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PARAMETRIZATION OF A FAMILY OF MINIMAL SURFACES BOUNDED BY THE BROKEN LINES IN $\mathbb{R}^3$

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Abstract. Consideration is given to a family of minimal surfaces bounded by the broken lines in $\mathbb{R}^3$ which are locally injectively projected onto the coordinate plane. The considered family is bijectively mapped by means of the Enepper–Weierstrass representation onto a set of circular polygons of a certain type. The parametrization of this set is constructed, and the dimension of the parameter domain is established.

The Dirichlet problem for an equation of minimal surfaces in the nonconvex domain does not always have a solution even under infinitely smooth boundary conditions. Geomerically, this means that a minimal surface bounded by a given curve injectively projected onto the coordinate plane is not always injectively projected onto the same plane. For references on this topic see [1] and [2]. One way to investigate the problem of projecting a minimal surface onto the plane is as follows: instead of studying the solvability of an individual Dirichlet boundary value problem one should consider a sufficiently well surveyable set of spatial curves and find out which part of this set is filled up by the curves for which the above problem is solvable.

The first step in this direction was made in [3] where it was shown that there exist no minimal surfaces which are bounded by four ribs of a tetrahedron of variable height and injectively projected onto the tetrahedron base.

In this paper, for some family of spatial broken lines we investigate a subfamily of such broken lines, which bound the minimal surfaces locally injectively projected onto the coordinate plane. The parametrization of this subfamily is constructed and the dimension of the parameter domain is established.

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§ 1. INTRODUCTION

Let S be a simply connected minimal surface parametrized by an infinitely smooth mapping $x : E = \{|z|^2 = \xi^2 + \eta^2 < 1\} \to \mathbb{R}^3$. We denote by π the orthogonal projection $\pi : \mathbb{R}^3 \to \Pi = \{x \in \mathbb{R}^3,\ x^3 = 0\}$ and call S the d-surface if $\pi \circ x$ is a homeomorphism, and the c-surface if $\pi \circ x$ is a local homeomorphism. Since $z \to (z, x(z))$ is a homeomorphism, S is the d-surface iff $\pi|_{x(E)}$ is injective. Assume that S is the c-surface, $x \in C(\overline{E})$, and the set of values of the restriction $\pi|_{\partial S}$ consists of boundary points of the bounded simply connected domain Q. We shall show that under these assumptions S is the d-surface. Indeed, it is easy to show that since $\pi \circ x : E \to \Pi$ is a local homeomorphism, we have $(\pi \circ x)(E) \cap (\Pi \backslash \overline{Q}) = \varnothing$ because otherwise x would be unbounded in E. Therefore $\pi \circ x$ is an unlimited non-ramified covering of the domain Q by a circle. The mapping $\pi \circ x$ is injective by virtue of the theorem on monodromy [4].

In this paper the mapping x will be represented by the following Enneper–Weierstrass formulas [2]:

$$
\begin{aligned}
2x^1(z) &= \operatorname{Re} \int_0^z F'(t)(1 - \omega^2(t))dt + c_1, \\
2x^2(z) &= \operatorname{Re}\ i \int_0^z F'(t)(1 + \omega^2(t))dt + c_2, \\
x^3(z) &= \operatorname{Re} \int_0^z F'(t)\omega(t)dt + c_3,
\end{aligned} \tag{1}
$$

where $F(z)$ and $\omega(z)$ are the holomorphic functions in E, c_j, $j = \overline{1,3}$, are the real constants.

If a minimal surface is the c-surface, then for any point $M \in S$ there exists a neighborhood $V_s(M)$ such that $\pi|_{V_s(M)}$ is injective, and therefore the surface $V_s(M)$ can be represented as $x^3 = u(x^1, x^2)$, $(x^1, x^2) = \pi(V_s(M))$ where $u \in C^2\big(\pi(V_s(M))\big)$ and

$$
(1 + u_{x^2}^2)u_{x^1x^1} - 2u_{x^1x^2}u_{x^1}u_{x^2} + (1 + u_{x^1}^2)u_{x^2x^2} = 0. \tag{2}
$$

Introducing the notation $p = u_{x^1}$, $q = u_{x^2}$, $W = \sqrt{1 + p^2 + q^2}$ for an appropriate orientation of the surface, we obtain [2] the following expression of the unit normal vector $\vec{\nu}(M) = (\nu_1(M), \nu_1(M), \nu_3(M))$:

$$
\begin{aligned}
\vec{\nu}(M) &= \Big(\frac{2 \operatorname{Re}\ \omega(z)}{|\omega(z)|^2 + 1}, \frac{2 \operatorname{Im} \omega(z)}{|\omega(z)|^2 + 1}, \frac{|\omega(z)|^2 - 1}{|\omega(z)|^2 + 1} \Big) = \\
&= \big[W(\pi(M))\big]^{-1} \cdot \big(p(\pi(M)), q(\pi(M)), -1\big).
\end{aligned} \tag{3}
$$

Let us show that for S to be the c-surface it is necessary and sufficient that $F'(z) \neq 0$, $|\omega(z)| \neq 1$, $z \in E$. We write $\pi \circ x$ in the form

$$\zeta(z) = x^1(z) + ix^2(z) = \frac{1}{2}\Big[\int\limits_0^z \overline{F'dt} - \int\limits_0^z F'\omega^2 dt + c_1 + ic_2\Big].$$

Hence $\overline{\zeta}_z = \dfrac{1}{2}F'(z)$, $\overline{\zeta}_z = -\dfrac{1}{2}F'(z)\omega^2(z)$ and $d\overline{\zeta} = F'(z)dz - F'(z)\omega^2(z)dz$, $|\overline{\zeta}_z|^2 - |\overline{\zeta}_{\overline{z}}|^2 = \dfrac{1}{4}|F'(z)|^2(1 - |\omega(z)|^2)$. The latter equality provides the sufficiency of the conditions. Let S be the c-surface and $|\omega(z_0)| = 1$. This equality can be fulfilled only when $W(\pi \circ x(z_0)) = \infty$, which contradicts the definition of the c-surface. If $|\omega(z_0)| < 1$ but $F'(z_0) = 0$, then on a sufficiently small circumference $z = z_0 + \varepsilon e^{i\theta}$ we shall have

$$\overline{\zeta}'_\theta = F'_\theta\big(1 - \overline{F}'_\theta(z_0 + \varepsilon e^{i\theta})(F'_\theta(z_0 + \varepsilon e^{i\theta}))^{-1}\overline{\omega^2}(z_0 + \varepsilon e^{i\theta})\big).$$

Hence, taking into account $|\overline{F}'_\theta\omega^2 \cdot (F'_\theta)^{-1}| < 1$, we obtain $\frac{1}{2\pi}\int_0^{2\pi} d\,\mathrm{Arg}\,\overline{\zeta}_\theta = \frac{1}{2\pi}\int_0^{2\pi} d\,\mathrm{Arg}\,F'_\theta \geq 2$, which also contradicts the fact that the mapping $\pi \circ x$ is a local homeomorphism.

In what follows, by a curve we shall mean both a class of the equivalence of continuous mappings of a segment and an individual representative of this class. The curve will be called an arc if it is an injective mapping. For $l : [a,b] \to \mathbb{R}^m$ we shall denote by l^0 the restriction of l on (a,b). The set of values of the curve l will be denoted by $[l]$ (if $a = b$, then $[l^0] = \varnothing$). By $l_1 \cdot l_2$ we shall mean the product of curves in the usual sense when the end point of the curve l_1 coincides with the initial point of the curve l_2. The notation $\pi\beta(l_1, l_2)$, $|\beta(l_1, l_2)| \leq 1$ will denote the angle between the positive tangent at the end point of l_1 and the positive tangent of l_2 at the initial point and counted from l_1. Finally, $|l|$ will denote the length of l.

Let $\widetilde{\Gamma}$ be a closed broken line in $\mathbb{R}^3$ not lying in one plane and satisfying the following condition: if $\pi(x_1) = \pi(x_2)$, $x_1 \in \widetilde{\Gamma}$, $x_2 \in \widetilde{\Gamma}$, then x_1 and x_2 belong to the same segment of $\widetilde{\Gamma}$. This condition immediately implies that $\pi[\widetilde{\Gamma}]$ cuts the plane into two components. We orient $\widetilde{\Gamma}$ so that the orientation induced on $\pi\circ\widetilde{\Gamma}$ would be positive with respect to the unbounded component. Let us number the vertices M_i of $\widetilde{\Gamma}$ according to the chosen orientation and denote by $\widetilde{\Gamma}_i$ the oriented segment of $\widetilde{\Gamma}$ whose initial point is M_i and whose end point is M_{i+1} ($M_{n+1} = M_1$). By $T_i = (X_i, Y_i Z_i)$ we denote the unit vector co-directed with $\widetilde{\Gamma}_i$. It is assumed that M_i, M_{i+1}, M_{i+2} are not collinear and therefore $T_i \neq T_{i+1}$. Moreover, since $\widetilde{\Gamma}$ does not lie in the plane, among the vectors T_i there are three noncomplanar vectors with successive indices, and without loss of generality we can consider T_{n-2}, T_{n-1}, T_n as such.

Denote by $\mathcal{G}$ a set of broken lines possessing the following property: if the initial point and orientation of Γ are appropriately chosen, then Γ_i is co-directed with $\widetilde{\Gamma}_i$. We denote by Λ a subspace of $\mathbb{R}^n$ defined by the equation $\sum_{i=1}^n \lambda_i T_i = 0$. Let $\lambda : \mathcal{G} \to \Lambda_+ \Lambda \cap \mathbb{R}^n_+$, $\lambda(\Gamma) = (|\Gamma_1|, |\Gamma_2|, \dots, |\Gamma_n|)$. Since T_{n-2}, T_{n-1}, T_n are noncoplanar, the mapping $\tau : \Lambda_+ \to \mathbb{R}^{n-3}_+$ $\tau(\lambda_1, \lambda_2, \dots, \lambda_n) = (\lambda_1, \lambda_2, \dots, \lambda_{n-3})$ is injective. The definition of Λ_+ readily implies that $\Lambda'_+ = (\tau \circ \lambda)(\mathcal{G})$ is an open connected convex subset in $\mathbb{R}^{n-3}_+$.

Since for $y \in \Lambda'_+$ and $t > 0$ we have $ty \in \Lambda'_+$, after introducing the notation $\widetilde{\Lambda} = \Lambda'_+ \cap S^{n-4}$, where S^{n-4} is the unit sphere in $\mathbb{R}^{n-3}$, we shall have the injective mapping $\mathfrak{p} : \mathcal{G} \to \widetilde{\Lambda}$

$$\mathfrak{p}(\Gamma) = \Big(\sum_{i=1}^{n-3} |\Gamma_i|^2 \Big)^{-\frac{1}{2}} \big(|\Gamma_1|, |\Gamma_2|, \dots, |\Gamma_{n-3}| \big).$$

We denote by $\mathcal{G}_c$ the subset of $\mathcal{G}$ consisting of broken lines bounding the minimal c-surfaces, and by $\mathfrak{S}$ the set of minimal c-surfaces bounded by the broken lines of the family $\mathcal{G}_c$. Let $P(S) = (\mathfrak{p} \circ \tau \circ \lambda)\partial S$, $\mathcal{P} = P(\mathfrak{S})$.

§ 2. ω-Images of Surfaces of the Family $\mathfrak{S}$

If a minimal surface is parametrized by formulas (1), g is the Gaussian mapping of the surface, and σ is the stereographic projection, then $\omega = \sigma \circ g \circ x$ [2].

Let $t_j = x^{-1}(M_j) = e^{i\theta_j}$, $0 < \theta_1 < \theta_2 < \dots < 2\pi$, $l_j = \{t = e^{i\theta}, \theta_j \le \theta \le \theta_{j+1}\}$. The function $\omega(z)$ is analytically continuable through l_j^0, $j = \overline{1, n}$, and, by virtue of (3), satisfies on l_j^0 the equation

$$2X_j \operatorname{Re}\, \omega(t) + 2Y_j \operatorname{Im} \omega(t) + Z_j(|\omega(t)|^2 - 1) = 0. \tag{4}$$

There exists the following representation [5]:

$$\begin{aligned} 2x^1(z) &= \operatorname{Re} \int_0^z \big[\Phi^2(t) - \Psi^2(t)\big] dt + c_1, \\ 2x^2(z) &= \operatorname{Re}\, i \int_0^z \big[\Phi^2(t) + \Psi^2(t)\big] dt + c_2, \\ x^3(z) &= \operatorname{Re} \int_0^z \Phi(t)\Psi(t) dt + c_3, \end{aligned} \tag{5}$$

where $\Phi^2(z)dz$ and $\Psi^2(z)dz$ are the holomorphic differentials in E. The following equalities are fulfilled in the neighborhood of t_j:

$$\begin{aligned}\Phi(z) &= C_1(z-t_j)^{\frac{1-\gamma_j}{2}}\Phi_1(z) + C_2(z-t_j)^{-\frac{1-\gamma_j}{2}}\Psi_1(z)\\ \Psi(z) &= C_3(z-t_j)^{\frac{1-\gamma_j}{2}}\Phi_2(z) + C_4(z-t_j)^{-\frac{1-\gamma_j}{2}}\Psi_2(z),\end{aligned} \tag{6}$$

where C_j, $j=\overline{1,4}$, are the constants; $C_1C_4 - C_2C_3 = 1$; $\Phi_1(z)$, $\Phi_2(z)$, $\Psi_1(z)$ and $\Psi_2(z)$ are a holomorphic in the neighborhood V_j of the point t_j, $0<\gamma_j<1$, while $(z-t_j)^{\pm\frac{1-\gamma_j}{2}}$ is holomorphic branch in $V_j\cap E$. The relation of $F(z)$ and $\omega(z)$ with $\Phi(z)$ and $\Psi(z)$ is given by the equalities $F'(z)=\Phi^2(z)$ and $\omega(z)=\Phi^{-2}(z)\cdot\Psi^2(z)$. This immediately implies that, firstly, $\omega'(z)$ can have only a finite number of zeros on l_j^0, $j=\overline{1,n}$, and, secondly, $\omega(z)$ has a finite or infinite limit $\omega(t_j)=\omega_j$ for $z\to t_j$. Hence on account of the chosen orientation of the surface and equation (4) we conclude that if $X_{j-1}Y_j - X_jY_{j-1}\neq 0$, then $\lim_{z\to t_j}\omega(z) = (\nu_1^{(j)}+i\nu_2^{(j)})(1-\nu_3^{(j)})^{-1}$, where $(\nu_1^{(j)},\nu_2^{(j)},\nu_3^{(j)}) = -\operatorname{sign}(X_{j-1}Y_j - X_jY_{j-1})\cdot|T_{j-1}\times T_j|^{-1}\cdot T_{j-1}\times T_j$. The case $X_{j-1}Y_j - X_jY_{j-1}=0$ will be considered below.

Let α_j, $|\alpha_j|<1$, $j=\overline{1,n}$, be the numbers defined by the equalities $\sin\pi\alpha_j = (T_{j-1},T_j,\nu^{(j)})$, $\cos\pi\alpha_j = T_{j-1}\cdot T_j$.

Lemma 1. *Let $S\in\mathfrak{S}$ and $\alpha_j<0$. Then the plane passing through $[\Gamma_{j-1}]$ and $[\Gamma_j]$ crosses S in any neighborhood of the point M_j.*

Proof. The formulation of the lemma implies that without loss of generality it can be assumed that Γ_{j-1} and Γ_j lie in the plane Π and $M_j=0$. By the symmetry principle the harmonic function $x^3(z) = \operatorname{Re}\int_0^z F'(t)\omega(t)dt + c_3$ continues through the arc $\theta_{j-1}<\theta<\theta_{j+1}$, while the holomorphic function $F'(z)$ in the semi-neighborhood $V(t_j)\cap E$ of the point t_j can be written in the form $F'(z)=(z-t_j)^{-\alpha_j}F_0(z)$, where $F_0(z)$ is holomorphic and non-vanishing [6]. Since in t_j the function $F'(z)\omega(z)$ cannot have zero of nonintegral order, by virtue of the boundedness of the harmonic function $\omega(z)$, $x^3(z)$ has zero of at least second order at the point t_j. Hence, on account of the familiar result ([7, Theorem 2.1]) we conclude that $x^3(z)$ changes its sign in $V(t_j)\cap E$. $\square$

Let now $S\in\mathfrak{S}$ and $X_{j-1}Y_j - X_jY_{j-1}=0$, i.e., the plane Q passing through $[\Gamma_{j-1}]$ and $[\Gamma_j]$ is orthogonal to Π. Then by the proven lemma, for any $\delta>0$ in $V_s(M_j,\delta)=B(M_j,\delta)\cap S$, where $B(M,\delta)$ is the ball with center M and radius δ, there are points lying in different half-spaces into which R^3 is divided by Q. We take two such points and connect them by the curve l, $[l]\subset V_s(M_j,\delta)$, intersecting Q in some point p_0 and let $p_1\in[\Gamma_{j-1}]\cup[\Gamma_j]$ $\pi(p_1)=\pi(p_0)$. Let $\Pi_1=\{x\in\mathbb{R}^3,\ ax^1+bx^2=c\}$ be the plane orthogonal to the plane Π and passing through p_0, $t'=e^{i\theta'}=x^{-1}(p_1)$, $z_0=x^{-1}(p_0)$,

l' $[l'] \subset E$ be the curve $\dfrac{a}{2}\Big[\operatorname{Re}\int_0^z F'(1-\omega^2)dt + c_1\Big] + \dfrac{b}{2}\Big[\operatorname{Re}\ i\int_0^z F'(1+\omega^2)dt + c_2\Big] = c$ connecting the points z_0 and t', and $l'' = x \circ l'$. Assuming l'' to be parametrized by the natural parameter $x = x(s)$, $0 \le s \le s_0$, $x(0) = p_0$, $x(s_0) = p_1$, we shall consider the function $d(s) = |\pi(p(s)) - \pi(p_1)|$ on $0 \le s \le s_0$. Since $d(s) \ge 0$, $d(s) \in C([0, s_0])$ and $d(0) = d(s_0) = 0$, there exists s^*, $0 < s^* < s_0$, such that $d(s^*)$ is a maximum value. It can be easily verified that the point $x(s^*)$ cannot have in S a neighborhood in which π is injective. If however $d(s) \equiv 0$, then l'' is a segment. Thus if $S \in \mathfrak{S}$, then it is necessary that $(T_{j-1}, T_j, \nu_j) > 0$, which uniquely fixes the position of ω_j for $|\omega_j| = 1$.

Denote by B_j a circumference described by equation (4). Let $\omega_j \neq \omega_{j+1}$. Denote by $\widetilde{b}_j$ an arc of B_j with the initial point ω_j, the end point ω_{j+1} and wholly lying in the closed unit circle. Such arcs are available, since $|\omega_j| \le 1$ and B_j are the stereographic images of circumferences of a sphere. If $\omega_j = \omega_{j+1}$, then $[\widetilde{b}_j] = \omega_j$. Let $\widetilde{b}_{j_1}, \widetilde{b}_{j_2}, \dots, \widetilde{b}_{j_{m_0}}$ be all the arcs of positive length.

It will always be assumed here that $\widetilde{b} = \prod_{k=1}^{m_0} \widetilde{b}_{j_k}$ is a Jordan curve.

Denote by D_0 the component $\mathbb{C}\backslash[\widetilde{b}]$ which lies in the unit circle.

Lemma 2. *If ω parametrizes $S \in \mathfrak{S}$, then $\omega(z)$ is univalent in D_0 and $\omega(E) \subseteq D_0$.*

Proof. Since $[b_j] \subset [B_j]$, where $b_j = \omega \circ l_j$, for $\omega \notin \bigcup_{j=1}^{n} [B_j]$ we have

$$1 \le \frac{1}{2\pi}\int_0^{2\pi} d\operatorname{Arg}(\omega(e^{i\theta}) - \omega_0) = \frac{1}{2\pi}\int_{\widetilde{b}} d\operatorname{Arg}(\omega - \omega_0)$$

which, by virtue of the Jordanian property of $\widetilde{b}$, implies

$$\operatorname{ind}_{\omega_0} \widetilde{b} = \frac{1}{2\pi}\int_{\widetilde{b}} d\operatorname{Arg}(\omega - \omega_0) = \begin{cases} 1, & \omega_0 \in D_0, & (7)\\ 0, & \omega_0 \in C\overline{D}_0. & (7') \end{cases}$$

The latter equality implies $\omega'(E)\backslash \bigcup_{j=1}^{n} [B_j] \subset D_0$. On assuming that there is a point $z' \in E$ such that $\omega(z') \in \bigcup_{j=1}^{n} [B_j] \cap C\overline{D}_0$ and recalling that the mapping ω is open, there will exist $z'' \in E$, $\omega(z'') \notin \bigcup_{j=1}^{n} [B_j]$ such that $\omega(z'') \in C\overline{D}_0$. $\square$

If $\mathfrak{S} \neq \varnothing$, then equality (7) holds for any $\omega_0 \in D_0$ and can be regarded as a necessary condition for the family $\mathfrak{S}$ to be nonempty. In this connection note that for the family of boundaries considered in [3] the topological indices corresponding to these boundaries are negative.

Let us discuss the degree of freedom of function $\omega(Z)$. Writing S in terms of isothermic coordinates

$$x^j(z) = \mathrm{Re}\int_0^z \varphi_j(t)dt + c_j, \quad j = \overline{1,3},$$

where $\varphi_j(z)$ are holomorphic in E, we obtain [2] $\omega(z) = \varphi_3(z)\cdot(\varphi_1 - i\varphi_2)^{-1}$. Therefore $\omega(z)$ is defined up to a conformal automorphism E and hence for uniqueness of $\omega(z)$ we should use some way of normalizing the mapping x. But since normalization of x is equivalent to the normalization of $\omega = \sigma \circ g \circ x$, we can use one of the standard ways of normalizing the conformal homeomorphism ω. Namely, we choose the points $\omega_0 \in D_0 \setminus \bigcup\limits_{j=1}^{n} [B_j]$ and $\omega_1 \in \widetilde{b}_n$ and normalize $\omega(z)$ by the condition

$$\omega(0) = \omega_0, \quad \omega(1) = \omega_1. \tag{8}$$

The function ω corresponding to the surface S and normalized by the condition (8) will be denoted by $\omega_s(z)$.

§ 3. Admissible Boundaries

Denote by Ω a class of functions $\omega(z)$ univalent holomorphic in E and continuous in $\overline{E}$, which are normalized by (8) and satisfy the following conditions:

(a) $|\omega(z)| \leq 1$, $z \in \overline{E}$;

(b) for each $\omega(z) \in \Omega$ there exist n points (depending on ω) $t_j(\omega) = e^{i\theta_j(\omega)}$, $j = \overline{1,n}$, $0 < \theta_1(\omega) < \cdots < \theta_n(\omega) < 2\pi$, such that $\omega(z)$ is analytically continuable through $l_j^0(\omega)$, where $l_j(\omega) = \{t = e^{i\theta},\ \theta_j(\omega) \leq \theta \leq \theta_{j+1}(\omega)\}$;

(c) $\omega(t) \in [B_j]$, $t \in l_j$, $j = \overline{1,n}$, $\theta_{n+1} = \theta_1$.

Denote by $\mu_\omega(z)$ the order of zero of ω' at the point z.

Lemma 3. *For $\omega \in \Omega$ we have*

(1) $\mu_\omega(z) \leq 1$, $z \in l_j^0$, $j = \overline{1,n}$;

(2) $\sum_{z \in l_j^0} \mu_\omega(z) \leq 2$, $j = \overline{1,n}$.

Proof. (1) Let $\mu(z_0) \geq 2$, $z_0 \in l_j^0$. Then in the neighborhood $U(z_0)$ of the point z_0 we have $\omega(z) - \omega(z_0) = (z - z_0)^{\mu+1} v(z)$, where $v(z)$ is a nonvanishing holomorphic function in $U(z_0)$. Denote by $v_1(z)$ an arbitrary regular branch of $\sqrt[\mu+1]{v(z)}$ in $U(z_0)$. As can be easily verified, $\zeta = (z - z_0)v_1(z) \cdot \zeta(z)$ is a univalent function in some neighborhood $U_1(z_0)$. Denoting by $z = h(\zeta)$, $\zeta \in \zeta(U_1(z))$ the inverse function of $\zeta(z)$, we obtain $\omega(h(z_1)) - \omega(z_0) = \zeta^{\mu+1}$, which means that $\omega(h(z))$ cannot be a univalent function in any component of $\zeta\big(U_1(z_0)\backslash[l_j]\big)$. Thus, taking into account that $\zeta(z)$ is univalent, we obtain a contradiction with the assumption that $\omega(z)$ is univalent.

(2) Let $t_{j,k} = e^{i\theta_{j,k}}$, $k = \overline{1, m_j - 1}$, $\theta_j = \theta_{j,0} < \theta_{j,1} < \theta_{j,2} < \theta_{j,3}, \dots < \theta_{j,m_j} = \theta_{j+1}$, be the critical points of ω on l_j^0. Let $l_{j,k} = \{t = e^{i\theta},\ \theta_{j,k} < \theta < \theta_{j,k+1}\}$, $k = \overline{0, m_j - 1}$, and $b_{j,k} = \omega \circ l_{j,k}$. Since $\omega(e^{i\theta}) \in [B_j]$, $e^{i\theta} \in l_j$ and, as it was proved, all critical points are simple, we obtain $\omega(U(e^{i\theta_{j,1}}) \cap E) \supset U(\omega(e^{i\theta_{j,1}})) \cap [b_{j,0}]$, and hence $[b_{j,1}^0] \cap [b_{j,0}^0] \neq \varnothing$. If we assume that $\omega(e^{i\theta_{j,2}}) \in [b_{j,0}]$, then we have $\omega(U(e^{i\theta_{j,2}})) \cap E) \supset U(\omega(e^{i\theta'}) \backslash [b_{j,1}]$, where $\theta_{j,0} < \theta' < \theta_{j,1}$, which contradicts the assumption that $\omega(z)$ is univalent. Therefore $\omega(e^{i\theta_{j,2}}) \notin [b_{j,0}]$. Since $\omega(U(e^{i\theta_{j,2}}) \cap E) \supset U(\omega(e^{i\theta_{j,2}})) \backslash [b_{j,2}^0]$, we have $[b_{j,3}^0] \cap [b_{j,2}^0] \neq \varnothing$ and therefore either $\omega(e^{i\theta_{j,3}}) \in [b_{j,2}^0]$ or $\omega(e^{i\theta_{j,1}}) \in [b_{j,3}^0]$. In both cases, repeating the above reasoning, we come to a contradiction with the property that $\omega(z)$ is univalent. $\square$

Lemmas 2 and 3 imply that if on l_j^0 there are two critical points, then $\omega(e^{i\theta_{j,2}}) \notin [b_{j,0}] \cup [\widetilde{b}_j]$. Denote by n_j the number of critical points $\omega(z)$ on l_j^0. Let $\omega \in \Omega$, $|\widetilde{b}_j| \neq 0$. If $n_j = 0$, then $b_j = \widetilde{b}_j$. If $n_j = 1$, then either $[b_{j,0}] \supset [\widetilde{b}_j]$, or $[b_{j,1}] \supset [\widetilde{b}_j]$. In the former case $b_j = \widetilde{b}_j b_{j,1}^{-1} \cdot b_{j,1}$, in the latter case $b_j = b_{j,0} \cdot b_{j,0}^{-1} \cdot \widetilde{b}_j$. If $n_j = 2$, then $b_j = b_{j,0} \cdot b_{j,0}^{-1} \cdot \widetilde{b}_j \cdot b_{j,2}^{-1} \cdot b_{j,2}$. For $|\widetilde{b}_j| = 0$, by the definition of the class Ω we have $n_j \geq 1$, and thus we obtain $b_j = b_{j,0} \cdot b_{j,0}^{-1}$ for $n_j = 1$ and $b_j = b_{j,0} \cdot b_{j,0}^{-1} \cdot b_{j,2}^{-1} \cdot b_{j,2}$ for $n_2 = 2$. Introducing the notation $b_{j,0} \cdot b_{j,0}^{-1} = p_{2j-1}$ and

$$p_{2j} = \begin{cases} b_{j,1}^{-1} \cdot b_{j,1} & \text{for} \quad n_j = 1, \\ b_{j,2}^{-1} \cdot b_{j,2} & \text{for} \quad n_j = 2 \end{cases}$$

in all cases we shall have $b_j = p_{2j-1} \widetilde{b}_j p_{2j}$, where each of the factors can have a zero length but $|b_j| \neq 0$. Note that if $|\widetilde{b}_j| \neq 0$, then $[p_{2j}] \cap [p_{2j-1}] = \varnothing$. If however $|\widetilde{b}_j| = 0$ and $|p_{2j-1}| \cdot |p_{2j}| = \varnothing$, then $[p_{2j-1}] \cap [p_{2j}] = \omega_j$.

Let $\omega_{j-1} \neq \omega_j = \omega_{j+1} = \cdots = \omega_{j+\nu_j} \neq \omega_{j+\nu_j+1}$, $\nu_j \geq 1$. We write it in the form $j \in T$ and introduce the notation $\widetilde{j} = \{j, j+1, \dots, j+\nu_j - 1\}$, $j^* = \{2j-2, 2j-1, \dots, 2j+2\nu_j - 1\}$.

Let $B'_{2k-1} : z = \varphi_{2k-1}(s)$, $0 \leq s \leq \delta_{2k-1}$, $B'_{2k} : z = \varphi_{2k}(s)$, $0 \leq s \leq \delta_{2k}$, $\varphi_{2k-1}(0) = \omega_k$, $\varphi_{2k}(0) = \omega_{k+1}$ be the arcs of the circumference B_k of positive length, satisfying the following conditions:

(a) If $\varphi_m(s') \notin D_0$, then $\varphi_m(s) \notin D_0$, $s' < s \leq \delta_m$;

(b) $[B'_{2k-1}] \cap [\widetilde{b}_k] = \omega_k$, $[B'_{2k}] \cap [\widetilde{b}_k] = \omega_{k+1}$, $[(B'_{2k-1})^0] \cap [(B'_{2k})^0] = \varnothing$;

(c) If $k \in \widetilde{j}$, then $\beta(\widetilde{b}_{j-1}, B'_{2k-1}) \geq 0$.

We denote the restriction of B'_m on $[0,s]$ by $B'_m(s)$. Let $\sigma_m = \max s : \{s \in [0,\delta_m], \varphi_m(s) \in \overline{D}_0$. Let V denote a subset of the set $\mathbb{N}_n = \{1,2,\dots,n\}$ such that $\sigma_{2m-2} > 0$, $m \in V$.

We denote by $\mathcal{L}$ the set of curves of the form

$$L = \prod_{k=1}^{n} b_k = \prod_{k=1}^{n} p_{2k-1}\widetilde{b}_k p_{2k}, \tag{9}$$

where $p_m = B'_m(s)\cdot(B'_m(s))^{-1}$, $0 \le s \le s_m < \sigma_m$ and for which there exists $\omega \in \Omega$ such that $L = \omega \circ \partial E$ when the initial point of ∂E coincides with $t_1(\omega)$. The curves contained in $\mathcal{L}$ are called admissible boundaries.

Theorem 1. *For the set $\mathcal{L}$ to be non-empty it is necessary and sufficient that, together with condition* (7)*, for each $j \in T$ there would exist a set P_j, $P_j \subset j^*$, such that*

(1) $\sigma_m > 0$ *for* $m \in P_j$*;*

(2) $P_j \cap \{2j+2k-1, 2j+2k\} \ne \varnothing$, $k = \overline{0,\nu_{j-1}}$*;*

(3) *for $k, m \in P_j$ and $k < m$ there holds*

$$\beta(\widetilde{b}_{j-1}B'_k) > \beta(\widetilde{b}_{j-1}B'_m). \tag{10}$$

Proof. The necessity of condition (7) is proved by repeating the arguments used in proving Lemma 2. Let $\mathcal{L} \ne \varnothing$, $L \in \mathcal{L}$, and $\omega_L(z) \in \Omega$ be the function conformally mapping E onto $D_0\backslash[L]$. Let $j \in T$. As P_j we shall choose a subset of $m \in j^*$ for which $|p_m| \ne 0$. Conditions (1) and (2) will be fulfilled by virtue of the definition of the class Ω. To prove (3) we choose, on l^0_k, a point z_k and connect it with $t_{j-1}(\omega)$ by a simple curve γ in E. Let D_γ be the domain lying in E and bounded by the arc γ and the arc of the unit circumferece from t_{j-1} to z_k and containing the point t_k. The assumption $\beta(\widetilde{b}_{j-1}, B'_k) < \beta(\widetilde{b}_{j-1}, B'_m)$ would imply $[b^0_m] \cap D_\gamma \ne \varnothing$, which obviously contradicts the fact that the function ω is univalent.

To prove the sufficiency we choose $\varepsilon > 0$ so small as to make the set $D_0 \backslash \bigcup\limits_{m=1}^{2n} [B'_m(\varepsilon)]$ connected. Next we consider a curve of form (9), where $0 < |p_m| < 2\varepsilon$, $m \in \bigcup\limits_{j\in T}(P_j)$, and $|p_m| = 0$ for $m \notin \bigcup\limits_{j\in T} P_j$. Let again $\omega_L(z)$ be the function conformally mapping E onto the domain $D_1 = D_0\backslash[L]$ and normalized by condition (8), $A_k = \varphi_k(\varepsilon)$, $k \in \bigcup\limits_{j\in T} P_j$, and $\varepsilon_1 > 0$ be so small that $U(A_k,\varepsilon_1)\backslash[B'_k(\varepsilon)] \subset D_1$. Let $a(\omega)$ be a homographic transformation corresponding to the rotation of the sphere mapping the point (X_k, Y_k, Z_k) into $(0,1,0)$. The function $f(z) = \sqrt{a(\omega(z)) - a(A_k)}$ maps $\omega^{-1}(U(A_k,\varepsilon_1))\backslash[B'_k(\varepsilon)]$ onto the semi-circle and therefore $f(z)$ analytically continues through the arc of the unit circumference. But $\omega(z) = a^{-1}(f^2(z) + a(A_k))$ and hence $\omega(z)$ is holomorphic in the neighborhood of $\omega^{-1}(A_k)$. Moreover, since L consists of a finite number of circumference arcs, then each point of L is an accessible point for D_1 and therefore $\omega(z)$ continues up to the homeomorphism $\omega^* : \overline{E} \to D_1 \cup L^*$ [8], where L^* denotes the set of prime ends of the domain D_1. By inequality (10) and the assumption that ω is univalent, the pre-image of the prime end ω_k will precede the pre-image of the prime end ω_m when L is described in the positive direction with respect to D_1. Denoting by l_k an arc of the unit circumference having initial point at $(\omega^*)^{-1}(\omega_k)$ and end at $(\omega^*)^{-1}(\omega_{k+1})$ and described counteclockwise, we shall have $\omega^* \circ l_k = B'_k(\varepsilon) \cdot (B'_k(\varepsilon))^{-1}$. $\square$

Theorem 1 (as follows from its proof) gives a criterion for of a curve of form (9) to belong to the set $\mathcal{L} : L \in \mathcal{L} \neq \varnothing$ iff $D_0 \backslash [L]$ is connected, $|b_n| \neq 0$, $k = \overline{1, n}$, and for any $k \in j^*$, $m \in j^*$, $\sigma_m > 0$, $\sigma_k > 0$, $k < m$ and $|p_k(L)|$, $|p_m(L)| \neq 0$ there holds

$$\beta(\widetilde{b}_{j-1}, p_k(L)) > \beta(\widetilde{b}_{j-1}, p_m(L)). \tag{11}$$

Moreover, in view of the equality $\beta(\widetilde{b}_{j-1}, B'_{2k}) = \beta(\widetilde{b}_{j-1}, B'_{2k-1}) - 1$, the proven theorem readily implies

Corollary 1. *If $L \in \mathcal{L} \neq \varnothing$ and for $k \in \widetilde{j}$, $|p_{2k-1}(L)| \neq 0$, then $|p_{2m}(L)| = 0$, $m = \overline{j, k-1}$. If $|p_{2k}(L)| \neq 0$, then $|p_{2m-1}(L)| = 0$, $m = \overline{k+1, j+\nu_j}$.*

Corollary 2. *If $L \in \mathcal{L} \neq \varnothing$ and $\beta_k(L) = \beta(b_{k-1}(L), b_k(L)) < 0$, then $\sigma_{2k-2} > 0$ and $|p_{2k-1}(L)| = |p_{2k-2}(L)| = 0$.*

§ 4. Parametrization of the Set $\mathcal{L}$

Let $I : \mathcal{L} \to \mathbb{N}_{2n} = \{1, 2, \dots, 2n\}$, where $I(L) = \{i_{k_1}, i_{k_2}, \dots, i_{k_{\rho(L)}}\}$ is the ordered set of indices of curves $p_{i_{k_s}}(L)$ of positive length contained in $L \in \mathcal{L}$. We introduce the notation $F_k = \{2k-1, 2k\}$ and assume that $i_{k_s} \in F_{k_s}$. Let further $\mathcal{I} = I(\mathcal{L})$, $\beta^-(L)$ be a subset of the set $\{\beta_1(L), \beta_2(L), \dots, \beta_n(L)\}$, consisting of negative numbers and $I^-(L) = \{i \in \mathbb{N}_n,\ \beta_i < 0\}$, $\varkappa(L) = \operatorname{card} \beta^-(L)$.

Two curves $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$ will be called equivalent ($L_1 \sim L_2$) if $I(L_1) = I(L_2)$. We denote by $\mathcal{L}_i$ the equivalence class $I(L) = i$ if $i \in \mathcal{I}$. Since $\rho(L_1) = \rho(L_2)$ and $\varkappa(L_1) = \varkappa(L_2)$ obviously hold for $L_1 \sim L_2$, the notations $\rho(i)$ and $\varkappa(i)$ are correct.

Let $a \subset \mathbb{N}_{2n}$, $i \in \mathcal{I}$, and $a \cap i = \varnothing$. We write $a \in \mathcal{E}(i)$ if $i \cup a \in \mathcal{I}$. Let $a \subset \mathbb{N}_{2n}$, $i \in \mathcal{I}$ and $a \subseteq i$. We write $a \in \mathcal{R}(i)$ if $i \backslash a \in \mathcal{I}$. $i \in \mathcal{I}$ is called maximal if $\mathcal{E}(i) = \varnothing$. The set of maximal i's is denoted by $\mathcal{I}'$. For $\mathcal{R}(i) \neq \varnothing$, i is called reducible.

Let $L = \prod_{k=1}^{n} p_{2k-1}\widetilde{b}_k p_{2k} \in \mathcal{L}$, $i_k \in I(L)$ and $t \geq 0$. Denote by $r_{i_k}^t(L)$ the curve defined by the following conditions:

$$|p_m(r_{i_k}^t(L))| = |p_m(L)|, \quad m \neq i_k$$
$$|p_{i_k}(r_{i_k}^t(L))| = t|p_{i_k}(L)|.$$

It is obvious that for any $L \in \mathcal{L}$ there exists $\varepsilon > 0$ such that $r_{i_k}^t(L) \in \mathcal{L}$, $1-\varepsilon < t < 1+\varepsilon$, $i_k \in I(L)$. From the definition of $\mathcal{R}(i)$ it follows that $a = \{i_{m_1}, i_{m_2}, \dots, i_{m_p}\} \in \mathcal{R}(i)$ iff $r_a(L) = r_{m_1}^0 \circ r_{m_2}^0 \circ \cdots r_{m_p}^0(L) \in \mathcal{L}$.

Let now $\beta_k(L) \in \beta^-(L) \neq \varnothing$, $L \in \mathcal{L}$. By virtue of Corollary 2 of Theorem 1 $\sigma_{2k-2} > 0$ and therefore there exists ε, $0 < \varepsilon < \min(\sigma_{2k-2}, \sigma_{2k-1})$, such that a curve $e_{2k}^{\varepsilon}(L)$ of form (9) defined by the conditions $|p_m(e_{2k-2}^{\varepsilon}(L))| = |p_m(L)|$, $m \neq 2k-2$, $|p_{2k-2}(e_{2k-2}^{\varepsilon}(L))| = \varepsilon$, belongs to $\mathcal{L}$. In a similar manner we define $e_{2k-1}^{\varepsilon}(L)$:

$$|p_m(e_{2k-1}^{\varepsilon}(L))| = |p_m(L)|, \quad m \neq 2k-1, \quad |p_{2k-1}(e_{2k-1}^{\varepsilon}(L))| = \varepsilon.$$

We introduce the notation $H_k = \{2k-2, 2k-1\}$. Let $i \in \mathcal{I}$, $\beta^-(i) = \{\beta_{k_1}, \beta_{k_2}, \dots, \beta_{\varkappa(i)}\}$, $\{k'_1, k'_2, \dots, k'_m\} \in I^-(L)$, $L \in \mathcal{L}_i$, $a = (a'_1, a'_2, \dots, a'_m) \in H_{k'_1} \times H_{k'_1} \times \cdots \times H_{k'_m}$. For $L \in \mathcal{L}_i$ we write the notation $e_a^{\varepsilon} = e_{a'}^{\varepsilon'} \circ e_{a'_2}^{\varepsilon_2} \circ \cdots \circ e_{a'_m}^{\varepsilon_m}(L)$. It can be easily verified that $r_a(i) = I(r_a^0(L))$ and $e_a(i) = I(e_a^{\varepsilon}(L))$ do not depend on an order of $r_{m_k}^0$ and $e_{a'_k}^{\varepsilon_k}$ in the definition of $r_a(L)$ and $e_a^{\varepsilon}(L)$.

Lemma 4. *If (a) $\sigma_{2k} = 0$ or (b) $\beta((B'_{2k-3})^{-1}, B_{2k}) > 0$, $2k-3 \in j^*$, $2k \in j^*$ holds for any $k \in \widetilde{j}$, $j \in T$, then $i \cap j^* = i' \cap j^*$ for any $i \in \mathcal{I}$, $i' \in \mathcal{I}$.*

Proof. In the case (a) we have $|p_{2k-1}(L)| \neq 0$, $L \in \mathcal{L}$ and therefore for $j \leq m \leq k$, by virtue of Corollary 1 of Theorem 1, we obtain $|p_{2m}(L)| = 0$ for any $L \in \mathcal{L}$. If $\sigma_{2m} > 0$ holds for some m, $k < m \leq j + \nu_j - 1$, then $\beta(\widetilde{b}_{j-1}, B'_{2m}) + 1 > \beta(\widetilde{b}_{j-1}, B'_{2k-1})$ and hence $|p_{2m-1}(L)| = 0$ for arbitrary m, $k < m \leq j + \nu_j - 1$, and any $L \in \mathcal{L}$. In the case (b), by Corollary 2 we have $|p_{2k-2}(L)| = |p_{2k-1}(L)| = 0$ and thus $|p_{2k-3}(L)| \cdot |p_{2k}(L)| \neq 0$ holds also for any $L \in \mathcal{L}$. $\square$

Lemma 5. *$i \in \mathcal{I}'$ iff $\varkappa(i) = 0$, $i \in \mathcal{I}$.*

Proof. Let $\varkappa(i) = 0$. As the preceding lemma suggests, it is of interest for us to consider only the case where $k \in \widetilde{j}$, $j \in T$, $\sigma_{2k-2} > 0$, $\beta((B'_{2k-3})^{-1}, B_{2k}) < 0$. Then either $F_{k-1} \subset i$ and $F_k \cap i = 2k$ and therefore $2k-1 \notin \mathcal{E}(i)$ or $F_k \subset i$ and $F_{k-1} \cap i = 2k-3$ and hence $2k-2 \notin \mathcal{E}(i)$, i.e.,

$i \in \mathcal{I}'$. Conversely, let $\beta_k < 0$. Then by Corollary 2 we have $2k-2 \in \mathcal{E}(i)$, $2k-1 \in \mathcal{E}(i)$ for $L \in \mathcal{L}_i$ and

$$\beta_k(e^{\varepsilon}_{2k-1}(L)) > 0, \quad \beta_k(e^{\varepsilon}_{2k-2}(L)) > 0. \qquad \square \tag{12}$$

From the definition of $r_a(i)$ and $e_a(i)$ and inequalities (12) it follows that

$$\begin{aligned} \varkappa(r_a(i)) &= \varkappa(i) + \operatorname{card} a, \quad \rho(r_a(i)) = \varkappa(i) - \operatorname{card} a, \\ \varkappa(e_a(i)) &= \varkappa(i) - \operatorname{card} a, \quad \rho(e_a(i)) = \varkappa(i) + \operatorname{card} a. \end{aligned} \tag{13}$$

Lemma 6. *If in $\mathcal{I}$ there exists a maximal non-reducible multi-index i, then $\mathcal{I} = i$.*

Proof. Let $i_{k_s} \in i$. Since i is non-reducible, we have $k_s \notin V$, $|\widetilde{b}_{k_s}| = 0$ and $i_{k_s} = F_{k_s} \cap i = 2k_s - 1$. Therefore $i_{k_s} \in i'$ for any $i' \in \mathcal{I}$, i.e., $i \subseteq i'$ for any $i \in \mathcal{I}$. But since i is maximal, we obtain $i = i'$. $\square$

Theorem 2. $\rho(i) + \varkappa(i) \equiv \rho = $ *const*, $i \in \mathcal{I}$.

Proof. First, we shall prove $\rho(i) \equiv$ *const*, $i \in \mathcal{I}'$. As suggested by Lemma 6, we should consider only the case with reducible $i \in \mathcal{I}'$. Let $i_{k_s} \in \mathcal{R}(i)$, $i \in \mathcal{I}'$. Then by Corollary 2 $\operatorname{card}(i \cap H_{k_s}) = 1$. Therefore $\rho(i) = \rho(i')$, $i, i' \in \mathcal{I}'$. If however $i \notin \mathcal{I}'$, then $\varkappa(i) > 0$ and therefore there exists $\beta_k(i) < 0$. But then $e_{2k-1}(i) \in \mathcal{I}$, and by (13) we have $\rho(e_{2k-1}(i)) + \varkappa(e_{2k-1}(i)) = \rho(i) + \varkappa(i)$. $\square$

Let $i \in \mathcal{I}'$, and $\tau_i : I' \to \mathbb{N}_\rho$ be the order preserving bijection. We define the mapping $\chi : \mathcal{L}' = \bigcup\limits_{i \in \mathcal{I}'} \mathcal{L}_i \to \mathbb{R}_\rho$ as follows:

$$\chi_m(L) = (-1)^{\tau_L^{-1}(m)} \cdot |p_{\tau_L^{-1}(m)}(L)|, \quad m \in \mathbb{N}_\rho, \quad \text{where} \quad \tau_L = \tau_{I(L)}.$$

Let us show that χ is injective. The statement is obvious for $L_1 \sim L_2$. If $I(L_1) \neq I(L_2)$, then since for $k \in \widetilde{j} \subset V$, $I(L_1) \cap H_k \neq I(L_2) \cap H_k$ there exists $m \in \mathbb{N}_\rho$ such that $\tau_{L_1}^{-1}(m) \in H_k$, $\tau_L^{-1}(m) \in H_k$, $\tau_{L_1}^{-1}(m) \neq \tau_{L_2}^{-1}(m)$ and therefore $\chi_m(L_1)$ and $\chi_m(L_2)$ will have different signs. Let now $L \in \mathcal{L}_i$, $i \notin \mathcal{I}'$, $a \in H_{m_1} \times H_{m_2} \times \cdots \times H_{m_{\varkappa(L)}}$, $m_j \in I^-(L)$, $j = \overline{1, \varkappa(L)}$. Then $I(e^{\varepsilon}_a(L)) \in \mathcal{I}'$. Define $\chi(L)$ by setting

$$\chi_m(L) = \begin{cases} \chi_m(e^{\varepsilon}_a(L)), & \tau^{-1}_{e^{\varepsilon}_a(L)}(m) \in I(L), \\ 0, & \tau^{-1}_{e^{\varepsilon}_a(L)}(m) \notin I(L). \end{cases} \tag{14}$$

It is obvious that χ defined in this manner does not depend on the choice of a. The injectivity of the mapping χ defined by (14) is proved similarly.

Theorem 3. $\chi(L)$ *is a domain.*

Proof. We shall show that $\chi(L)$ is open. Let $\chi(L) = (\chi_1(L), \dots, \chi_\rho(L))$, $L \in \mathcal{L}$, and $\chi_k(L) \neq 0$. We introduce the notation $U_k = \{\chi_k = \chi_k(r^t(L)), 1 - \varepsilon_k < t < 1 + \varepsilon_k\}$, where $\varepsilon_k > 0$ is so small that $D_0 \backslash [r^t_{\tau_L(k)}(L)]$ is connected for any t, $|t-1| < \varepsilon$. If $\chi_k(L) = 0$, then, taking $\varepsilon > 0$ so small that $D_0 \backslash [e^\varepsilon_a(L)]$, $a \in H_{m_1} \times H_{m_2} \times \dots \times H_{m_{\varkappa(L)}}$, is connected, we find that $\tau^{-1}_{e^\varepsilon_a(L)}(k)$ is equal either to $2m_j - 2$ or to $2m_j - 1$, $m_j \in I^-(L)$. We write

$$\begin{aligned}
U_k^+ &= \{\chi_j = \chi_j(L),\ j \neq k,\ \chi_k = \chi_k(e^t_{2m_j-2}(L)),\ 0 \le t < \varepsilon\},\\
U_k^- &= \{\chi_j = \chi_j(L),\ j \neq k,\ \chi_k = \chi_k(e^t_{2m_j-1}(L)),\ 0 < t < \varepsilon\},\\
U_k &= U_k^+ \cup U_k^-.
\end{aligned}$$

It can be easily verified that $U(L) = U_1 \times U_2 \times \cdot \times U_\rho \subset \chi(\mathcal{L})$ is the neighborhood of $\chi(L)$.

Now we shall show that $\chi(L)$ is connected. Fix $\varepsilon_0 > 0$ such that $D_0 \backslash \bigcup\limits_{m=1}^{2n} [B'_m(\varepsilon_0)]$ is connected. Let $p(L) = \min\limits_{i_s \in I(L)} |p_{i_s}(L)|$ and assume that $L_1 \sim L_2$, $I(L_1) = i \in \mathcal{I}'$. Fix the number ε_1, $0 < \varepsilon_1 < \min\big(\varepsilon_0, |p(L_1)|, |p(L_2)|\big)$. Let $\psi_s(t)$ be a monotonically decreasing function on $[s-1, s]$, $\psi_s(s-1) = 1$, $\psi_s(s) = \varepsilon_1 \cdot |p_{\tau_i^{-1}(s)}(L_k)|^{-1}$, $s = \overline{1, \rho}$. Denote by $g_s(L_k)$ a segment $g_s(L_k) = \{\chi_m = \chi_m(L_k), m \neq \tau_i(i_s);\ \chi_{\tau_i(i_s)} = \chi_{\tau_i(i_s)}(r^{\psi_s(t)}_{i_s}(L_k))$. By construction, $[g_s(L_k)] \subset \chi(\mathcal{L})$, $k = 1, 2$, $s = \overline{1, \rho}$, and therefore the graph of the broken line $g(L_1, L_2) = g(L_1) \cdot (g(L_2))^{-1}$, where $g(L_k) = \prod_{s=1}^\rho g_s(L_k)$ and which connects $\chi(L_1)$ and $\chi(L_2)$, is contained in $\chi(\mathcal{L})$.

Let now $I(L_1) \in \mathcal{I}'$, $I(L_2) \in \mathcal{I}'$, $I(L_1) \neq I(L_2)$. We write $L_k^{\varepsilon_1} = \chi^{-1}(g(L_k)(\rho))$, $k = 1, 2$. As suggested by Lemma 4, it is of interest for us to consider only the case where $I(L_1) \cap j^* \neq I(L_2) \cap j^*$, $\sigma_{2k} > 0$, $k \in \widetilde{j} \subset V$, and $\beta((B'_{2k-3})^{-1}, B'_{2k}) < 0$ for any $k \in \widetilde{j}$, $2k-3 \in j^*$, $2k \in j^*$.

Let $F_{k_1} \subset I(L_1)$, $F_{k_2} \subset I(L_2)$, $k_1 \in \widetilde{j}$, $k_2 \in \widetilde{j}$, $k_1 < k_2$. Then by Corollary 1 of Theorem 1 we have $2m-1 \in I(L_1)$, $j \le m \le k_1$, $2m \notin I(L_1)$, $j \le m \le k_1 - 1$; $2m-1 \notin I(L_1)$, $k_1 + 1 \le m < j + \nu_j - 1$; $2m \in I(L_1)$, $k_1 + 1 \le m < j + \nu_j - 1$; $2m-1 \in I(L_2)$, $j \le m \le k_2$; $2m \in I(L_2)$, $j \le m \le k_2 - 1$; $2m-1 \notin I(L_2)$, $k_2 + 1 \le m \le j + \nu_j - 1$; $2m \in I(L_2)$, $k_2 + 1 \le m \le j + \nu_j - 1$. Since $2k_1 \in \mathcal{R}(I(L_1))$, the segment

$$\begin{aligned}
h_{2k_1}(L_1) = \big\{\chi_q = \chi_q(L_1^\varepsilon),\ q \neq \tau_{L_1}(2k_1);\ \chi_{\tau_{L_1}(2k_1)} = \\
= \chi_{\tau_{L_1}(2k_1)}(r^{\varphi_{2k_1}(t)}_{2k_1}(L_1^{\varepsilon_1}))\big\},
\end{aligned}$$

where $\varphi_{2k_1}(t)$ is a decreasing function on $[0,1]$, $\varphi_{2k_1}(0) = 1$, $\varphi_{2k_1}(1) = 0$, is contained in $\chi(\mathcal{L})$, including the point $h_{2k_1}(1)$, and $\beta_{k_1}(r^{\varphi_{2k_1}(1)}_{2k_1}(L_1^{\varepsilon_1})) =$

$\beta_{k_1}(r^0_{2k_1}(L_1^{\varepsilon_1}))<0$. Therefore $2k_1+1\in\mathcal{E}(r^0_{2k_1}(L_1^{\varepsilon_1}))$ and hence the segment

$$h^*_{2k_1+1}=\big\{\chi_q=\chi_q(r^0_{2k_1}(L_1^{\varepsilon})),\ q\neq\tau_{L_1}(2k_1),$$
$$\chi_{\tau_{L_1}(2k_1)}=\chi_{\tau_{L_1}(2k_1)}(e_{2k_1+1}^{\varphi^*_{2k_1+1}(t)}(r^0_{2k_1}(L_1^{\varepsilon_1}))\big\},$$

where $\varphi^*_{2k_1+1}(t)$ is an increasing function on $[1,2]$, $\varphi^*_{2k_1+1}(1)=0$, $\varphi^*_{2k_1+1}(2)=\varepsilon_1$, is contained in $\chi(\mathcal{L})$, including the end point. By construction, we have $F_{k_1+1}\subset I\big(e_{2k_1+1}^{\varphi^*_{2k_1+1}(2)}(r^0_{2k_1}(L_1^{\varepsilon_1})\big)\in\mathcal{I}'$. Continuing the process, after a finite number of steps we shall obtain the broken line $h=h_{2k_1}\cdot h^*_{2k_1+1}\cdot h_{2k_1+2}\cdot h^*_{2k_1+3}\cdots h_{2k_2-2}\cdot h^*_{2k_2-1}$ connecting the points $\chi(L_1)$ and $\chi(L_2)$ in $\chi(\mathcal{L})$.

Finally, let $L\in\mathcal{L}$, $I(L)\notin\mathcal{I}'$, and $I^-(L)=\{m_1,m_2,\dots,m_{\varkappa(L)}\}$, $a\in H_{m_1}\times H_{m_2}\times\ldots\times H_{m_{\varkappa(L)}}$. Then the broken line $f=f_1\cdot f_2\cdots f_{\varkappa(L)}$, where f_s is a segment in $\chi(\mathcal{L}):f_s=\{\chi_q=\chi_q(L),\ q\neq\tau_{e_a^{\varepsilon_1}(L)}(a_s)$; $\chi_{\tau_{e^{\varepsilon_1}(L)}(a_s)}=\chi_{\tau_{e^{\varepsilon_1}(L)}(a_s)}\big(e_{a_s}^{\sigma_s(t)}(e_{a_{s-1}}^{\varepsilon_1}\circ\cdots e_{a_1}^{\varepsilon_1}(L))\big)$, where $\sigma_s(t)$ is an increasing function on $[s-1,s]$, $\sigma(s-1)=0$, $\sigma(s)=\varepsilon_1$, is also contained in $\chi(\mathcal{L})$. By construction, $I(e^{\varepsilon_1}_{a_{\varkappa(L)}}\circ e^{\varepsilon_1}_{a_{\varkappa(L)-1}}\circ\cdots e^{\varepsilon_1}_{a_1})\in\mathcal{I}'$ and we have thus proved the connectedness of $\chi(L)$. $\square$

§ 5. Construction of a Minimal Surface by a Given Admissible Boundary

Let $S\in\mathfrak{S}$ be represented by formulas (1). By differentiating (1) with respect to θ, $e^{i\theta}\in l_k^0$, $k=\overline{1,n}$, we obtain

$$\begin{aligned}(x^1)_\theta&=-\frac{1}{2}\operatorname{Im}F'(e^{i\theta})(1-\omega^2(e^{i\theta}))e^{i\theta},\\ (x^2)_\theta&=-\frac{1}{2}\operatorname{Re}F'(e^{i\theta})(1+\omega^2(e^{i\theta}))e^{i\theta},\\ (x^3)_\theta&=-\operatorname{Im}F'(e^{i\theta})\omega(e^{i\theta})e^{i\theta}.\end{aligned}\tag{15}$$

Since $S\in\mathfrak{S}$ we have $((x^1)_\theta,(x^2)_\theta,(x^3)_\theta)=d_k(\theta)T$, $d_k>0$ for $e^{i\theta}\in l_k^0$. Consider (15) as a system of equations with respect to $\operatorname{Re}F'(e^{i\theta})$ and $\operatorname{Im}F'(e^{i\theta})$. The compatibility condition of system (15) is expressed by equation (4) and hence is fulfilled automatically. By solving (15) we obtain

$$F'(e^{i\theta})=-2i\frac{X_k-iY_k+Z_k\overline{\omega}(e^{i\theta})}{1+|\omega(e^{i\theta})|^2}e^{i\theta}d_k(\theta)\tag{16}$$

or

$$\operatorname{Re}(Q(t)tF'(t))=0,\tag{17}$$

where

$$Q(t)=X_k+iY_k+Z_k\omega(t),\quad t\in e^{i\theta}\in l_k,\quad k=\overline{1,n}.\tag{18}$$

Thus $F'(z)$ is a solution of the Riemann–Hilbert boundary value problem (17) with a piecewise-continuous coefficient $Q(t)$ which is analytic on each arc l_k^0. Moreover, as follows from [6], $F'(z)$ is bounded in the neighborhood of the points $t_{k_j}(\omega)$, $j=\overline{1,\alpha^-}$, corresponding to the negative values of α_j or, speaking in terms of [9], $F'(z)$ belongs to the class $h(t_{k_1}, t_{k_2}, \dots, t_{k_{\alpha^-}})$. The number α^- of points in whose neighborhood $F'(z)$ is bounded is defined by the initial broken line $\widetilde{\Gamma}$ and does not depend on $L \in \mathcal{L}$.

We rewrite (17) as

$$Q(t)tF'(t) + \overline{Q}(t)\overline{t}\overline{F}'(t) = 0.$$

Since $F^-(z) = \overline{F}(1/\overline{z})$ is bounded at infinity, $(F^-(z))'$ has zero of second order at infinity. Moreover, for $z = 1/\overline{\zeta}$ we have

$$\left(\overline{\frac{dF}{dz}}\right) = -\zeta^2 \frac{dF^-}{d\zeta} \tag{19}$$

and therefore on the unit circumference the boundary values $\dfrac{dF^+}{dt} = \dfrac{dF}{dt}$ and $\dfrac{dF^-}{dt}$ will be connected through the relation

$$\overline{\frac{dF^+}{dt}} = -t^2 \frac{dF^-}{dt} \tag{20}$$

which makes it possible to rewrite the boundary conditon (17) as

$$\frac{dF^+}{dt} = e^{2i \operatorname{Arg} \overline{Q}(t)} \frac{dF^-}{dt}. \tag{21}$$

By direct calculations we ascertain that $\sin \operatorname{Arg}(\overline{Q}(t_k - 0))/\overline{Q}(t_k+0)) = -\sin \pi\alpha_k$, $\cos \operatorname{Arg}(\overline{Q}(t_k-0))/\overline{Q}(t_k+0)) = \cos \pi\alpha_k$, $k = \overline{1,n}$, $t_{n+1} = t_1$.

On each arc l_k we choose a branch $\operatorname{Arg} \overline{Q}(t)$ such that

$$\frac{1}{\pi}\big[\operatorname{Arg} \overline{Q}(t_k - 0) - \operatorname{Arg}(t_k + 0)\big] = -\alpha_k.$$

By simple calculations we obtain $\frac{1}{\pi}\big[\operatorname{Arg}\overline{Q}(t_1 - 0) - \operatorname{Arg}(t_1+0)\big] = 2 - \alpha_1$. Let us rewrite the boundary condition (21) as

$$\frac{dF^+}{dt} = \frac{e^{2i \operatorname{Arg} \overline{Q}(t)}}{t^2} t^2 \frac{dF^-}{dt} \tag{22}$$

Introducing the notations $\Phi^+(z) = \frac{dF^+}{dz}$ and $\Phi^-(z) = z^2 \frac{dF^-}{dz}$ and recalling that $z^2 \frac{dF^-}{dz}$ is bounded at infinity, we find that the index of the boundary value problem in the class of functions bounded at infinity and belonging to $h(t_{k_1}, t_{k_2}, \dots, t_{k_{\alpha^-}})$ is equal to zero. Thus a general solution of problem (22) is given by the formula

$$\Phi(z) = C\exp\frac{1}{\pi}\int_0^{2\pi}\frac{\operatorname{Arg}\overline{Q}(e^{i\theta})-\theta}{e^{i\theta}-z}de^{i\theta}, \tag{23}$$

where C is an arbitrary complex constant. From (23) we have

$$\begin{aligned} \frac{dF^+}{dz} &= C\exp\frac{1}{\pi}\int_0^{2\pi}\frac{\operatorname{Arg}\overline{Q}(e^{i\theta})-\theta}{e^{i\theta}-z}de^{i\theta}, \quad |z|<1, \\ \frac{dF^-}{dz} &= \frac{1}{z^2}C\exp\frac{1}{\pi}\int_0^{2\pi}\frac{\operatorname{Arg}\overline{Q}(e^{i\theta})-\theta}{e^{i\theta}-z}de^{i\theta}, \quad |z|>1. \end{aligned} \tag{24}$$

Condition (19) is fulfilled by an appropriate choice of the constant C. Using simple transformations [9], we find that C must satisfy the equality

$$\overline{C}\exp\Big(-\frac{i}{\pi}\int_0^{2\pi}\big[\operatorname{Arg}\overline{Q}(e^{i\theta})-\theta\big]d\theta = -C. \tag{25}$$

Assuming $C=\lambda_0 e^{i\theta}$, $\lambda_0 \geq 0$ we obtain

$$\alpha = -\frac{1}{2\pi}\int_0^{2\pi}\big[\operatorname{Arg}\overline{Q}(e^{i\theta})-\theta\big]d\theta \pm \frac{\pi}{2}. \tag{26}$$

The substitution of (25) into (24) gives

$$\frac{dF}{dz} = \pm\lambda_0\exp\Big(-\frac{1}{2\pi i}\int_0^{2\pi}\big[\operatorname{Arg}\overline{Q}(e^{i\theta})-\theta\big]\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\Big). \tag{27}$$

Let now S^ω be a minimal surface defined by (1), where $F'(z)$ and $Q(t)$ are given by (27) and (18), respectively. In (27) we take the sign "$-$" to show that $S^\omega \in \mathfrak{S}$. For $t \in l_k^0$ (27) implies

$$\operatorname{Arg}F'(t) = \operatorname{Arg}(X_k - iY_k + Z\overline{\omega}(t)) - \frac{\pi}{2} - \theta. \tag{28}$$

On account of (28), (4), and the equality $|X_k - iY_k + Z\overline{\omega}| = 1$ we obtain by direct calculations

$$\begin{aligned} (x^1)_\theta &= \frac{\lambda_0}{2}X_k|F'(e^{i\theta})|(1+|\omega(e^{i\theta})|^2), \\ (x^2)_\theta &= \frac{\lambda_0}{2}Y_k|F'(e^{i\theta})|(1+|\omega(e^{i\theta})|^2), \end{aligned}$$

$$(x^3)_\theta = \frac{\lambda_0}{2} Z_k |F'(e^{i\theta})|(1 + |\omega(e^{i\theta})|^2).$$

Thus $S^\omega \in \mathfrak{S}$ and heence we have the bijection $\delta : \mathfrak{S} \to \varphi$.

Denote by Γ^ω the boundary S^ω. Passing to the limit in (27) as $z \to t \neq t_k$, we obtain [9]

$$|F'(e^{i\psi})| = \lambda_0 \exp \frac{1}{2\pi} \int_0^{2\pi} \big[\operatorname{Arg} \overline{Q}(e^{i\theta}) - \theta \big] ctg \frac{\theta - \psi}{2} d\theta,$$

which implies [2]

$$|\Gamma_k^\omega| = \frac{\lambda_0}{2} \int_{\theta_k}^{\theta_{k+1}} \exp \Big\{ \frac{1}{2\pi} \int_0^{2\pi} \big[\operatorname{Arg} \overline{Q}(e^{i\theta}) - \theta \big] \operatorname{ctg} \frac{\theta - \psi}{2} d\theta \Big\} (1 + |\omega| e^{i\theta}))^2) d\psi.$$

Fix λ_0 so that $\sum_{k=1}^{n-3} |\Gamma_k^\omega| = 1$. We obtain the bijective mapping $P \circ \delta \circ \chi^{-1}(\mathcal{L}) : \chi(\mathcal{L}) \to \mathcal{P}$ and hence the parametrization of the set $\mathcal{P}$ by the domain $\chi(\mathcal{L})$.

§ 6. Dimension of the Domain $\chi(\mathcal{L})$.

Let $L \in \mathcal{L}$, $\omega_L(z)$ and $F'_\omega(z)$ be the functions constructed in the preceding sections. For each k, $k \in \mathbb{N}_n$ the vector $(\omega_L(e^{i\theta}))_\theta$ is oriented along the tangent to circumference (4) at the point $t = e^{i\theta}$ and hence $X_k + iY_k + Z_k\omega(t) = if_k(t)(\omega_L(e^{i\nu}))_\theta$, where $f_k(t)$, $t \in l_k$, is the real function. This gives us

$$0 = \operatorname{Re} \big(X_k + iY_k + Z_k \omega(t) \big) tF'(t) = f_k(t) \operatorname{Im} tF'(t)(\omega_L(e^{i\theta}))_\theta$$

so that $\operatorname{Im} F'(t)t(\omega_L(t))_\theta d\theta^2 = 0$, from which in view of $(\omega_L(t))_\theta = t\dfrac{d\omega_L(t)}{dt}$ we obtain

$$\operatorname{Im} F'(t)\omega'(t)t^2 d\theta^2 = -\operatorname{Im} F'(t)\omega'(t)dt^2 = 0.$$

Thus the quadratic differential $\eta(z) = F'(z)\omega'(z)dz^2$ is analytically continuable onto the entire Riemann sphere if for the point z^* symmetric to z with respect to the unit circumference we set [10] $\eta(z^*) = \overline{\eta(z)}$. Taking into account that $\omega'_L(z) \neq 0$, $|z| < 1$, and in the neighborhood of the point t_k we have $\omega'_L(z) = (z - t_k)^{-\beta_k(L)}\omega_k(z)$, where $\omega_k(z)$ is holomorphic and nonvanishing, in this neighborhood [11], and recalling the behavior of $F'(z)$ in this neighborhood [6], we conclude that zeros and poles of the differential $\eta(z)$ are located on the unit circumference. Moreover, by the definition of α_k and β_k and also on account of Lemma 3 the poles and zeros of $\eta(z)$ can be only simple. Let $I(L) \in \mathcal{I}'$. Then by virtue of Lemma 5 $\varkappa(L) = 0$ and therefore $\eta(z)$ cannot have zero at any of the points $t_k(\omega)$, $k = \overline{1, n}$. The poles of $\eta(z)$ coincide with t_k corresponding to $\alpha_k > 0$. Hence the number $P(\eta)$ of the poles of $\eta(z)$ on the unit circumference (and on the entire sphere) is equal to $n-\alpha^-$. Furthermore, the differential $\eta(z)$ cannot have zero at any of the points $t_k(\omega)$ and hence the number $N(\eta)$ of zeros of $\eta(z)$ coincides with the number of critical points $\omega_L(z)$ on $\bigcup\limits_{k=1}^{n} l_k^0$ or, as follows from Theorem 2, with the number ρ. On the other hand, we have the equality [10] $\operatorname{ord}\eta(z)=N(\eta)-P(\eta)=-4$. Recalling that ρ is constant on I and substituting $N(\eta)=\rho$ and $P(\eta)=n-\alpha^-$ into the above equality, we obtain

Theorem 4. *If $\mathcal{L}\neq\varnothing$, then $\rho=n-\alpha^- - 4$.*

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