

**ON THE INTEGRABILITY OF THE ERGODIC HILBERT
TRANSFORM FOR A CLASS OF FUNCTIONS WITH
EQUAL ABSOLUTE VALUES**

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ABSTRACT. It is proved that for an arbitrary non-atomic finite measure space with a measure-preserving ergodic transformation there exists an integrable function f such that the ergodic Hilbert transform of any function equal in absolute values to f is non-integrable.

Let T be a one-to-one measure-preserving ergodic transformation of a finite measure space (X, \mathbb{S}, μ) . For a real integrable function f , $f \in L(X)$, the ergodic Hilbert transform is defined by

$$H(f)(x) = \sum'_{k=-\infty}^{\infty} \frac{f(T^k x)}{k}, \quad x \in X$$

(the mark ' indicates that the 0-th term in the sum is omitted). It exists for almost all $x \in X$ (in the sense of measure μ) and, furthermore, the operator H satisfies a weak-type $(1, 1)$ inequality

$$\mu(|H(f)| > \lambda) \leq \frac{c}{\lambda} \|f\|_1, \quad \lambda > 0, \quad f \in L(X), \quad (1)$$

where c is an absolute constant (see, e.g., [2]).

It is well known that the condition $f \in L \lg L$ implies the integrability of $H(f)$ (see[1]) but, in general, it is rather difficult to give by means of any integral classes an exact description of the set of functions f for which $H(f) \in L(X)$. Therefore the theorem below seems to be interesting for investigation of this set.

Theorem. *Let (X, \mathbb{S}, μ) be a non-atomic finite measure space, and let T be a one-to-one measure-preserving ergodic transformation on X . Then*

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there exists $f \in L(X)$ such that for any function g with the same absolute values, $|g| = |f|$, we have

$$H(g) \notin L(X).$$

Analogous questions as a general problem of “kernel and hull” were formulated by Prof. O. Tsereteli (see [3],[4]). He also proved the following theorem on the conjugate operator: for any integrable function f on the unit circle there exists a function g with the same absolute values such that \tilde{g} is integrable (see [5]).

The proof of the following lemma is simple and we give it for the sake of completeness.

Lemma. *Let T be a one-to-one measure-preserving ergodic transformation of a non-atomic finite measure space (X, \mathbb{S}, μ) . Then there exists a set A of positive measure, $\mu(A) > 0$, such that*

$$\mu\left(X \setminus \bigcup_{k=0}^n T^k A\right) > 0 \quad (2)$$

for all positive integers n .

Proof. Let us construct successively a sequence of imbedded measurable sets $(A_n)_{n=0}^\infty$. Let A_0 be a set of positive not full measure, i.e., $\mu(A_0) > 0$ and the inequality

$$\mu\left(X \setminus \bigcup_{k=0}^n T^k A_n\right) > 0 \quad (3)$$

holds for $n = 0$. Assume now that $A_0 \supset A_1 \supset \dots \supset A_n$ have already been chosen, $\mu(A_n) > 0$, and inequality (3) holds for A_n . Choose $A_n^0 \subset X \setminus \bigcup_{k=0}^n T^k A_n$ such that

$$0 < \mu(A_n^0) < \frac{\mu(A_0)}{2^{n+2}},$$

and assume $A_{n+1} = A_n \setminus T^{-(n+1)} A_n^0$. Then

$$A_n^0 \subset X \setminus \left(\bigcup_{k=0}^{n+1} T^k A_{n+1}\right). \quad (4)$$

Let

$$A = \bigcap_{n=0}^{\infty} A_n.$$

Obviously,

$$\begin{aligned}
 \mu(A) &= \mu(A_0) - \sum_{n=0}^{\infty} \mu(A_n \setminus A_{n+1}) \geq \\
 &\geq \mu(A_0) - \sum_{n=0}^{\infty} \mu(T^{-(n+1)}A_n^0) = \mu(A_0) - \sum_{n=0}^{\infty} \mu(A_n^0) \geq \\
 &\geq \mu(A_0) - \sum_{n=0}^{\infty} \frac{\mu(A_0)}{2^{n+2}} = \frac{\mu(A_0)}{2} > 0,
 \end{aligned}$$

and (2) also holds for each positive integer n because of (4). \square

Proof of the Theorem. Let A be a set of positive measure with property (2), and let us represent the space X as a skyscraper construction with the base A , i.e., assume

$$X = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^n T^k B_n,$$

where $\bigcup_{n=0}^{\infty} B_n = A$ and $T^i B_n \cap T^j B_m = \emptyset$ when $n \neq m$, $0 \leq i \leq n, 0 \leq j \leq m$. Because of (2), there exists arbitrarily large n for which $\mu(B_n) > 0$.

Let us choose successively an increasing sequence of positive integers m_n , for which $\mu(B_{m_n}) > 0$, and decreasing to 0 sequences δ_n and ε_n , $n = 1, 2, \dots$, which satisfy the inequalities

$$\begin{aligned}
 \delta_1 &< \frac{1}{6}, \quad \sum_{n=1}^{\infty} \varepsilon_n < 1, \\
 \delta_n \cdot \max_{1 \leq i \leq n-1} \left(\frac{\varepsilon_i}{\mu(B_{m_i})} \right) &< \frac{1}{4}, \\
 \varepsilon_n \cdot \log(\delta_n \cdot m_n) &> 3, \\
 \frac{\varepsilon_n}{\mu(B_{m_n})} \cdot c \sum_{k=n+1}^{\infty} \varepsilon_k &< 1, \quad n = 1, 2, \dots,
 \end{aligned} \tag{5}$$

where c is the constant appearing in inequality (1). After a suitable choice of ε_n , δ_{n-1} and m_{n-1} one can take δ_n small enough, m_n large enough and ε_{n+1} small enough again to satisfy the second, the third, and the fourth inequalities, respectively.

To simplify our expressions let us use the following notations: $\bar{n} = \lfloor \frac{m_n}{2} \rfloor$ ($\lfloor a \rfloor$ is the nearest integer less than or equal to a) and $\beta_n = \mu(B_{m_n})$. Define f_n , $n = 1, 2, \dots$, by the equalities

$$f_n(x) = \begin{cases} \frac{\varepsilon_n}{\beta_n}, & \text{when } x \in T^{\bar{n}} B_{m_n} \\ 0, & \text{when } x \in X \setminus T^{\bar{n}} B_{m_n} \end{cases}.$$

Note that $\|f_n\| = \varepsilon_n$. Thus the function f ,

$$f = \sum_{n=1}^{\infty} f_n,$$

is integrable and let us show that it has the property required in the theorem.

Let g be any function equal in absolute value to f . Assume

$$g = \sum_{n=1}^{\infty} g_n,$$

where $|g_n| = f_n$, $n = 1, 2, \dots$. Partial sums of this series will be denoted by h_n ,

$$h_n = \sum_{k=1}^n g_k.$$

The relations below are assumed to hold for almost all x , in particular, for those x for which the Hilbert transform of the corresponding functions exists.

Temporarily, suppose n to be fixed. Let us show that the inequality

$$|H(h_n)(T^i x) - H(h_n)(T^{-i} x)| \geq \frac{2\varepsilon_n}{i\beta_n} - \frac{12\varepsilon_n}{m_n\beta_n} - 8\delta_n \|h_{n-1}\|_{\infty}, \quad (6)$$

$x \in T^{\bar{n}}B_{m_n}$, holds for each i , $0 < i < \delta_n m_n$. Indeed, because of the linearity of the Hilbert transform,

$$\begin{aligned} & |H(h_n)(T^i x) - H(h_n)(T^{-i} x)| \geq \\ & \geq |H(g_n)(T^i x) - H(g_n)(T^{-i} x)| - |H(h_{n-1})(T^i x) - H(h_{n-1})(T^{-i} x)| = \\ & = S_1 - S_2. \end{aligned}$$

Taking into account that $T^k x \in T^{\bar{n}}B_{m_n}$ implies $T^{k+j} x \notin T^{\bar{n}}B_{m_n}$, $j = 1, 2, \dots, m_n$, we have

$$\begin{aligned} S_1 &= \left| -\frac{g_n(x)}{i} + \sum'_{k \neq -i} \frac{g_n(T^{k+i} x)}{k} - \frac{g_n(x)}{i} - \sum'_{k \neq i} \frac{g_n(T^{k-i} x)}{k} \right| = \\ &= \left| -\frac{2g_n(x)}{i} + \sum_{|k| > m_n/2} \left(\frac{g_n(T^{k+i} x)}{k} - \frac{g_n(T^{k+i} x)}{k+2i} \right) \right| \geq \\ &\geq \frac{2f_n(x)}{i} - \sum_{|k| > m_n/2} f_n(T^{k+i} x) \left(\frac{1}{k} - \frac{1}{k+2i} \right) \geq \\ &\geq \frac{2f_n(x)}{i} - 2f_n(x) \sum_{k=0}^{\infty} \left(\frac{1}{m_n/2 + km_n} - \frac{1}{m_n/2 + km_n + 2i} \right) \geq \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{2f_n(x)}{i} - \frac{4if_n(x)}{m_n^2} \sum_{k=0}^{\infty} \frac{1}{(1/2+k)^2} \geq \frac{2f_n(x)}{i} - \frac{12\delta_n f_n(x)}{m_n} = \\
 &= \frac{2\varepsilon_n}{i\beta_n} - \frac{12\delta_n \varepsilon_n}{m_n \beta_n}
 \end{aligned}$$

and

$$\begin{aligned}
 S_2 &= \left| \sum_{|k| > (1-\delta_n)m_n} \left(\frac{h_{n-1}(T^{k+i}x)}{k} - \frac{h_{n-1}(T^{k+2i}x)}{k+2i} \right) \right| \leq \\
 &\leq 2\|h_{n-1}\|_{\infty} \sum_{k > (1-\delta_n)m_n} \left(\frac{1}{k} - \frac{1}{k+2i} \right) \leq 4i\|h_{n-1}\|_{\infty} \sum_{k > (1-\delta_n)m_n} \frac{1}{k^2} \leq \\
 &\leq 4\delta_n m_n \|h_{n-1}\|_{\infty} \frac{1}{(1-\delta_n)m_n - 1} \leq 8\delta_n \|h_{n-1}\|_{\infty}.
 \end{aligned}$$

Thus (6) holds and therefore the inequality

$$\begin{aligned}
 &\max(|H(h_n)(T^i x)|, |H(h_n)(T^{-i} x)|) \geq \\
 &\geq \frac{\varepsilon_n}{i\beta_n} - \frac{6\delta_n \varepsilon_n}{m_n \beta_n} - 4\delta_n \|h_{n-1}\|_{\infty} \equiv \eta_n(i)
 \end{aligned} \tag{7}$$

is true.

Denote by the same letter η_n the function on X defined by the equalities: $\eta_n(T^i x) = \eta_n(|i|)$ whenever $x \in T^{\bar{n}}B_{m_n}$ and $|H(h_n)(T^i x)| \geq \eta_n(|i|)$, $0 < |i| < \delta_n m_n$; $\eta_n(x) = 0$ otherwise.

Obviously,

$$|H(h_n)| \geq \eta_n.$$

Let $E_n = \bigcup_{\bar{n}-\delta_n m_n < i < \bar{n}+\delta_n m_n} T^i B_{m_n}$ and

$$M_n = \left\{ x \in X : \left| H \left(\sum_{k=n+1}^{\infty} g_k \right) \right| > 1 \right\}.$$

Because of inequality (1),

$$\mu(M_n) \leq c \cdot \left\| \sum_{k=n+1}^{\infty} g_k \right\| = c \cdot \sum_{k=n+1}^{\infty} \varepsilon_k.$$

Taking into account (7), we have

$$\begin{aligned}
 \int_{E_n} |H(g)| d\mu &\geq \int_{E_n} \left| |H(h_n)| - \left| H \left(\sum_{k=n+1}^{\infty} g_k \right) \right| \right| d\mu \geq \\
 &\geq \int_{E_n \setminus M_n} (|H(h_n)| - 1) d\mu \geq \int_{E_n \setminus M_n} \eta_n d\mu - \mu(E_n) \geq
 \end{aligned}$$

$$\begin{aligned}
&\geq \int_{E_n} \eta_n d\mu - \|\eta_n\|_\infty \mu(M_n) - \mu(E_n) \geq \\
&\geq \sum_{i=1}^{[\delta_n m_n]} \eta_n(i) \beta_n - \frac{\varepsilon_n}{\beta_n} \mu(M_n) - \mu(E_n) \geq \\
&\geq \varepsilon_n \log(\delta_n m_n) - \frac{6\delta_n \varepsilon_n [\delta_n m_n]}{m_n} - 4\delta_n \|h_{n-1}\|_\infty \mu(E_n) - \\
&- \frac{\varepsilon_n}{\beta_n} c \sum_{k=n+1}^{\infty} \varepsilon_k - \mu(E_n).
\end{aligned}$$

Thus, by virtue of (5),

$$\int_{E_n} |H(g)| d\mu \geq 1 - 2\mu(E_n),$$

and since E_n , $n = 1, 2, \dots$, are pairwise disjoint, we can conclude that

$$\|H(g)\| = \infty. \quad \square$$

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