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NUMERICAL SOLUTIONS TO THE DARBOUX PROBLEM WITH THE FUNCTIONAL DEPENDENCE

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Abstract. The paper deals with the Darboux problem for the equation $D_{xy}z(x,y) = f(x,y,z_{(x,y)})$ where $z_{(x,y)}$ is a function defined by $z_{(x,y)}(t,s) = z(x+t,y+s)$, $(t,s) \in [-a_0,0] \times [-b_0,0]$. We construct a general class of difference methods for this problem. We prove the existence and uniqueness of solutions to implicit functional difference equations by means of a comparison method; moreover we give an error estimate. The convergence of explicit difference schemes is proved under a general assumption that given functions satisfy nonlinear estimates of the Perron type. Our results are illustrated by a numerical example.

§ 1. Introduction

Given any two metric spaces X and Y, we denote by $C(X,Y)$ the class of all continuous functions from X into Y. Take $a, b > 0$ and $a_0, b_0 \in R_+$, $R_+ = [0, +\infty)$. Define

$$E = (0,a] \times (0,b], \quad E^0 = \big([-a_0,a] \times [-b_0,b]\big) \backslash E$$

and $B = [-a_0,0] \times [-b_0,0]$. Given a function $z : E^0 \cup E \to R$ and a point $(x,y) \in E$, we define the function $z_{(x,y)} : B \to R$ by

$$z_{(x,y)}(t,s) = z(x+t,y+s), \quad (t,s) \in B. \tag{1}$$

Suppose that $f : \overline{E} \times C(B,R) \to R$ and $\phi : E^0 \to R$ are given functions. (Here $\overline{E}$ is the closure of E.) Consider the Darboux problem

$$D_{xy}z(x,y) = f(x,y,z_{(x,y)}), \quad (x,y) \in E, \tag{2}$$

$$z(x,y) = \phi(x,y) \quad \text{for} \quad (x,y) \in E^0, \tag{3}$$

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where $D_{xy}z = \frac{\partial^2 z}{\partial x \partial y}$. We consider classical solution to problem (2), (3). A function $v \in C(E^0 \cup E, R)$ is regarded as a solution of (2), (3) if $D_{xy}v$ exists on E, $D_{xy}v \in C(E, R)$, and v satisfies (2), (3). Sufficient conditions for the existence and uniqueness of a solution to (2), (3) are given in [1], see also [2].

For a few recent years, a certain number of papers concerning numerical methods for functional partial differential equations have been published.

Difference methods for nonlinear parabolic functional differential problems were considered in [3]–[5]. The main problem in these investigations is to find a difference approximation which is stable and satisfies consistency conditions with respect to the original problem. A method of difference inequalities or simple theorems on linear recurrent inequalities are used in the investigation of stability.

The semidiscretizations in space variables of linear parabolic Volterra integral-differential equations (the method of lines, Galerkin or collocation techiques) lead to large systems of stiff ordinary integral-functional equations. The analysis of spatial and temporal disretizations of linear integral-functional parabolic problems has received considerable attention during the last years [6]–[11]. Most of these contributions seem to focus on the convergence theory. There are very few numerical studies.

Difference methods for first order functional differential equations with initial or initial-boundary conditions were studied in [12], [13]. The proofs of the convergence were based either on functional difference inequalities or on a general theorem on error estimates for approximate solutions to functional difference equations of the Volterra type with initial or initial-boundary conditions and with unknown function of several variables.

The convergence of difference methods for functional hyperbolic systems in the Schauder canonic form was studied in [14].

For further biblioghaphical information concerning numerical methods for functional partial differential equations we suggest to see the survey papers [15] and [13].

The paper is organized as follows. In Section 2 we construct a general class of difference schemes for (2), (3). This leads to implicit functional difference problems. The existence and uniqueness of solutions to such problems are considered in Section 3. The comparison method of investigation of functional difference equations is used. The next section deals with a theorem on the convergence of explicit difference schemes with nonlinear estimates for given functions. We assume that increment functions satisfy nonlinear estimates of the Perron type with respect to the functional variable. In Section 5, we establish an error estimate for implicit difference methods. We give a numerical example.

Differential equations with a deviated argument and integral-differential problems can be obtained from (2), (3) by a specification of given operators.

§ 2. Discretization

Given any two sets X and Y, we denote by $F[X,Y]$ the class of all functions defined on X and taking values in Y. We will denote by $\mathbf{N}$ and $\mathbf{Z}$ the sets of natural numbers and integers, respectively.

We construct a mesh in $E^0 \cup E$ in the following way. Suppose that $(h,k) \in (0,a] \times (0,b]$ stand for steps of the mesh. Write

$$x_i = ih, \quad y_j = jk, \quad x_{i+\frac{1}{2}} = ih + \frac{h}{2}, \quad y_{j+\frac{1}{2}} = jk + \frac{k}{2}\ , \ \ i,j \in \mathbf{Z}.$$

Denote by I_0 the set of all $(h,k) \in (0,a] \times (0,b]$ such that there exist $M_0, N_0 \in \mathbf{N}$ such that $M_0 h = a_0$, $N_0 k = b_0$. We assume that $I_0 \neq \varnothing$ and there is a sequence $\{(h_n,k_n)\}$, $(h_n,k_n) \in I_0$, such that $\lim_{n\to\infty}(h_n,k_n) = (0,0)$. For $(h,k) \in I_0$ we put $Z_{hk} = \{(x_i,y_j) : i,j \in \mathbf{Z}\}$, and $E^0_{hk} = Z_{hk} \cap E^0$, $E_{hk} = Z_{hk} \cap E$. There are $M, N \in \mathbf{N}$ such that $Mh \leq a < (M+1)h$, $Nk \leq b < (N+1)k$. Let

$$A_{hk} = \big\{(x_i,y_j) :\ 0 \leq i \leq M-1,\ 0 \leq j \leq N-1\big\}.$$

Using the above definitions of M and N, we have

$$E_{hk} = \big\{(x_i,y_j) :\ 1 \leq i \leq M,\ 1 \leq j \leq N\big\}.$$

Let $K, L \in \mathbf{Z}$ be fixed and assume that $-M_0 \leq K \leq 1$, $-N_0 \leq L \leq 1$. Write

$$D_{hk} = \big\{(x_i,y_j) :\ -M_0 \leq i \leq K,\ -N_0 \leq j \leq L\big\}.$$

For $z \in F[E^0_{hk} \cup E_{hk}, R]$ we write $z^{(i,j)} = z(x_i,y_j)$, $(x_i,y_j) \in E^0_{hk} \cup E_{hk}$. In the same way we define $w^{(i,j)}$ for $w \in F[D_{hk}, R]$.

We will need a discrete version of the restriction operator given by (1). If $z : E^0_{hk} \cup E_{hk} \to R$ and $0 \leq i \leq M-1$, $0 \leq j \leq N-1$ then the function $z_{[i,j]} : D_{hk} \to R$ is defined as follows:

$$z_{[i,j]}(t,s) = z(x_i + t, y_j + s), \quad (t,s) \in D_{hk}.$$

We consider the difference operator δ given by

$$\delta z^{(i,j)} = \frac{1}{hk}\big[z^{(i+1,j+1)} - z^{(i+1,j)} - z^{(i,j+1)} + z^{(i,j)}\big].$$

Suppose that

$$F_{hk} : A_{hk} \times F[D_{hk}, R] \to R, \quad \phi_{hk} : E^0_{hk} \to R$$

are given functions. Consider the problem

$$\delta z^{(i,j)} = F_{hk}(x_i, y_j, z_{[i,j]}), \quad (x_i,y_j) \in A_{hk}, \tag{4}$$

$$z^{(i,j)} = \phi^{(i,j)}_{hk} \quad \text{for} \quad (x_i,y_j) \in E^0_{hk}. \tag{5}$$

Remark 2.1. If $K = 1$ and $L = 1$ then problem (4), (5) turns out to be an implicit difference method. If $K \le 0$ or $L \le 0$, then problem (4), (5) represents a simple functional difference equation of the Volterra type. It is obvious that in this case there exists exactly one solution $u_{hk} : E^0_{hk} \cup E_{hk} \to R$ of (4), (5).

Example 1. Suppose that $F : \overline{E} \times R \to R$ and $\varphi : [0,a] \to R$, $\psi : [0,b] \to R$ are given functions. We assume that $\varphi(0) = \psi(0)$. Put $a_0 = 0$, $b_0 = 0$ and $f(x,y,w) = F(x,y,w(0,0))$, $(x,y,w) \in \overline{E} \times C(B,R)$. Then (2), (3) reduces to the classical Darboux problem

$$D_{xy} z(x,y) = F(x,y,z(x,y)), \quad (x,y) \in E, \tag{6}$$

$$z(x,0) = \varphi(x) \ \text{ for } \ x \in [0,a], \quad z(0,y) = \psi(y) \ \text{ for } \ y \in [0,b]. \tag{7}$$

One of the implicit difference schemes for (6), (7) takes the form

$$\delta z^{(i,j)} = F\Big(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, \frac{1}{4}\big(z^{(i,j)} + z^{(i+1,j)} + z^{(i,j+1)} + z^{(i+1,j+1)}\big)\Big), \qquad (x_i, y_j) \in A_{hk},$$

$$z^{(i,0)} = \varphi(x_i) \ \text{ for } \ 0 \le i \le M, \quad z^{(0,j)} = \psi(y_j) \ \text{ for } \ 0 \le j \le N.$$

The most natural explicit difference method for (6), (7) takes the form

$$\delta z^{(i,j)} = F(x_i, y_j, z^{(i,j)}), \quad (x_i, y_j) \in A_{hk},$$

with the above boundary condition.

Example 2. Suppose that $a_0 < 0$, $b_0 < 0$. For the same F we put

$$f(x,y,w) = F\Big(x, y, \int_B w(t,s)\,dt\,ds\Big), \quad (x,y,w) \in \overline{E} \times C(B,R).$$

Then problem (2), (3) is equivalent to the integral-differential equation

$$D_{xy} z(x,y) = F\Big(x, y, \int_B z(x+t, y+s)\,dt\,ds\Big), \quad (x,y) \in E, \tag{8}$$

with boundary condition (3). Now, we construct an explicit difference method for (8), (3). Let

$$B_{hk} = \big\{(x_i, y_j) : \ -M_0 \le i \le 0, \ -N_0 \le j \le 0, \big\}.$$

We define the operator $T_{hk} : F[B_{hk}, R] \to F[B, R]$ in the following way. Suppose that $w \in F[B_{hk}, R]$ and $(t,s) \in B$. Then there is $(x_i, y_j) \in B_{hk}$

such that $i < 0$, $j < 0$ and $x_i \le t \le x_{i+1}$, $y_j \le s \le y_{j+1}$. We put

$$T_{hk}w(t,s) = w^{(i,j)}\Big[1 - \frac{t-x_i}{h}\Big]\Big[1 - \frac{s-y_j}{k}\Big] + \\ + w^{(i+1,j+1)}\,\frac{t-x_i}{h}\,\frac{s-y_j}{k} + \\ + w^{(i+1,j)}\,\frac{t-x_i}{h}\Big[1 - \frac{s-y_j}{k}\Big] + \\ + w^{(i,j+1)}\Big[1 - \frac{t-x_i}{h}\Big]\frac{s-y_j}{k}\,. \tag{9}$$

It is easy to see that $T_{hk} : F[B_{hk}, R] \to C(B, R)$. We assume that $D_{hk} = B_{hk}$, i.e., $K = 0$, $L = 0$. We will approximate solutions to problem (8), (3) by means of solutions to the equation

$$\delta z^{(i,j)} = F\Big(x_i, y_j, \int_B T_{hk} z_{[i,j]}(t,s)dt\,ds\Big), \quad (x_i, y_j) \in A_{hk}, \tag{10}$$

with boundary condition (5).

Example 3. Suppose that

$$F : \overline{E} \times R \to R, \quad \varphi : \overline{E} \to R, \quad \psi : \overline{E} \to R, \quad \phi : E^0 \to R$$

are given functions, and

$$-a_0 \le \varphi(x,y) - x \le 0, \quad -b_0 \le \psi(x,y) - y \le 0 \quad \text{for } (x,y) \in \overline{E}.$$

Let

$$f(x,y,w) = F\big(x, y, w(\varphi(x,y) - x, \psi(x,y) - y)\big), \quad (x,y,w) \in \overline{E} \times C(B,R).$$

Then problem (2), (3) reduces to the differential equation with a deviated argument

$$D_{xy}z(x,y) = F\big(x, y, z(\varphi(x,y), \psi(x,y))big), \quad (x,y) \in E,$$

with boundary condition (3). It is easy to construct difference methods for the above equation using the ideas from Example 2. Further examples of the operator F_{hk} are given in Sections 4 and 5.

§ 3. Existence and Uniqueness of Solutions of Functional Difference Problems

The existence and uniqueness of a solution of problem (4), (5) is investigated by the comparison method. This method is based on the association of the operator F_{hk} with another operator σ_{hk}, which is followed by thorough analysis of a comparison equation. If the latter equation possesses adequate properties, then problem (4), (5) has exactly one solution which is the limit of a sequence of successive approximations. We obtain the simplest case of the operator σ_{hk} corresponding to equation (4) if the function F_{hk} satisfies the Lipschitz condition. The comparison problem is linear in this case.

The following property of the operator δ is important:

Lemma 3.1. *Problem* (4)*,* (5) *is equivalent to*

$$z^{(i,j)} = hk\sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} F_{hk}(x_\mu, y_\nu, z_{[\mu,\nu]}) + \phi_{hk}^{(i,0)} + \phi_{hk}^{(0,j)} - \phi_{hk}^{(0,0)}, \qquad (11)$$

$$1 \le i \le M, \ \ 1 \le j \le N,$$

$$z^{(i,j)} = \phi_{hk}^{(i,j)} \ \ \text{for} \ \ (x_i, y_j) \in E_{hk}^0. \qquad (12)$$

We omit the simple proof of the lemma.

Let $\eta_{hk} : E_{hk}^0 \cup E_{hk} \to R$ be a function given by

$$\eta_{hk}(x,y) = \phi_{hk}(x,y) \ \text{ for } \ (x,y) \in E_{hk}^0 \ \text{ and } \ \eta_{hk}(x,y) = 0 \ \text{ for } \ (x,y) \in E_{hk}.$$

We define a sequence $\{z_n\}$, $z_n : E_{hk}^0 \cup E_{hk} \to R$, in the following way:

$$z_0^{(i,j)} = \eta_{hk}^{(i,j)} \quad \text{for} \quad (x_i, y_j) \in E_{hk}^0 \cup E_{hk}, \qquad (13)$$

and

$$z_{n+1}^{(i,j)} = hk\sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} F_{hk}(x_\mu, y_\nu, (z_n)_{[\mu,\nu]}) + \phi_{hk}^{(i,0)} + \phi_{hk}^{(0,j)} - \phi_{hk}^{(0,0)}, \quad (14)$$

$$1 \le i \le M, \quad 1 \le j \le N,$$

$$z_{n+1}^{(i,j)} = \phi_{hk}^{(i,j)} \quad \text{for} \quad (x_i, y_j) \in E_{hk}^0, \qquad (15)$$

where $n = 0, 1, 2, \ldots$.

We prove that, under suitable assumptions on the function F_{hk}, the sequence $\{z_n\}$ converges to the unique solution of problem (4), (5).

For $w \in F(D_{hk}, R)$ we put

$$\|w\|_{hk} = \max\left\{|w^{(i,j)}| : \ (x_i, y_j) \in D_{hk}\right\}.$$

For the above w we define the function $|w|_{hk} : D_{hk} \to R_+$ by

$$|w|_{hk}(x,y) = |w(x,y)|, \quad (x,y) \in D_{hk}.$$

We will consider a comparison function $\sigma_{hk} : A_{hk} \times F[D_{hk}, R_+] \to R_+$ coresponding to the function F_{hk}.

Assumption H$_1$. Suppose that

1^0 for each $(x,y) \in A_{hk}$ the function $\sigma_{hk}(x,y,\cdot) : F[D_{hk}, R_+] \to R_+$ is nondecreasing, and $\sigma_{hk}(x,y,\Theta_{hk}) = 0$ for $(x,y) \in A_{hk}$, where $\Theta_{hk}(x,y) = 0$ for $(x,y) \in D_{hk}$,

2^0 for $(x,y,w) \in A_{hk} \times F[D_{hk},R]$, $\overline{w} \in F[D_{hk},R]$ we have

$$|F_{hk}(x,y,w) - F_{hk}(x,y,\overline{w})| \le \sigma_{hk}(x,y,|w-\overline{w}|_{hk}), \tag{16}$$

3^0 there exists a function $\overline{g}_{hk} : E^0_{hk} \cup E_{hk} \to R_+$ which is a solution to the problem

$$\omega^{(i,j)} \ge hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu, y_\nu, \omega_{[\mu,\nu]}) + \overline{\eta}^{(i,j)}_{hk}, \tag{17}$$
$$1 \le i \le M, \quad 1 \le j \le N,$$

$$\omega^{(i,j)} = |\phi^{(i,j)}_{hk}| \quad \text{for} \quad (x_i,y_j) \in E^0_{hk}, \tag{18}$$

with the function $\overline{\eta}_{hk}$ satisfying the condition

$$\overline{\eta}^{(i,j)}_{hk} \ge hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \big|F_{hk}(x_\mu, y_\nu, (\overline{\eta}_{hk})_{[\mu,\nu]})\big| + $$
$$+ |\phi^{(i,0)}_{hk}| + |\phi^{(0,j)}_{hk}| + |\phi^{(0,0)}_{hk}|, \tag{19}$$

where $1 \le i \le M$, $1 \le j \le N$,

4^0 the function $\omega(x,y) = 0$, $(x,y) \in E^0_{hk} \cup E_{hk}$, is the unique solution to the problem

$$\omega^{(i,j)} = hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu, y_\nu, \omega_{[\mu,\nu]}), \quad 1 \le i \le M, \quad 1 \le j \le N, \tag{20}$$

$$\omega^{(i,j)} = 0 \quad \text{for} \quad (x_i,y_j) \in E^0_{hk} \tag{21}$$

in the class of functions satisfying the condition $0 \le \omega(x,y) \le \overline{g}_{hk}(x,y)$, $(x,y) \in E_{hk}$.

Theorem 3.2. *If Assumption H_1 is satisfied then there exists a solution $\overline{z} : E^0_{hk} \cup E_{hk} \to R$ of problem* (4), (5)*. The solution is unique in the class of functions $z : E^0_{hk} \cup E_{hk} \to R$ satisfying the condition*

$$|z(x,y)| \le \overline{g}_{hk}(x,y) \quad \textit{for} \quad (x,y) \in E_{hk}.$$

Proof. Consider the sequence $\{g_n\}$, $g_n : E^0_{hk} \cup E_{hk} \to R_+$ given by

$$g_0 = \overline{g}_{hk},$$

and

$$g^{(i,j)}_{n+1} = hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu, y_\nu, (g_n)_{[\mu,\nu]}), \quad 1 \le i \le M, \quad 1 \le j \le n, \tag{22}$$

$$g^{(i,j)}_{n+1} = 0 \quad \text{for} \quad (x_i,y_j) \in E^0_{hk}, \tag{23}$$

where $n = 0, 1, 2, \dots$. We prove that

$$g_{n+1}(x,y) \le g_n(x,y) \quad \text{for} \quad (x,y) \in E_{hk}, \ \ n = 0,1,2,\dots, \tag{24}$$

$$\lim_{n\to\infty} g_n(x,y) = 0, \quad (x,y) \in E_{hk}, \tag{25}$$

$$|z_n(x,y)| \le \overline{g}_{hk}(x,y), \quad (x,y) \in E_{hk}, \ \ n = 0,1,2,\dots, \tag{26}$$

$$|z_{n+r}(x,y) - z_n(x,y)| \le g_n(x,y), \ \ (x,y) \in E_{hk}, \ \ n, r = 0,1,2,\dots. \tag{27}$$

It follows from condition 3^0 of Assumption H_1 that $g_1(x,y) \le g_0(x,y)$ for $(x,y) \in E_{hk}$. Assume that, for fixed $n \in \mathbf{N}$, we have $g_n(x,y) \le g_{n-1}(x,y)$, $(x,y) \in E_{hk}$. It follows from the monotonicity of σ_{hk} with respect to the functional variable and from (22) that $g_{n+1}(x,y) \le g_n(x,y)$ for $(x,y) \in E_{hk}$. Then we have (24) by induction on $n \in \mathbf{N}$.

Since $0 \le g_n(x,y)$ for $(x,y) \in E_{hk}$, $n \in \mathbf{N}$, there exists

$$\overline{g}(x,y) = \lim_{n\to\infty} g_n(x,y), \quad (x,y) \in E_{hk}.$$

It follows from (22), (23) that $\overline{g}$ is a solution to problem (20), (21). Condition 4^0 of Assumption H_1 implies $\overline{g}(x,y) = 0$ for $(x,y) \in E_{hk}$.

Now we prove (26). This relation is obvious for $n = 0$. If we assume that $|z_n(x,y)| \le \overline{g}_{hk}(x,y)$, $(x,y) \in E_{hk}$, then we deduce from (16), (19) and the monotonicity of σ_{hk} that

$$\begin{aligned}
|z_{n+1}^{(i,j)}| &\le hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \big|F_{hk}(x_\mu, y_\nu, (z_n)_{[\mu,\nu]}) - F_{hk}(x_\mu, y_\nu, (\eta_{hk})_{[\mu,\nu]})\big| + \\
&+ hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \big|F_{hk}(x_\mu, y_\nu, (\eta_{hk})_{[\mu,\nu]})\big| + |\phi_{hk}^{(i,0)}| + |\phi_{hk}^{(0,j)}| + |\phi_{hk}^{(0,0)}| \le \\
&\le hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \sigma_{hk}\big(x_\mu, y_\nu, (\overline{g}_{hk})_{[\mu,\nu]}\big) + \\
&+ |\phi_{hk}^{(i,0)}| + |\phi_{hk}^{(0,j)}| + |\phi_{hk}^{(0,0)}| + \overline{\eta}_{hk}^{i,j)} \le \overline{g}_{hk}^{(i,j)},
\end{aligned}$$

where $1 \le i \le M$, $1 \le j \le N$; and it is seen that inequality (26) is obtained by induction on $n \in \mathbf{N}$.

We prove (27) by induction on $n \in \mathbf{N}$. Estimate (27) for $n = 0$ follows from (26). If we assume that for fixed $n \in \mathbf{N}$ we have

$$|z_{n+r}^{(i,j)} - z_n^{(i,j)}| \le g_n^{(i,j)}, \quad (x_i, y_j) \in E_{hk}, \ \ r = 0,1,2,\dots,$$

then, applying (14), (16), (22) and the monotonicity of σ_h, we get

$$\big|z_{n+r+1}^{(i,j)} - z_{n+1}^{(i,j)}\big| \le hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \big|F_{hk}(x_\mu, y_\nu, (z_{n+r})_{[\mu,\nu]}) -$$

$$
\begin{aligned}
&- F_{hk}(x_\mu, y_\nu, (z_n)_{[\mu,\nu]})\big| \le \\
&\le hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \sigma_{hk}\big(x_\mu, y_\nu, |(z_{n+r})_{[\mu,\nu]} - (z_n)_{[\mu,\nu]}|_{hk}\big) \le \\
&\le hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu, y_\nu, (g_n)_{[\mu,\nu]}) = g_{n+1}^{(i,j)},
\end{aligned}
$$

which completes the proof of assertion (27).

It follows from (25), (27) that there is $\overline{z} : E^0_{hk} \cup E_{hk} \to R$ such that

$$\overline{z}(x,y) = \lim_{n\to\infty} z_n(x,y), \quad (x,y) \in E^0_{hk} \cup E_{hk}.$$

Relations (14), (15) imply the function $\overline{z}$ is a solution to problem (11), (12).

Suppose that $\overline{u} : E^0_{hk} \cup E_{hk} \to R$ is another solution to problem (11), (12), and that $|\overline{u}(x,y)| \le \overline{g}_{hk}(x,y)$ for $(x,y) \in E_{hk}$. Then we obtain by induction on $n \in \mathbf{N}$ the relation

$$|\overline{u}(x,y) - z_n(x,y)| \le g_n(x,y) \quad \text{for} \quad (x,y) \in E_{hk}, \ \ n = 0,1,2,\dots .$$

It follows from (25) that $\overline{u} = \overline{z}$, which completes the proof of Theorem 1.2. □

Now, we prove a result on the global uniqueness of solution to (11), (12).

Lemma 3.3. *Suppose that Assumption H_1 is satisfied and the function $\overline{\omega}(x,y) = 0$ for $(x,y) \in E^0_{hk} \cup E_{hk}$ is the only solution of the problem*

$$\omega^{(i,j)} \le hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu, y_\nu, \omega_{[\mu,\nu]}), \quad 1 \le i \le M, \ \ 1 \le j \le N, \tag{28}$$

$$\omega^{(i,j)} = 0 \quad \textit{for} \quad (x_i, y_j) \in E^0_{hk}. \tag{29}$$

Then the solution $\overline{z} : E^0_{hk} \cup E_{hk} \to R$ to problem (11), (12) is unique.

Proof. If $\overline{z}$, $\overline{u} : E^0_{hk} \cup E_{hk} \to R$ are solutions to (11), (12), then $\widetilde{\omega} = \overline{z} - \overline{u}$ satisfies (28), (29), and the assertion follows. □

Now, we give sufficient conditions for the uniqueness of the solution $\overline{\omega} = 0$ to problem (28), (29).

Lemma 3.4. *Suppose that the function σ_{hk} satisfies the conditions:*

1^0 *for each function* $\lambda_{hk} : E_{hk} \to R_+$ *there exists a solution to the problem*

$$\omega^{(i,j)} \geq hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu, y_\nu, \omega_{[\mu,\nu]}) + \lambda_{hk}^{(i,j)}, \qquad (30)$$

$$1 \leq i \leq M, \quad 1 \leq j \leq N,$$

$$\omega^{(i,j)} = 0 \quad \textit{for} \quad (x_i, y_j) \in E_{hk}^0, \qquad (31)$$

2^0 *the function* $\overline{\omega} = 0$ *is a unique solution to problem* (20), (21).

Under these assumptions, the function $\overline{\omega}(x, y) = 0$, $(x, y) \in E_{hk}^0 \cup E_{hk}$, *is the only solution to problem* (28), (29).

Proof. Suppose that $\widetilde{\omega} : E_{hk}^0 \cup E_{hk} \to R_+$ is a solution to problem (28), (29). Consider the sequence $\{\omega_n\}$, $\omega_n : E_{hk}^0 \cup E_{hk} \to R_+$ given by

(i) ω_0 is a solution of (30), (31) for $\lambda_{hk} = \widetilde{\omega}$,

(ii) if ω_n is a given function then

$$\omega_{n+1}^{(i,j)} = hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu, y_\nu, (\omega_n)_{[\mu,\nu]}), \quad 1 \leq i \leq M, \quad 1 \leq j \leq N, \quad (32)$$

$$\omega_{n+1}^{(i,j)} = 0 \quad \text{for} \quad (x_i, y_j) \in E_{hk}^0. \qquad (33)$$

We obtain

$$\widetilde{\omega}(x, y) \leq \omega_n(x, y) \quad \text{for} \quad (x, y) \in E_{hk}, \quad n = 0, 1, 2, \dots,$$

$$0 \leq \omega_{n+1}(x, y) \leq \omega_n(x, y) \quad \text{for} \quad (x, y) \in E_{hk}, \quad n = 0, 1, 2, \dots.$$

The above relations can be proved by induction on $n \in \mathbf{N}$.

Let $\overline{\omega} : E_{hk}^0 \cup E_{hk} \to R_+$ be defined by

$$\overline{\omega}(x, y) = \lim_{n \to \infty} \omega_n(x, y).$$

It follows from (32), (33) that $\overline{\omega} = 0$. Since $\widetilde{\omega} \leq \overline{\omega}$, the assertion follows. □

§ 4. Convergence of Difference Methods with Nonlinear Estimates for Increment Functions

In this section we consider the particular case of the set D_{hk}. We assume that $K = 0$, $L = 0$. We will use the following comparison lemma:

Lemma 4.1. *Suppose that* $K = 0$ *and* $L = 0$ *in the definition of* D_{hk} *and*

1^0 *the function* $G_{hk} : A_{hk} \times F[B_{hk}, R] \to R$ *is non-decreasing with respect to the functional variable,*

2^0 *the functions* $u, v : E^0_{hk} \cup E_{hk} \to R$ *satisfy the relations*

$$u^{(i,j)} - hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} G_{hk}(x_\mu, y_\nu, u_{[\mu,\nu]}) \leq$$
$$\leq v^{(i,j)} - hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} G_{hk}(x_\mu, y_\nu, v_{[\mu,\nu]}),$$

whenever $1 \leq i \leq M$, $1 \leq j \leq N$; *and* $u^{(i,j)} \leq v^{(i,j)}$ *on* E^0_{hk}.

Then $u^{(i,j)} \leq v^{(i,j)}$ *for* $(x_i y_j) \in E_{hk}$.

We omit the simple proof of the lemma.

Denote by Ξ the class of all functions $\alpha : I_0 \to R_+$ such that

$$\lim_{(h,k)\to(0,0)} \alpha(h,k) = 0.$$

Assumption H$_2$. Suppose that $K = 0$, $L = 0$ and

1^0 conditions 1^0, 2^0, 4^0 of Assumption H$_1$ are satisfied,

2^0 the solution $\overline{\omega}(x_i, y_j) = 0$, $(x_i, y_j) \in E^0_{hk} \cup E_{hk}$, of the problem

$$\begin{aligned} \delta\omega^{(i,j)} &= \sigma_{hk}(x_i, y_j, \omega_{[i,j]}), \quad (x_i, y_j) \in A_{hk}, \\ \omega^{(i,j)} &= 0 \quad \text{for} \quad (x_i, y_j) \in E^0_{hk}, \end{aligned} \tag{34}$$

is stable in the following sense: if $\omega_{hk} : E^0_{hk} \cup E_{hk} \to R_+$ is the solution of the problem

$$\delta\omega^{(i,j)} = \sigma_{hk}(x_i, y_j, \omega_{[i,j]}) + \alpha(h,k), \quad (x_i, y_j) \in A_{hk}, \tag{35}$$

$$\omega^{(i,j)} = \alpha_0(h,k) \quad \text{for} \quad (x_i, y_j) \in E^0_{hk}, \tag{36}$$

where α, $\alpha_0 \in \Xi$ then there is $\beta \in \Xi$ such that $\omega^{(i,j)}_{hk} \leq \beta(h,k)$ for $(x_i, y_j) \in E_{hk}$.

Theorem 4.2. *Suppose that Assumption H$_2$ is satisfied, and*

1^0 $u_h : E^0_{hk} \cup E_{hk} \to R$ *is a solution to problem* (4), (5) *and there is* $\alpha_0 \in \Xi$ *such that*

$$|\phi^{(i,j)} - \phi^{(i,j)}_{hk}| \leq \alpha_0(h,k) \quad \textit{for} \quad (x_i, y_j) \in E^0_{hk}; \tag{37}$$

2^0 $v : E^0 \cup E \to R$ *is a solution to problem* (2), (3) *and* v *is of class* C^3, *on* $\overline{E}$;

3^0 *the following compatibility condition is satisfied: there is* $\widetilde{\alpha} \in \Xi$ *such that*

$$\begin{aligned} \big|F_{hk}(x_i, y_j, (v_{hk})_{[i,j]}) - f(x_i, y_j, v_{(x_i,y_j)})\big| &\leq \widetilde{\alpha}(h,k), \\ &(x_i, y_j) \in A_{hk}, \end{aligned} \tag{38}$$

where the function v_{hk} is the restriction of the function v to the set $E^0_{hk} \cup E_{hk}$.

Under these assumptions there exists $\beta \in \Xi$ such that

$$\left|u^{(i,j)}_{hk} - v^{(i,j)}_{hk}\right| \le \beta(h,k), \quad (x_i,y_j) \in E_{hk}. \tag{39}$$

Proof. Let $\Gamma_{hk} : A_{hk} \to R$ be defined by

$$\delta v^{(i,j)}_{hk} = F_{hk}(x_i,y_j,(v_{hk})_{[i,j]}) + \Gamma^{(i,j)}_{hk}, \quad (x_i,y_j) \in A_{hk}. \tag{40}$$

It follows from assumption 2^0 that there is $\alpha_1 \in \Xi$ such that

$$\left|\delta v^{(i,j)}_{hk} - D_{xy}v^{(i,j)}\right| \le \alpha_1(h,k), \quad (x_i,y_j) \in A_{hk}. \tag{41}$$

From the above inequality and from the compatibility condition (39) we deduce that there is $\alpha \in \Xi$ such that $|\Gamma^{(i,j)}_{hk}| \le \alpha(h,k)$ for $(x_i,y_j) \in A_{hk}$. Let $\omega_{hk} = |u_{hk} - v_{hk}|$. Then the function ω_{hk} satisfies the relations

$$\omega^{(i,j)}_{hk} \le hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu,y_\nu,(\omega_{hk})_{[\mu,\nu]}) + ih\,jk\,\alpha(h,k), \tag{42}$$
$$1 \le i \le M, \quad 1 \le j \le N,$$
$$\omega^{(i,j)}_{hk} \le \alpha_0(h,k) \quad \text{for} \quad (x_i,y_j) \in E^0_{hk}. \tag{43}$$

Let $\widetilde{\omega} : E^0_{hk} \cup E_{hk} \to R_+$ be a solution of the problem

$$\omega^{(i,j)} = hk \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} \sigma_{hk}(x_\mu,y_\nu,(\omega)_{[\mu,\nu]}) + ab\alpha(h,k), \tag{44}$$
$$1 \le i \le M, \quad 1 \le j \le N,$$
$$\omega^{(i,j)} = \alpha_0(h,k) \quad \text{for} \quad (x_i,y_j) \in E^0_{hk}. \tag{45}$$

Relations (42), (43) and Lemma 4.1 imply $\omega^{(i,j)}_{hk} \le \widetilde{\omega}^{(i,j)}_{hk}$ for $(x_i,y_j) \in E_{hk}$. Now we obtain the assertion of our theorem from the stability of the functional difference problem (34). □

Remark 4.3. If the assumptions of Theorem 4.2 are satisfied then we have the following estimate for the error of approximate solutions to problem (2), (3):

$$\left|u^{(i,j)}_{hk} - v^{(i,j)}_{hk}\right| \le \widetilde{\omega}^{(i,j)}_{hk}, \quad (x_i,y_j) \in E_{hk},$$

where the function $\widetilde{\omega}_h$ is the only solution to problem (44), (45) with α_0 given by (37), $\alpha = \widetilde{\alpha} + \alpha_1$, and $\widetilde{\alpha}$, α_1 are defined by (39), (41).

Now, we consider problem (2), (3) and the difference method

$$\begin{aligned} \delta z^{(i,j)} &= f(x_i, y_j, T_{hk} z_{[i,j]}), \quad (x_i, y_j) \in A_{hk}, \\ z^{(i,j)} &= \phi_{hk}^{(i,j)} \quad \text{for} \quad (x_i, y_j) \in E_{hk}^0, \end{aligned} \tag{46}$$

where the operator T_{hk} is defined in Example 2. It is obvious that there exists exactly one solution to problem (46).

Assumption H$_3$. Suppose that the function $f : \overline{E} \times C(B, R) \to R$ is continuous, and there is a function $\sigma : \overline{E} \times R_+ \to R_+$ such that

1^0 σ is continuous, and $\sigma(x, y, 0) = 0$ for $(x, y) \in \overline{E}$;

2^0 σ is nondecreasing with respect to all variables and the function $\overline{\omega}(x, y) = 0$, $(x, y) \in \overline{E}$ is the unique solution to the problem

$$\begin{aligned} D_{xy} z(x, y) &= \sigma(x, y, z(x, y)), \quad (x, y) \in \overline{E}, \\ z(x, 0) = 0 \ \text{ for } \ x \in [0, a], &\quad z(0, y) = 0 \ \text{ for } \ y \in [0, b]; \end{aligned}$$

3^0 the estimate

$$|f(x, y, w) - f(x, y, \overline{w})| \le \sigma(x, y, \|w - \overline{w}\|_B)$$

is satisfied on $\overline{E} \times C(B, R)$.

Theorem 4.4. *Suppose that Assumption H_3 is satisfied, and*

1^0 *$u_{hk} : E_{hk}^0 \cup E_{hk} \to R$ is a solution to problem* (46)*, and there is $\alpha_0 \in \Xi$ such that estimate* (37) *holds;*

2^0 *$v : E^0 \cup E \to R$ is a solution to problem* (2)*,* (3)*, and v is of class C^3 on $\overline{E}$.*

Then there is $\beta \in \Xi$ such that

$$\big| u_{hk}^{(i,j)} - v_{hk}^{(i,j)} \big| \le \beta(h, k), \quad (x_i, y_j) \in E_{hk}, \tag{47}$$

where v_{hk} is the restriction of the function v to the set $E_{hk}^0 \cup E_{hk}$.

Proof. We apply Theorem 4.2 in the proof of assertion (47). Put $D_{hk} = B_{hk}$ and

$$F_{hk}(x, y, w) = f(x, y, T_{hk} w), \quad (x, y, w) \in A_{hk} \times F[D_{hk}, R]. \tag{48}$$

Then we have

$$\begin{aligned} \big| F_{hk}(x, y, w) - F_{hk}(x, y, \overline{w}) \big| &\le \sigma\big(x, y, \|T_{hk}(w - \overline{w})\|_B\big) = \\ &= \sigma\big(x, y, \|w - \overline{w}\|_{hk}\big) \quad \text{on} \quad A_{hk} \times F[D_{hk}, R]. \end{aligned}$$

Consider the problem

$$\delta \omega^{(i,j)} = \sigma(x_i, y_j, \omega^{(i,j)}), \quad 0 \le i \le M - 1, \ 0 \le j \le N - 1, \tag{49}$$

$$\omega^{(i,j)} = 0 \quad \text{for} \quad (x_i, y_j) \in E_{hk}^0. \tag{50}$$

We prove that the solution $\overline{\omega}(x,y)=0$, $(x,y)\in E^0_{hk}\cup E_{hk}$, to the problem

$$\delta\omega^{(i,j)}=\sigma(x_i,y_j,\omega^{(i,j)}),\ \ (x_i,y_j)\in A_{hk},\quad \omega^{(i,j)}=0\ \text{ for }\ (x_i,y_j)\in E^0_{hk},$$

is stable in the sense of condition 2^0 of Assumption H_2.

Let $\overline{\omega}_{hk}:E^0_{hk}\cup E_{hk}\to R_+$ be a solution to the problem

$$\delta\omega^{(i,j)}=\sigma(x_i,y_j,\omega^{(i,j)})+\alpha(h,k),\quad (x_i,y_j)\in A_{hk}, \tag{51}$$

$$\omega^{(i,j)}=\alpha_0(h,k)\quad\text{for}\quad (x_i,y_j)\in E^0_{hk}, \tag{52}$$

where α $\alpha_0\in\Xi$. Consider the Darboux problem

$$D_{xy}z(x,y)=\sigma(x,y,z(x,y))+\alpha(h,k),\quad (x,y)\in\overline{E},$$
$$z(x,0)=\alpha_0(h,k)\ \text{ for }\ x\in[0,a],\quad z(0,y)=\alpha_0(h,k)\ \text{ for }\ y\in[0,b].$$

Since there is $\varepsilon_0>0$ such that if $h+k\le\varepsilon_0$, there is also a solution $z_{hk}:\overline{E}\to R$ to the above problem, and

$$\lim_{(h,k)\to(0,0)} z_{hk}(x,y)=0\quad\text{uniformly on}\quad \overline{E}. \tag{53}$$

It follows from the monotonicity of σ that for $h+k\le\varepsilon_0$, $(x_i,y_j)\in A_{hk}$, we have the relations

$$z^{(i,j)}_{hk}=\int_0^{x_i}\int_0^{y_j}[\sigma(t,s,z_{hk}(t,s))+\alpha(h,k)]dt\,ds+\alpha_0(h,k)\ge$$
$$\ge hk\sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1}[\sigma(x_\mu,y_\nu,z^{(\mu,\nu)}_{hk})+\alpha(h,k)]+\alpha_0(h,k).$$

Then the function z_{hk} satisfies the difference inequality

$$z^{(i,j)}_{hk}\ge hk\sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1}[\sigma(x_\mu,y_\nu,z^{(\mu,\nu)}_{hk})+\alpha(h,k)]+\alpha_0(h,k),$$
$$1\le i\le M,\quad 1\le j\le N.$$

The function $\overline{\omega}_{hk}$ satisfies the equation

$$\overline{\omega}^{(i,j)}_{hk}=hk\sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1}[\sigma(x_\mu,y_\nu,\overline{\omega}^{(\mu,\nu)}_{hk})+\alpha(h,k)]+\alpha_0(h,k),$$
$$1\le i\le M,\quad 1\le j\le N.$$

It follows from Lemma 4.1 that $\overline{\omega}^{(i,j)}_{hk}\le z^{(i,j)}_{hk}$ for $(x_i,y_j)\in E_{hk}$. Thus the stability of problem (49), (50) follows from condition (53).

Now, we prove the consistency condition for equation (48). We will use the following property of the operator T_{hk} ([12]): if $w\in F[B,R]$, w is of class C^3 and w_{hk} is the restriction of w to the set B_{hk}, then there is $C > 0$ such that

$$\|T_{hk}w_{hk} - w\|_B \le C(h^2 + k^2).$$

It follows from the above property of T_{hk} and from assumption 2^0 that the operator F_{hk} given by (48) satisfies condition (39) with $\widetilde{\alpha} \in \Xi$. □

§ 5. Convergence of Implicit Difference Methods

In this section, we consider a general class of difference problems consistent with (2), (3) which satisfy Assumption H_1 and are convergent. We formulate a functional difference equation.

Let $K = 1$, $L = 1$ in the definition of D_{hk}. We define the operator

$$\widetilde{T}_{hk} : F[D_{hk}, R] \to F[\,[-a_0, h] \times [-b_0, k], R\,]$$

in the following way. Let $w \in F[D_{hk}, R]$ and $(t, s) \in [-a_0, h]\times[-b_0, k]$. Then there is (x_i, y_j) such that $-M_0 \le i < 1$, $-N_0 \le j < 1$ and $x_i \le t \le x_{i+1}$, $y_j \le j \le y_{j+1}$. We define $(\widetilde{T}_{hk}w)(t, s)$ as the right-hand side of formula (9). Denote by $S_{hk} : F[D_{hk}, R] \to F[B, R]$ the operator given by

$$(S_{hk}w)(t, s) = (\widetilde{T}_{hk}w)\Big(t + \frac{h}{2}, s + \frac{k}{2}\Big), \quad (t, s) \in B.$$

The function $S_{hk}w$ is the restriction of the function $\widetilde{T}_{hk}w$ to the set $[-a_0 + \frac{h}{2}, \frac{h}{2}] \times [-b_0 + \frac{k}{2}, \frac{k}{2}]$ which is shifted to the set B.

Consider problem (2), (3) and the difference equation

$$\delta z^{(i,j)} = f(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, S_{hk}z_{[i,j]}), \quad 0 \le i \le M-1, \;\; 0 \le j \le N-1, \quad (54)$$

with boundary condition (5).

Assumption H$_4$. Suppose that the function $f : \overline{E} \times C(B, R) \to R$ is continuous and there is $\widetilde{L} \in R_+$ such that

$$|f(x, y, w) - f(x, y, \overline{w})| \le \widetilde{L}\|w - \overline{w}\|_D \quad \text{on} \quad \overline{E} \times C(B, R).$$

Theorem 5.1. *Suppose that Assumption H_4 is satisfied and*

1^0 *$v : E^0 \cup E \to R$ is a solution to problem* (2), (3), *and v is of class C^4 on $\overline{E}$,*

2^0 *$ab\widetilde{L} < 1$, and there is $\alpha_0 \in \Xi$ such that inequality* (37) *holds true.*

Then there exists exactly one solution $u_{hk} : E^0_{hk} \cup E_{hk} \to R$ to problem (54), (5), *and there is $C \in R_+$ such that the following error estimate holds:*

$$\big|u_{hk}^{(i,j)} - v^{(i,j)}\big| \le \frac{Cx_iy_j(h^2 + k^2) + 2\alpha_0(h, k)}{1 - x_iy_j\widetilde{L}}, \quad (x_i, y_j) \in E_{hk}. \quad (55)$$

Proof. We put

$$F_{hk}(x_i, y_j, w) = f\big(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, S_{hk}w\big), \quad (x_i, y_j, w) \in A_{hk} \times F[D_{hk}, R].$$

Then F_{hk} satisfies the Lipschitz condition with respect to the functional variable with the constant $\widetilde{L}$. Put

$$\sigma(x, y, w) = \widetilde{L}\|w\|_{hk}, \quad (x, y, w) \in A_{hk} \times F[D_{hk}, R_+]. \tag{56}$$

Then equation (20) is equivalent to

$$\omega^{(i,j)} = hk \sum_{\mu=1}^{i} \sum_{\nu=1}^{j} \widetilde{L}\|\omega_{[\mu-1,\nu-1]}\|_{hk}, \quad 1 \le i \le M, \ \ 1 \le j \le N.$$

The above equation with boundary condition (21) is equivalent to the problem

$$\begin{gathered} \omega^{(i,j)} = hk \sum_{\mu=1}^{i} \sum_{\nu=1}^{j} \widetilde{L}\omega^{(\mu,\nu)}, \quad 1 \le i \le M, \ \ 1 \le j \le N, \\ \omega^{(i,j)} = 0 \quad \text{for} \quad (x_i, y_j) \in E^0_{hk}. \end{gathered} \tag{57}$$

It follows from assumption 2^0 that problem (57) satisfies conditions 3^0, 4^0 of Assumption H_1 and that the unique solution to problem (28) (29) with σ given by (56) is $\omega(x, y) = 0$. Then there exists exactly one solution $u_{hk} : E^0_{hk} \cup E_{hk} \to R$ to problem (54), (5).

Let $\Gamma_{hk} : A_{hk} \to R$ be defined by

$$\delta v^{(i,j)}_{hk} = f\big(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, S_{hk}(v_{hk})_{[i,j]}\big) + \Gamma^{(i,j)}_{hk}, \quad (x_i, y_j) \in A_{hk}.$$

There is $\widetilde{C} \in R_+$ such that we have

$$\big|\delta v^{(i,j)}_{hk} - D_{xy}v(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})\big| \le \widetilde{C}(h^2 + k^2), \quad (x_i, y_j) \in A_{hk},$$

and

$$\big|\widetilde{T}_{hk}(v_{hk})_{[i,j]}(t, s) - v_{(x_{i+1}, y_{j+1})}(t, s)\big| \le \widetilde{C}(h^2 + k^2),$$

where $(t, s) \in [-a_0, h] \times [-b_0, k]$, $0 \le i \le M - 1$, $0 \le j \le N - 1$. Then there is $C \in R_+$ such that $|\Gamma^{(i,j)}_{hk}| \le C(h^2 + k^2)$ for $(x_i, y_j) \in A_{hk}$. Let $\omega_{hk} = u_{hk} - v_{hk}$. Obviously, the function ω_{hk} satisfies the inequalities

$$|\omega^{(i,j)}_{hk}| \le hk \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \big[\widetilde{L}(\|(\omega_{hk})_{[\mu,\nu]}\|_{hk} + C(h^2 + k^2)\big] + 2\alpha_0(h, k),$$

$$1 \le i \le M, \quad 1 \le j \le N,$$

$$\omega^{i,j)}_{hk} \le \alpha_0(h, k) \quad \text{for} \quad (x_i, y_j) \in E^0_{hk}.$$

The function

$$\widetilde{\omega}_{hk}^{(i,j)} = \frac{Cx_iy_j(h^2+k^2)+2\alpha_0(h,k)}{1-x_iy_j\widetilde{L}}\ , \quad (x_i,y_j)\in E_{hk},$$
$$\widetilde{\omega}_{hk}^{(i,j)} = 2\alpha_0(h,k) \quad \text{for} \ \ (x_i,y_j)\in E^0_{hk},$$

satisfies the inequalities

$$\omega_{hk}^{(i,j)} \geq hk\sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1}\left[\widetilde{L}\|(\omega)_{[\mu,\nu]}\|_{hk}+C(h^2+k^2)\right]+2\alpha_0(h,k),$$
$$1\leq i\leq M, \quad 1\leq j\leq N,$$
$$\omega^{(i,j)} \geq \alpha_0(h,k) \quad \text{for} \ \ (x_i,y_j)\in E^0_{hk}.$$

Consequently, we obtain assertion (55) from Lemma 4.1. □

Numerical example. Define $E=(0,1]\times(0,1]$, $E^0=([-\frac{1}{2},1]\times[\frac{1}{2},1])\backslash E$ and $B=[-\frac{1}{2},0]\times[-\frac{1}{2},0]$. Consider the Darboux problem

$$D_{xy}z(x,y) = 2(x+y)\big(z(x-0.25,y-0.25)-z(x,y)\big) - \\ -(x+y)\int_{-\frac{1}{2}}^{0}\int_{-\frac{1}{2}}^{0} z(x+t,y+s)dt\,dx + f(x,y), \qquad (58)$$
$$(x,y)\in\overline{E},$$
$$z(x,y) = \sin(1+x+y) \quad \text{for} \quad (x,y)\in E^0, \qquad (59)$$

where

$$f(x,y) = (x+y-1)\sin(1+x+y)-(x+y)\sin(x+y).$$

Let M_0, N_0, M, N be natural numbers which satisfy

$$M_0h=0.5, \quad N_0k=0.5, \quad M=2M_0, \quad N=2N_0.$$

Assume that M_0 and N_0 are even numbers. Consider the difference equation corresponding to equation (58)

$$\delta z^{(i,j)} = 2(x_{i+\frac{1}{2}}+y_{j+\frac{1}{2}})\big(S_{hk}z_{(i,j)}(-0.25,-0.25)-S_{hk}z_{(i,j)}(0,0)\big) - \\ -(x_{i+\frac{1}{2}}+y_{j+\frac{1}{2}})\int_{-\frac{1}{2}}^{0}\int_{-\frac{1}{2}}^{0} T_{hk}z_{(i,j)}(t,s)dt\,ds + f^{(i,j)},$$
$$0\leq i\leq M-1, \quad 0\leq j\leq N-1.$$

Let $m_0=\frac{1}{2}M_0$ and $n_0=\frac{1}{2}N_0$. Then we have

$$S_{hk}z_{(i,j)}(-0.25,-0.25) = I^{(-)}[i,j], \quad S_{hk}z_{(i,j)}(0,0) = I^{(0)}[i,j],$$

where

$$I^{(-)}[i,j] = \frac{1}{4}\big(z^{(i-m_0,j-n_0)} + z^{(i-m_0+1,j-n_0)} + \\ + z^{(i-m_0,j-n_0+1)} + z^{(i-m_0+1,j-n_0+1)}\big),$$

$$I^{(0)}[i,j] = \frac{1}{4}\big(z^{(i,j)} + z^{(i+1,j)} + z^{(i,j+1)} + z^{(i+1,j+1)}\big).$$

Let $w \in F(B_{hk}, R)$. Then

$$\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} T_{hk} w(t,s) dt\, ds = \frac{hk}{4}\big(w^{(i,j)} + w^{(i+1,j)} + w^{(i,j+1)} + w^{(i+1,j+1)}\big)$$

and consequently

$$\int_{-\frac{1}{2}}^{0} \int_{-\frac{1}{2}}^{0} T_{hk} z_{(i,j)}(t,s) dt\, ds = I[i,j],$$

where

$$I[i,j] = \frac{hk}{4}\big(z^{(i-M_0,j-N_0)} + z^{(i,j-N_0)} + z^{(i-M_0,j)} + z^{(i,j)}\big) + \\ + \frac{hk}{2}\sum_{i'=1}^{M_0-1}\big(z^{(i-M_0+i',j-N_0)} + z^{(i-M_0+i',j)}\big) + \\ + \frac{hk}{2}\sum_{j'=1}^{N_0-1}\big(z^{(i-M_0,j-N_0+j')} + z^{(i,j-N_0+j')}\big) + \\ + hk\sum_{i'=1}^{M_0-1}\sum_{j'=1}^{N_0-1} z^{(i-M_0+i',j-N_0+j')}.$$

We approximate the solution $v : E^0 \cup E \to R$ of problem (58), (59) by means of solutions of the implicit difference equation

$$z^{(i+1,j+1)} - z^{(i+1,j)} - z^{(i,j+1)} + z^{(i,j)} = \\ = -hk(x_{i+\frac{1}{2}} + y_{j+\frac{1}{2}})I[i,j] + \\ + 2hk(x_{i+\frac{1}{2}} + y_{j+\frac{1}{2}})\big(I^{(-)}[i,j] - I^{(0)}[i,j]\big) + hkf^{(i,j)}, \quad (60) \\ 0 \le i \le M-1, \quad 0 \le j \le N-1,$$

with the boundary condition

$$z^{(i,j)} = \sin(1 + x_i + y_j), \quad (x_i, y_j) \in E^0_{hk}. \quad (61)$$

The function $v(x,y) = \sin(1+x+y)$, $(x,y) \in E^0 \cup E$, is the solution to problem (58), (59). Let $u_{hk} : E^0_{hk} \cup E_{hk} \to R$ be a solution to problem (60), (61), and $\varepsilon_{hk} = v_{hk} - u_{hk}$, where u_{hk} is the restriction of the function v to the set $E^0_{hk} \cup E_{hk}$. Some values of $\varepsilon^{(i,j)}_{hk}$ are listed in the table for $h = k = 10^{-3}$.

TABLE OF ERRORS

	$x = 0.80$	$x = 0.85$	$x = 0.90$	$x = 0.95$	$x = 1$
$y = 0.80$	$1.325\,10^{-4}$	$1.597\,10^{-4}$	$1.898\,10^{-4}$	$2.229\,10^{-4}$	$2.593\,10^{-4}$
$y = 0.85$	$1.587\,10^{-4}$	$1.905\,10^{-4}$	$2.244\,10^{-4}$	$2.618\,10^{-4}$	$3.028\,10^{-4}$
$y = 0.90$	$1.898\,10^{-4}$	$1.244\,10^{-4}$	$2.627\,10^{-4}$	$3.048\,10^{-4}$	$3.510\,10^{-4}$
$y = 0.95$	$2.229\,10^{-4}$	$2.618\,10^{-4}$	$3.048\,10^{-4}$	$3.521\,10^{-4}$	$4.041\,10^{-4}$
$y = 1$	$2.593\,10^{-4}$	$3.028\,10^{-4}$	$3.510\,10^{-4}$	$3.641\,10^{-4}$	$4.625\,10^{-4}$

The computation was performed by the computer IBM AT.

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