections of integers $1 \le k_1 < k_2 < \cdots < k_n \le 2n$.

(b) *The right-hand side inequality takes place in general, without the assumption $\sum_1^{2n} a_k = 0$ (may be with a different absolute constant C).*

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* A general source for this article is the reference [4]; its first author is Professor N. Vakhania to whom this work is dedicated.

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Proof. First let us prove the left-hand side inequality of (a). Denote by μ the uniform distribution on the set Π of all permutations of $\{1,\ldots,2n\}$. Then we have for any collection of signs $\theta=(\theta_1,\ldots,\theta_n)$

$$\mu\left[\Big\|\sum_1^n a_{\pi(k)}\Big\|>t\right]=\mu\left[\Big\|\sum_1^n (a_{\pi(k)}-a_{\pi(n+k)})\theta_k\Big\|>2t\right]$$

(since $\sum\limits_1^{2n} a_i=0$). This gives

$$\mu\left[\Big\|\sum_1^n a_{\pi(k)}\Big\|>t\right]=P\times\mu\left[\Big\|\sum_1^n (a_{\pi(k)}-a_{\pi(n+k)})r_k\Big\|>2t\right]$$
$$\le P\times\mu\left[\Big\|\sum_1^n a_{\pi(k)}r_k\Big\|>t\right]+P\times\mu\left[\|\sum_1^n a_{\pi(n+k)}r_k\Big\|>t\right]$$

(due to the Lévy inequality and Fubini theorem)

$$P\times\mu\left[\Big\|\sum_1^n a_{\pi(k)}r_k\Big\|>t\right]\;+\;P\times\mu\left[\Big\|\sum_1^n a_{\pi(k)}r_k\Big\|>t\right]$$
$$\le 4P\times\mu\left[\Big\|\sum_1^{2n} a_{\pi(k)}r_k\Big\|>t\right].$$

Now the use of the obvious equality

$$\mu\left[\Big\|\sum_1^n a_{\pi(k)}\Big\|>t\right]\;=\;m\left[\Big\|\sum_1^n a_{k_l}\Big\|>t\right],\quad t>0,$$

proves the left-hand side inequality of (1) with $c=\frac{1}{4}$.

To prove the right-hand side inequality of (a), denote

$$\xi(\omega)=\sum_1^{2n} a_k r_k(\omega)\;,\quad \eta(\pi,\omega)=\sum_1^n (a_{\pi(2k-1)}-a_{\pi(2k)})r_{\pi(2k-1)}(\omega)\;,\tag{2}$$

where $\pi:\{1,\ldots,2n\}\to\{1,\ldots,2n\}$ is a permutation. One can easily verify that

$$\frac{1}{(2n)!}\sum_\pi \eta(\pi,\omega)=\frac{n}{2n-1}\xi(\omega)\,.\tag{3}$$

As above, μ denotes the uniform distribution on the set Π of all permutations. Then (3) means that $E_\mu\,\eta(\pi,\omega)=\frac{n}{2n-1}\,\xi(\omega)$, where E_Q denotes the expectation w.r.t. the probability measure Q; we omit the index when Q is assumed.

Let us exploit the following Kahane inequality for a positive random variable Y:

$$P\,[Y>\lambda EY]\;\ge\;(1-\lambda)^2\frac{(EY)^2}{EY^2}\,,$$

where $0 < \lambda < 1$ is any real (see [3, Ch. I]). Then for any λ we can write

$$\mu\Big\{\pi : \|\eta(\pi,\omega)\| > \lambda\frac{n}{2n-1}\|\xi(\omega)\|\Big\} \geq \mu\Big\{\pi : \|\eta(\pi,\omega)\| > \lambda E_\mu\|\eta(\pi,\omega)\|\Big\}$$
$$\geq \frac{(1-\lambda)^2(E_\mu\,\|\eta(\pi,\omega)\|)^2}{E_\mu\,\|\eta(\pi,\omega)\|^2} \geq \frac{(1-\lambda)^2\,\|\xi(\omega)\|^2}{4E_\mu\,\|\eta(\pi,\omega)\|^2}\,. \tag{4}$$

On the other hand, for any $s > 0$, by the Markov inequality,

$$P\Big\{\omega : (E_\mu\|\eta(\pi,\omega)\|^2)^{1/2} I_{[\|\xi(\omega)\|>t]} > s\Big\}$$
$$\leq \frac{(E_P E_\mu\,\|\eta\|^2)^{1/2} P^{1/2}[\|\xi(\omega)\| > t]}{s}\,. \tag{5}$$

According to the definition of $\eta(\pi,\omega)$ (see (2)), and the triangle inequality,

$$(E_P\,\|\eta(\pi,\omega)\|^2)^{1/2} \leq \Big(E_P\,\Big\|\sum_1^n a_{\pi(2k-1)} r_{\pi(2k-1)}\Big\|^2\Big)^{1/2}$$
$$+\Big(E_P\,\Big\|\sum_1^n a_{\pi(2k)} r_{\pi(2k)}\Big\|^2\Big)^{1/2}$$

(by the Levy inequality)

$$\leq 2\Big(2E_P\Big\|\sum_1^{2n} a_k r_k\Big\|^2\Big)^{1/2},$$

which results in

$$E_P\,\|\eta(\pi,\omega)\|^2 \leq 8E_P\,\Big\|\sum_1^{2n} a_k r_k\Big\|^2.$$

Now we can go on in (5):

$$\leq \frac{1}{s}\,8(E\|\xi\|^2)^{1/2}\,P^{1/2}[\|\xi\| > t]\,. \tag{6}$$

Assume that $E\|\xi\|^2 = 1$ and put $s = c_0 P^{-1/2}[\|\xi\| > t]$. Then (6) gives

$$P\Big[(E_\mu\|\eta(\pi,\omega)\|^2)^{1/2} > s\,;\|\xi(\omega)\| > t\Big] \leq \frac{8}{c_0}P\,[\|\xi\| > t]\,.$$
$$P\Big[(E_\mu\|\eta(\pi,\omega)\|^2)^{1/2} \leq s\,;\|\xi(\omega)\| > t\Big] \geq P\,[\|\xi(\omega)\| > t\Big]$$
$$-\frac{8}{c_0}\,P\,[\|\xi(\omega)\| > t] = \Big(1-\frac{8}{c_0}\Big)P\,[\|\xi(\omega)\| > t]\,. \tag{7}$$

Let us now introduce the event

$$A = \Big\{\omega : E_\mu\|\eta(\pi,\omega)\|^2 \leq s\,;\|\xi(\omega)\| > t\Big\}\,.$$

According to (7),

$$P(A) \geq \Big(1-\frac{8}{c_0}\Big)P[\|\xi(\omega)\| > t]\,,$$

and according to (4) we have, for any $\omega \in A$,

$$\mu\Big\{\pi : \|\eta(\pi,\omega)\| > \frac{\lambda t}{2}\Big\} \ \geq\ \frac{(1-\lambda)^2 t^2}{2s^2} \ =\ \frac{(1-\lambda)^2 t^2}{2c_0^2} P[\|\xi\| > t]\,.$$

Integrating both sides with respect to ω we get

$$P \times \mu\left\{(\pi,\omega) : \|\eta(\pi,\omega)\| > \frac{\lambda t}{2}\right\} \ \geq\ \frac{(1-\lambda)^2 t^2}{2c_0^2} P^2[\|\xi\| > t]$$

(using another Kahane inequality $P[\|\sum a_k r_k\| > 2t] \ \leq\ 4P^2[\|\sum a_k r_k\| > t]$ from [2, Ch. 2])

$$\geq\ C_1(1-\lambda)^2 t^2 P[\|\xi\| > 2t]\,,$$

where $C_1 = \frac{1}{8}(1-\frac{8}{c_0})\frac{1}{c_0^2} \leq \frac{1}{3756}$.

The last inequality gives us that for each $t > 0$, $0 < \lambda < 1$,

$$\begin{aligned} P[\|\xi\| > t] &\leq \frac{C_2}{(1-\lambda)^2 t^2} P \times \mu\left[\|\eta\| > \frac{\lambda t}{4}\right] \\ &= \frac{C_2}{(1-\lambda)^2 t^2}\, m\left[\|a_{k_1} + \cdots + a_{k_n}\| > \frac{\lambda t}{8}\right], \end{aligned} \tag{8}$$

where C_2 is an absolute constant (e.g., $C_2 = 3756 \times 4 = 15024$ fits). The inequality is fine for large t' s. We have to get rid of t^{-2} for small t' s. Fix any $0 < t_0 < 1$. Then (8) implies that for any $t > t_0$

$$P \times \mu\left[\|\eta\| > \frac{\lambda t}{2}\right] \ \geq\ \frac{(1-\lambda)^2 t_0^2}{C_2}\, P[\|\xi\| > t]. \tag{9}$$

Now let $t \leq t_0$ and use the Kahane inequality to show that

$$\begin{aligned} P \times \mu\left[\|\eta\| > tE\|\eta(\pi,\omega)\|\right] \ &\geq\ (1-t)^2 \frac{(E\|\eta(\pi,\omega)\|)^2}{E\|\eta(\pi,\omega)\|^2} \\ &\geq (1-t)^2 \Big(\frac{n}{2n-1}\Big)^2 \frac{(E\|\xi\|)^2}{4E\|\xi\|^2} \end{aligned} \tag{10}$$

(the Khinchine–Kahane inequality $\frac{E\|\xi\|}{(E\|\xi\|^2)^{1/2}} \geq 2^{-1/2}$, see [3])

$$\geq\ \frac{1}{32}(1-t_0)^2 P[\|\xi\| > t]\,.$$

Further, for left-hand side in (10) we have

$$\begin{aligned} P \times \mu\left[\|\eta\| > tE\|\eta(\pi,\omega)\|\right] \ &\leq\ P \times \mu\left[\|\eta\| > t\frac{n}{2n-1}E\|\xi\|\right] \\ \leq P \times \mu\Big[\|\eta\| > \frac{t}{2}\frac{E\|\xi\|}{(E\|\xi\|^2)^{1/2}}\Big] \ &\leq\ P \times \mu\left[\|\eta\| > \frac{t}{2\sqrt{2}}\right]. \end{aligned}$$

So, for any $t \leq t_0 < 1$

$$P[\|\xi\| > t] \ \leq\ \frac{32}{(1-t_0)^2} P \times \mu\left[\|\eta\| > \frac{t}{2\sqrt{2}}\right]. \tag{11}$$

Now (9) and (11) imply the existence of an absolute constant C such that the inequality

$$P\Big[\|\xi\| > t\Big] \le CP \times \mu\Big[\|\eta\| > \frac{2t}{C}\Big] = Cm\left[\|a_{k_1} + \cdots + a_{k_n}\| > \frac{t}{C}\right]$$

holds true for any $t > 0$.

So, inequality (1) is proved under the condition that $E\|\xi\|^2 = 1$. For a general collection $(a_1, \dots, a_{2n}) \subset X$ we apply (1) to $(a_1/(E\|\xi\|^2)^{1/2}, \dots, a_{2n}/(E\|\xi\|^2)^{1/2})$ and use the fact that (1) is true for any $t > 0$. The part (a) of the Theorem is proved.

Proof of part (b). Let $(a_1, \dots, a_{2n}) \subset X$ be an arbitrary collection. In order to reduce the issue to a), consider the collection $(b_1, \dots, b_{2n})$, where $b_k = a_k - \bar{a}$, $\bar{a} = \frac{1}{2n}(a_1 + \cdots + a_{2n})$. Since $\sum_1^{2n} b_k = 0$, we can write according to (a):

$$\begin{aligned}
P &\left[\Big\|\sum_1^{2n} a_k r_k\Big\| - \Big\|\sum_1^{2n} a_k\Big\| > t\right] \\
&\le Cm\left[\|a_{k_1} + \cdots + a_{k_n}\| + \frac{1}{2}\|a_1 + \cdots + a_{2n}\| > \frac{t}{C}\right] \\
&\le Cm\Bigg[\|a_{k_1} + \cdots + a_{k_n}\| + \frac{1}{2}\|a_{k_1} + \cdots + a_{k_n}\| \\
&\quad + \frac{1}{2}\|a_{k'_1} + \cdots + a_{k'_n}\| > \frac{t}{C}\Bigg] \\
&\le C\left(m\left[3\,\|a_{k_1} + \cdots + a_{k_n}\| > \frac{t}{C}\right] + m\left[\|a_{k'_1} + \cdots + a_{k'_n}\| > \frac{t}{C}\right]\right) \\
&\le 2C\, m\left[\|a_{k_1} + \cdots + a_{k_n}\| > \frac{t}{3C}\right].
\end{aligned}$$

So that as a new absolute constant in (b) one can choose $3C$. □

2. Corollaries

Estimates for moments of the norm of Rademacher sums.

Corollary 1. *Let $(a_1, \dots, a_{2n})$ be any collection of a normed space X, $\Phi : R^+ \to R^+$ be any continuous increasing function, $\Phi(0) = 0$. Then*

(a)

$$E\Phi\left(\Big\|\sum_1^{2n} a_k r_k\Big\|\right) \le C\frac{(n!)^2}{(2n)!}\sum_{k_1<\cdots<k_n} \Phi(C\|a_{k_1} + \cdots + a_{k_n}\|),$$

where C is an absolute constant.

(b) *If $\sum_1^{2n} a_k = 0$, then*

$$c\,\frac{(n!)^2}{(2n)!}\sum_{k_1<\cdots<k_n} \Phi(c\|a_{k_1} + \cdots + ak_n\|) \le E\Phi\left(\Big\|\sum_1^{2n} a_k r_k\Big\|\right),$$

where c is another absolute constant.

Proof. (a). According to the part (b) of the Theorem and the integration by parts formula,

$$E\,\Phi\Big(\Big\|\sum_1^{2n} a_k r_k\Big\|\Big) = \int_0^\infty P\Big[\Big\|\sum_1^{2n} a_k r_k\Big\| > t\Big]\,\mu\Phi(t)$$

$$\leq \int_0^\infty Cm\Big[\|a_{k_1} + \cdots + a_{k_n}\| > \frac{t}{C}\Big]\,d\Phi(t)$$

$$= CE\Phi(C\|a_{k_1} + \cdots + a_{k_n}\|) = C\frac{(n!)^2}{(2n)!}\sum_{k_1<\cdots<k_n}\Phi(C\|a_{k_1} + \cdots + a_{k_n}\|)\,.$$

The part (b) can be derived from Theorem (a) in a similar way. □

A maximal inequality for exchangeable systems of random variables. Let X be a separable Banach space. A system $(\xi_1, \ldots, \xi_n)$ of X-valued random variables is called exchangeable if for any permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ the distribution of $(\xi_{\pi(1)}, \ldots, \xi_{\pi(n)})$ is the same as that of $(\xi_1, \ldots, \xi_n)$. Concerning the issues of probability in a Banach space the reader is referred to [5].

In [1] we have proved the following maximal inequality for exchangeable systems of random variables with $\sum\limits_1^n \xi_k = 0$:

$$cP\Big[\Big\|\sum \xi_k r_k\| > \frac{t}{c}\Big] \leq P\Big[\max_{k\leq n}\|\xi_1 + \cdots + \xi_k\| > t\Big] \leq CP\Big[\Big\|\sum \xi_k r_k\Big\| > \frac{t}{C}\Big], \quad (12)$$

where (r_n) is a system of Rademacher r.v.'s independent of ξ's, $t > 0$ is arbitrary, c and C are absolute constants.

As before μ stands for the uniform distribution on the set Π of all permutations $\pi : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$.

Given a collection $(a_1, \ldots, a_{2n}) \subset X$, we can introduce the following system of exchangeable random variables $\xi_k : \Pi \to X :\quad \xi_k(\pi) = a_{\pi(k)}\,,\quad k = 1, \ldots, 2n$. If $\sum\limits_1^{2n} a_k = 0$, then the right-hand side of (1) for this system takes the form

$$\mu\Big[\max_{k\leq 2n}\|a_{\pi(1)} + \cdots + a_{\pi(k)}\| > t\Big] \leq CP\Big[\Big\|\sum_1^{2n} a_k r_k\Big\| > \frac{t}{C}\Big]$$

(and using Theorem)

$$\leq C_1\, m\Big[\|a_{k_1} + \cdots + a_{k_n}\| > \frac{t}{C_1}\Big] = C_1\mu\Big[\|a_{\pi(1)} + \cdots + a_{\pi(n)}\| > \frac{t}{C_1}\Big]. \quad (13)$$

Corollary 2 (Pruss, [4]).

(a) *Let* $(a_1, \ldots, a_{2n})$ *be any collection of vectors of a normed space* X (*not necessarily with* $\sum\limits_1^{2n} a_k = 0$). *Then for any* $t > 0$

$$\mu\Big[\max_{k\leq 2n}\|a_{\pi(1)} + \cdots + a_{\pi(k)}\| > t\Big] \leq C\,\mu\Big[\|a_{\pi(1)} + \cdots + a_{\pi(n)}\| > \frac{t}{C}\Big] \quad (14)$$

for some absolute constant C.

(b) *For any exchangeable system $(\xi_1, \dots, \xi_{2n})$ of X-valued random variables and any $t > 0$,*

$$P\Big[\max_{k\le 2n} \|\xi_1 + \dots + \xi_k\| > t\Big] \le CP\Big[\|\xi_1 + \dots + \xi_n\| > \frac{t}{C}\Big],$$

where C is an absolute constant.

Proof. According to (13), (a) holds true when $\sum_1^{2n} a_k = 0$. Now let $(a_1, \dots, a_{2n})$ be an arbitrary collection. Then for the new collection $(b_1, \dots, b_{2n})$, $b_k = a_k - \bar{a}$, $k = 1, \dots, 2n$, (14) is applicable, i.e.,

$$\begin{aligned}
&\mu\left[\max_{k\le 2n} \|(a_{\pi(1)} - \bar{a}) + \dots + (a_{\pi(k)} - \bar{a})\| > t\right] \\
&\quad\le C\,\mu\Big[\Big\|a_{\pi(1)} + \dots + a_{\pi(n)} - (a_1 + \dots + a_{2n})\frac{1}{2}\Big\| > \frac{t}{C}\Big] \\
&\quad\le C\,\mu\Big[\Big\|\frac{1}{2}(a_{\pi(1)} + \dots + a_{\pi(n)})\Big\| + \|a_{\pi(n+1)} + \dots + a_{\pi(2n)}\| > \frac{t}{C}\Big] \\
&\quad\le C\Big(\mu\Big[\|a_{\pi(1)} + \dots + a_{\pi(n)}\| > \frac{t}{C}\Big] + \mu\Big[\|a_{\pi(n+1)} + \dots + a_{\pi(2n)}\| > \frac{t}{2C}\Big)\\
&\quad\le 2C\mu\Big[\|a_{\pi(1)} + \dots + a_{\pi(n)}\| > \frac{t}{2C}\Big]. \qquad (15)
\end{aligned}$$

Let us transform the left-hand side of (15)

$$\begin{aligned}
&\mu\Big[\max_{k\le 2n} \|(a_{\pi(1)} - \bar{a}) + \dots + (a_{\pi(k)} - \bar{a})\| > t\Big] \\
&\ge \mu\Big[|\max_{k\le 2n} \|a_{\pi(1)} + \dots a_{\pi(k)}\| - \|a_1 + \dots + a_{2n}\|| > t\Big] \\
&\ge \mu\Big[\max_{k\le 2n} \|a_{\pi(1)} + \dots + a_{\pi(k)}\| > t\Big] - \mu\Big[\|a_1 + \dots + a_{2n}\| > \frac{t}{2}\Big] \\
&\ge \mu\Big[\max_{k\le 2n} \|a_{\pi(1)} + \dots + a_{\pi(k)}\| > t\Big] \\
&- \mu\Big[\|a_{\pi(1)} + \dots + a_{\pi(n)}\| > \frac{t}{4}\Big] - \mu\Big[\|a_{\pi(n+1)} + \dots + a_{\pi(2n)}\| > \frac{t}{4}\Big].
\end{aligned}$$

This along with (15) proves (a).

(b) We have

$$P\Big[\max_{k\le 2n} \|\xi_1 + \dots + \xi_k\| > t\Big]$$

(due to the exchangeability)

$$= P \times \mu\Big[\max_{k\le 2n} \|\xi_{\pi(1)} + \dots + \xi_{\pi(k)}\| > t\Big]$$

(due to the part (a))

$$\le CP \times \mu\Big[\|\xi_{\pi(1)} + \dots + \xi_{\pi(n)}\| > \frac{t}{C}\Big]$$

(due to the Fubini theorem and exchangeability)

$$= CP\Big[\|\xi_1 + \cdots + \xi_n\| > \frac{t}{C}\Big]. \qquad \square$$

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