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# On the fine spectra of some averaging operators

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**Abstract**

The aim of this text is the study of the fine spectra for a class of Cesàro generalized operators, Rhaly operators, when those are defined on the spaces $l^p$, $p > 1$.

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The averaging operators $A$ are determined by relations

$$\inf_{x\in X} f(x) \leq A(f) \leq \sup_{x\in X} f(x)\ ,$$

$\forall\ f \in F = \{f \mid f : X \to \mathbb{R}\}$, where $\varnothing \neq X \subset \mathbb{R}$.

$A(f)$ is the mean of $f$ for the operator $A$.

For $a = (a_n) \in s$, Rhaly operator $R_a : s \to s$

$$(R_a f)(n) = a_n \sum_{i=0}^{n} f(i)\ , \qquad n \in \mathbb{N},$$

for every $f = (f(n))_{n\in\mathbb{N}} \in s = \{g = (g(n))_{n\in\mathbb{N}} : g(n) \in \mathbb{C}\}$.

In this case, Rhaly operator $R_a$ determines and is determined by an infinite matrix, lower triangular, noted also with $R_a$:

$$R_a = \begin{vmatrix} a_0 & 0 & 0 & \cdots & \\ a_1 & a_1 & 0 & \cdots & \\ a_2 & a_2 & a_2 & \cdots & \\ & & & \ddots & \\ a_n & a_n & a_n & a_n & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

The space $s$ may by replaced with the spaces of sequences $l^p (p > 1)$;

$$l^p = \left\{ f \in s : \sum_{n=0}^{\infty} |f(n)|^p < \infty \right\}.$$

The dual of an Rhaly operator $R_a : l^p \to l^p$ is the operator $R_a^* : l^q \to l^q$, where $q$ is the conjugated index of $p$, to which is associated by the infinite matrix:

$$R_a^* = \begin{vmatrix} \overline{a}_0 & \overline{a}_1 & \overline{a}_2 & \ldots & \overline{a}_n & \ldots \\ 0 & \overline{a}_1 & \overline{a}_2 & \ldots & \overline{a}_n & \ldots \\ 0 & 0 & \overline{a}_2 & \ldots & \overline{a}_n & \ldots \\ \vdots & \vdots & \vdots & \ddots & & \end{vmatrix}$$

For $a = \left( \frac{1}{n+1} \right)_{n \in \mathbb{N}} \in s$ one obtains the discrete Cesàro operator and for $a = \left( \frac{1}{(n+1)^z} \right)_{n \in \mathbb{N}}$, with $z \in \mathbb{C}$, one obtains the $z$-Cesáro operator. If $a_n = \frac{p_n}{P_n}$, with $p_0 > 0$, $p_n \geq 0$ and $P_n = \sum_{k=0}^{n} p_k$, Rhaly operator $R_a$ is an example of operator called weighted mean matrices.

G. Leibowitz [2] studies the algebraic - topological structure for the set of the Rhaly operators, continuity and compactness of these operators, defined on the spaces $l^p$, $p > 1$. Also, he investigate the continuity of these operators when they are defined an the spaces of sequence $c_0$ and $l^\infty$.

H. C. Rhaly [5] studies the spectrum and point spectrum for $R_a : l^2 \to l^2$.

In a recent book "Weighted mean operator", K. G. Grosse-Erdmann studies the spectra for weighted mean matrices (in 1998).

In this text $I$ present some results concerning the spectra of $R_a : l^p \to l^p$ $(p > 1)$, where:

$\rho(R_a, l^p) = \{\lambda \in \mathbb{C} : \lambda I - R_a$ is bijective and $(\lambda I - R_a)^{-1}$ is continuous$\}$;

$\sigma(R_a, l^p) = \mathbb{C} \setminus \rho(R_a, l^p)$;

$\sigma_p(R_a, l^p) = \{\lambda \in \mathbb{C} : \lambda I - R_a$ is not injective$\}$

$\sigma_c(R_a, l^p) = \{\lambda \in \mathbb{C} : \lambda I - R_a$ is injective, is not surjective and $\overline{(\lambda I - R_a)(l^p)} = l^p\}$

$\sigma_r(R_a, l^p) = \{\lambda \in \mathbb{C} : \lambda I - R_a$ is injective, and $\overline{(\lambda I - R_a)(l^p)} \neq l^p\}$.

$A$ Rhaly operator $R_a : l^p \to l^p$ $(p > 1)$ is correctly defined if the sequence $((n+1)a_n)$ is bounded and $R_a$ is continuous.

Let $S = \overline{\{a_n : n \in \mathbb{N}\}}$.

**Theorem 1.**

a) *If* $((n+1)a_n)$ *is bounded, then* $R_a \in B(l^p)$*for any* $p > 1$ *and*

$$\|R_a\| \leq \frac{p}{p-1} \sup |(n+1)a_n|.$$

b) *If* $\lim\limits_{n\to\infty} (n+1)a_n = 0$, *then* $R_a$ *is compact in* $l^p$ *for any* $p > 1$.

c) *If* $\lim\limits_{n\to\infty} |(n+1)a_n| = \infty$, *then* $R_a$ *isn't continuous,* $\forall\ p > 1$.

**Proof.** In the article [4].

**Lemma 1.** *Let* $R_a$ *be a Rhaly matrix* $(a \in s)$, $C = \lambda I - R_a$, *such that*

$c_{jj} \neq 0 \ \ \forall\, j \in \mathbb{N}$. *Then* $C^{-1}$ *has the entries:*

$$c_{jj} = \frac{1}{\lambda - a_j} \quad \forall\, j \in \mathbb{N}$$
$$c_{ij} = 0 \quad \forall\, i, j \in \mathbb{N}\ , \quad i < j$$
$$c_{ij} = c_{j+r,j} = \lambda^{r-1} a_{j+r} \left[ \prod_{k=j}^{j+r} (\lambda - a_k) \right]^{-1} \quad \forall\, r \in \mathbb{N}^*\ ; \ \ i, j \in \mathbb{N}\ , \quad i = j + r.$$

(1)

**Proof.** The method of the mathematical induction is used.

To calculate the entries $c_{jj}$ are used determinants of some matrices lower triangular with entries of diagonal $\lambda - a_k$, $k \neq j$.

Obviously $c_{00}$ has form $\frac{1}{\lambda - a_0}$.

Supposing that for $j \in \mathbb{N}^*$, $c_{i0}, c_{i1}, ..., c_{i,i-1}$ have the specified form, can be prove that entries $c_{i+1,0}, c_{i+1,1}, ..., c_{i+1,i}$ are gived by the specified relations, too.

The condition $c_{i+1,i}(\lambda - a_i) + c_{i+1,i+1}(-a_{i+1}) = 0$ implies

$$c_{i+1,i} = \frac{a_{i+1}}{(\lambda - a_i)(\lambda - a_{i+1})}.$$

The condition

$$c_{i+1,i-1}(\lambda - a_{i-1}) + c_{i+1,i}(-a_i) + c_{i+1,i+1}(-a_{i+1}) = 0 \Longleftrightarrow$$
$$\Longleftrightarrow c_{i+1,i-1} = \frac{a_i}{\lambda - a_{i-1}} \cdot \frac{a_{i+1}}{\prod_{k=i}^{i+1} (\lambda - a_k)} + \frac{a_{i+1}}{\lambda - a_{i-1}} \cdot \frac{1}{\lambda - a_{i+1}}$$

implies

$$c_{i+1,i-1} = \frac{\lambda a_{i+1}}{(\lambda - a_{i-1})(\lambda - a_i)(\lambda - a_{i+1})}.$$

From the equality

$$c_{i+1,i-2}(\lambda - a_{i-2}) + c_{i+1,i-1}(-a_{i-1}) + c_{i+1,i}(-a_i) + c_{i+1,i+1}(-a_{i+1}) = 0$$

result that

$$c_{i+1,i-2} = \frac{\lambda a_{i-1}a_{i+1}}{(\lambda-a_{i-2})(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})}+$$
$$+\frac{a_i a_{i+1}(\lambda-a_{i-1})}{(\lambda-a_{i-2})(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})}+$$
$$+\frac{a_{i+1}(\lambda-a_{i-1})(\lambda-a_i)}{(\lambda-a_{i-2})(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})}=$$
$$=\frac{\lambda^2 a_{i+1}}{(\lambda-a_{i-2})(\lambda-a_{i-1})(\lambda-a_i)(\lambda-a_{i+1})},\ \text{too.}$$

In the same way are found the expressions for $c_{i+1,i-3}$, $c_{i+1,i-4}$, ..., $c_{i+1,1}$, $c_{i+1,0}$.

**Theorem 2.** *Consider $R_a$ as an operator on $l^p$, $p>1$, such that $((n+1)a_n)$ is bounded. Then,*

$$\sigma(R_a, l^p) \subseteq \left\{\lambda : \max_{k\in\mathbb{N}}\left|\frac{\lambda}{\lambda-a_k}\right| \geq 1\right\}.$$

**Proof.** From the hypothesis the sequence $(|a_n|)$ is bounded. So, $\exists m_1$, $m_2 \in \mathbb{R}$, such that: $m_1 \leq |a_n| \leq m_2 \quad \forall n \in \mathbb{N}$.

Let $\lambda \in \mathbb{C}^*$ with $\max_{k\in\mathbb{N}}\left|\frac{\lambda}{\lambda-a_k}\right| < 1$. $m_2$ is choose such that $|\lambda| \neq |m_2|$.

For $i=0$,

$$\sum_{j=0}^{\infty}|c_{ij}| = \frac{1}{|\lambda-a_i|} + |\lambda|^{i-1}|a_i|\left[\prod_{k=0}^{i}|\lambda-a_k|\right]^{-1} + |\lambda|^{i-2}|a_i|\left[\prod_{k=1}^{i}|\lambda-a_k|\right]^{-1} + ...+$$

$$+|\lambda|^0|a_i|\left[\prod_{k=i-1}^{i}|\lambda-a_k|\right]^{-1} \leq \left|\frac{1}{|\lambda|-|m_2|}\right| + \frac{|m_2|}{|\lambda|^2}\left|\frac{|\lambda|^{i+1}}{\prod_{k=0}^{i}|\lambda-a_k|} + \frac{|\lambda|^i}{\prod_{k=1}^{i}|\lambda-a_k|} + ...+\right.$$

$$\left.+\frac{|\lambda|^2}{\prod_{k=i-1}^{i}|\lambda-a_k|}\right| \leq \left|\frac{1}{|\lambda|-|m_2}\right| + \frac{|m_2|}{|\lambda|^2}\cdot\frac{|\lambda|^2}{|\lambda-a_{i-1}||\lambda-a_i|}\cdot\frac{1}{1-\max_{k\in\mathbb{N}}\left|\frac{\lambda}{\lambda-a_k}\right|}.$$

So $C \in B(l^p)$, $\forall p > 1$. Result that for $\lambda$ with the property that $\left|\dfrac{\lambda}{\lambda - a_k}\right| < 1$, $\lambda \in \rho(R_a, l^p)$.

**Theorem 3.** *Let $R_a$ be a Rhaly matrix with $a = (a_n)$ a sequence of real numbers such that $((n+1)a_n)$ is bounded. Then, for any $p > 1$:*

$$\sigma(R_a, l^p) \supseteq \left\{\lambda \in \mathbb{C} \ : \ \left|\lambda - \frac{q}{2}\sup(n+1)|a_n|\right| \leq \frac{q}{2}\sup(n+1)|a_n|\right\} \cup S.$$

**Proof.** The sequence $(|a_n|)$ being a real sequence can be defined $\limsup |a_n|$ and $\liminf |a_n|$. Is noted with $\delta = \limsup |a_n|$ and $\delta = \liminf |a_n|$. Then, obviously that $|\lambda - \delta| \leq |\lambda|$.

$$|c_{ij}| = |\lambda|^{r-1}|a_{j+r}|\frac{1}{\prod\limits_{k=j}^{j+r}|\lambda - a_k|} = \frac{|a_{j+r}|}{|\lambda|^2}\cdot\frac{1}{\prod\limits_{k=j}^{j+r}\left|1 - \dfrac{a_k}{j}\right|} \ ; \quad r \in \mathbb{N}^* \ , \quad i = j + r.$$

$\left|1 - \dfrac{a_k}{\lambda}\right| \leq 1 \Leftrightarrow |1 + a_k(\alpha + i\beta)| \leq 1 \Leftrightarrow (1 + a_k\alpha)^2 + a_k^2\beta^2 \leq 1,$ where $-\dfrac{1}{\lambda} = \alpha + i\beta$.

But,

$$\left|1 - \frac{\delta}{\lambda}\right| \leq 1 \Leftrightarrow (1 + \delta\alpha)^2 + \delta^2\beta^2 \leq 1 \Rightarrow$$

$$\Rightarrow \exists N \in \mathbb{N} \ , \quad \forall n \geq N : \left(1 + \alpha \sup_{i \geq n} a_i\right)^2 + \beta^2\left(\sup_{i \geq n} a_i\right)^2 \leq 1.$$

Results that $(1 + \alpha a_k)^2 + \beta^2 a_k^2 \leq 1 \ \forall k \in \mathbb{N}$. It is obtained that for $i > N$,

$$\sum_{j=N}^{i}|c_{ij}| \geq \sum_{k=0}^{r}\frac{|a_{j+k}|}{|\lambda|} \geq |m_1||\lambda|^{-2}(r+1),$$

where $|m_1|$ is a non - null lower limit of sequence $(|a_n|)$. So $C \notin B(l^p)$, $\forall p > 1$.

If, $\lambda = a_n$, $n \in \mathbb{N}$, det $(\lambda I - R_a) = 0$. Results that in this cases $\lambda$ is from $\sigma(R_a, l^p)$.

**Theorem 4.** *For any Rhaly matrix with* $a \in s$*,* $a_n \neq 0\ \forall n \in \mathbb{N}$ *and* $((n+1)|a_n|)$ *is bounded,* $0 \in fr\ \sigma(R_a, l^p)$*,* $p > 1$*.*

**Proof.** From the theorem of the consistence of the spectre, $\sigma(R_a, l^p) \neq \varnothing$. Supposing that $0 \notin fr\ (R_a, l^p)$, results that $0 \in int\ \sigma(R_a, l^p)$ and $\sigma(R_a, l^p) \in \mathcal{V}(0)$.

$\Rightarrow \exists \varepsilon > 0$, $[-\varepsilon, \varepsilon] \subseteq \sigma(R_a, l^p) \Rightarrow \varepsilon \in \sigma(R_a, l^p)$, $\varepsilon > 0$.

Considering the operator $C = \varepsilon I - R_a$, $C^{-1}$ has the entries:

$$c_{jj} = \frac{1}{\varepsilon - a_j} \quad \forall j \in \mathbb{N} \qquad c_{ij} = 0 \qquad \forall i, j \in \mathbb{N}\ , \quad i < j$$

$$c_{ij} = c_{i+r,j} = \lambda^{r-1} a_{j+r} \left[ \prod_{k=r}^{j+r} (\varepsilon - a_k) \right]^{-1} \qquad \forall r \in \mathbb{N}^*\ ; \quad i, j \in \mathbb{N}\ , \quad i = j + r.$$

It can be choose $\varepsilon_1 \leq \varepsilon$ such that $\max\limits_{h \in \mathbb{N}} \left| \dfrac{\varepsilon_1}{\varepsilon_1 - a_k} \right| < 1$. Then $\varepsilon \in \rho(R_a, l^p)$ (from theorem 2) and for $\varepsilon \to 0$ it is obtained that $0 \geq 1$, what is false. $\Rightarrow 0 \in fr\ \sigma(R_a, l^p)$.

**Theorem 5.** *Let* $R_a$ *be a Rhaly operator such that the sequence* $((n+1)a_n)$ *is bounded, with positive numbers. If* $\lambda \in \sigma(R_a, l^p)$ *and* $\lambda \notin S$*, then* $\lambda \in \sigma_c(R_a, l^p)$*.*

**Proof.** We must proof that:

a) $\lambda I - R_a$ is injective

b) $\overline{(\lambda I - R_a)(l^p)} = l^p$ ($\Leftrightarrow \lambda I - R_a^*$ is injective)

c) $(\lambda I - R_a)^{-1}$ isn't continuous ($\Leftrightarrow \lambda I - R_a^*$ isn't surjective)

a) $\lambda \notin S$ implies that $\lambda I - R_a$ is injective operator

b) It is proved that the equation $(\lambda I - R_a^*) f = 0$ hasn't solution $f \in l^q$, $f$ non-null

From $(\lambda - a_n)f(n) - \sum_{k=n+1}^{\infty} a_k f(k) = 0 \ \ \forall n \in \mathbb{N}$ it is obtained the recurrent relation:

$$f(n+1) = \frac{\lambda}{\lambda - a_n} f(n). \tag{2}$$

Results the equalities

$$f(n+1) = \frac{\lambda^{n+1}}{(\lambda - a_n)(\lambda - a_{n-1})...(\lambda - a_0)} f(0) \ , \qquad n \geq 1.$$

Supposing that $f(0) \neq 0$ ($f(0) = 0 \Rightarrow f = 0$), it can be written

$$\left| \frac{f(n+1)}{f(n)} \right|^q = \frac{|\lambda|^q}{|\lambda - a_n|^q} \geq 1.$$

So $\sum |f(n)|^q$ is divergente, that mean that equation $(\lambda I - R_a^*)f = 0$ hasn't solution $f \in l^q$, $f \neq 0$

c) It will be proof that the operator $\lambda I - R_a^*$ isn't surjective, that mean the equation $(\lambda I - R_a^*)f = g$ hasn't solution $f \in l^q$ for any $g$.

Let $f \in l^q$. We consider the equation $(\lambda I - R_a^*)f = g$, $f \in s$. It can be choose $f(0) = f(1) = 0$ (this option is good for calculation). We obtained:

$$\lambda f(n) = g(n) - g(0) - \sum_{k=1}^{n} a_k f(k) \Rightarrow$$

$$f(n) = \lambda^{-1} \left( g(n) - g(0) - \sum_{k=1}^{n-1} a_k f(k) \right) \qquad \forall n \geq 2. \tag{3}$$

The equation 3 can be written as a system with the form $f = Bg$, where

B has the entries:

$$b_{20} = -\lambda^{-1} \qquad b_{21} = 0 \qquad b_{22} = \lambda^{-1}$$

$$b_{n,n-1} = -a_{n-1}\lambda^{-2} \qquad n \geq 3$$

$$b_{n,n-k} = -\lambda^{-1}a_{n-1}\left[1 + \sum_{r=1}^{k-1}(-1)^r \left(\sum_{n-k<i_1<\ldots<i_r<n} a_{i1}a_{i2}\ldots a_{ir}\right)\lambda^{-(r+1)}\right],$$

$$n \geq k+2 \quad b_{n,p} = -\lambda^{-1}\left[1 + \sum_{r=1}^{n-1}(-1)^r \left(\sum_{2\leq i_1<\ldots<i_r\leq n-1} a_{i1}a_{i2}\ldots a_{ir}\right)\lambda^{-r}\right],$$

$n \geq 2.$

(4)

It remains to proove that $\sup_n \sum_{n=m}^{\infty} |b_{nm}|^q$ isn't finite.

For $m = 0$,

$$\sum_{n=2}^{\infty} |b_{nm}|^q = \sum_{n=2}^{\infty} |\lambda^q| \left|1 + \sum_{r=1}^{n-2}(-1)^r \left(\sum_{2\leq i_1<\ldots<i_r\leq n-1} a_{i1}a_{i2}\ldots a_{ir}\right)\lambda^{-r}\right|^q.$$

For $m = 1$, $\sum_{n=2}^{\infty} |b_{n1}|^q = 0.$

For $m = 2$,

$$\sum_{n=m+2}^{\infty} |b_{nm}|^q + |b_{mm}|^q + |b_{m+1,m}|^q = |\lambda^{-1}|^q + |a_m|^q|\lambda^{-2}|^q + \sum_{n=m+2}^{\infty} |\lambda^{-1}|^q|a_m|^q \Bigg| 1 +$$

$$+ \sum_{r=1}^{n-m-1}(-1)^r \left(\sum_{m<i_1<\ldots<i_n<n} a_{i1}a_{i2}\ldots a_{ir}\right)\lambda^{-(r+1)}\Bigg|^q |\lambda^{-1}|^q|m_1|^q \geq |\lambda^{-1}|^q + |m_1|^q|\lambda^{-2}|^q +$$

$$+|\lambda^{-1}|^q|\, m_1|^q + \sum_{n=m+2}^{\infty}\left|1 + \sum_{r=1}^{n-m-1}(-1)^r \left(\sum_{m<i_1<\ldots<i_r<n} |m_1|^r\right)\lambda^{-(r+1)}\right|^q \geq |\lambda^{-1}|^2 +$$

$$+|m_1|^q|\lambda^{-2}|^q + |\lambda^{-1}|^q|m_1|^q \sum_{n=m+2}^{\infty}\left|\left(\sum_{r=1}^{n-m-1}(n-m-1)|m_1|^r\right)\lambda^{-(r+1)} - 1\right|^q,$$

where $0 < |m_1| \leq |a_n|$, for all $n$.

From criterion of the ratio,

$$\sum_{n=m+2}^{\infty}\left|\left(\sum_{r=1}^{n-m-1}(n-m-1)|m_1|^r\right)\lambda^{-(r+1)}-1\right|^q$$

is divergente, so $\sup_n \sum_{n=m}^{\infty}|b_{nm}|^q$ isn't finite. $\lambda I - R_a^*$ isn't surjective implies $\lambda \in \sigma_c(R_a, l^p)$.

**Theorem 6.** *Let $R_a$ be a Rhaly operator with $a_i \neq a_j$ for $i \neq j$ and the sequence $((n+1)a_n)$ is bounded. If $\lambda = a_n$, $n \in \mathbb{N}^*$ then $\lambda \in \sigma_r(R_a, l^p)$, and $\lambda = a_0 \in \sigma_p(R_a, l^p)$.*

**Proof.** We must proof that:

a) $\lambda I - R_a$ is injective

b) $\overline{(\lambda I - R_a)(l^p)} \neq l^p$.

Let $j \geq 1$ arbitrary fixed.

a) It is prove that $(\lambda I - R_a)f = 0$, $f \in l^p \Rightarrow f = 0$

$$(\lambda I - R_a) = o \Rightarrow \sum_{k=0}^{n-1} a_n f(k) = (\lambda - a_n)f(k)\ , \qquad n \geq j.$$

If $\lambda = aj$ we obtain:

$$a_j - a_n = a_n \sum_{k=0}^{n-1} f(k)\ , \qquad n \geq j.$$

So, $\sum_{n=0}^{\infty}|f(n)|^p < \infty \Leftrightarrow |a_j|^p \sum_{n=j}^{\infty}\left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right|^p < \infty$.
Let $m < -1$ be a lower limit for the sequence $(a_n)$. Then

$$|a_j|^p \sum_{n=j}^{\infty}\left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right|^p \leq 2|a_j|^p \sum_{n=j}^{\infty}\left|\frac{1}{a_n}\right|^p < 2|a_j|^p \sum_{n=j}^{\infty}\left|\frac{1}{m}\right|^p < \infty.$$

If $\exists a_n = 0,\ n \geq j,\ a_j - a_n = a_n \sum_{k=0}^{n-1} f(k) \Rightarrow a_j = 0 \Rightarrow \lambda = 0$, but from theorem 4, $0 \in \sigma_p(R_a, l^p),\ p > 1$. So we can calculate in hypothesis that $a_n \neq 0, \forall n \geq j$. $\Rightarrow$ the operator $\lambda I - R_a$ isn't injective.

b) Obviously $a_j I - R_a^*$ isn't injective, so $\overline{(\lambda I - R_a)(l^p)} = \overline{(a_j I - R_a)(l^p)} \neq l^p, \forall j \in \mathbb{N}$, what remains for prooving because $\lambda \in \sigma_r(R_a, l^p)$.

Supposing that $\lambda = a_0$, the operator $(\lambda I - R_a)(e_0) = 0$, $e_0 = (1, 0, 0, ...)$, is this situation the operator $\lambda I - R_a$ isn't injective, too $\Rightarrow \lambda \in \sigma_p(R_a, l^p)$.

**Theorem 7.** *Let $R_a$ be a Rhaly operator with the property that the sequence of its main diagonal elements doesn't have distinct elements, doesn't have an infinity of equal terms, and the sequence $((n+1)a_n)$ is bounded. If $\lambda = a_n, n \in \mathbb{N}$, then $\lambda \in \sigma_r(R_a l^p)$.*

**Proof.** The restriction for $\lambda$ implies the fact that aren't put in discution the entries null of diagonal.

Let $a_j \neq 0$ a element that apeares more then two times on diagonal of $R_a$ and $k, r \in \mathbb{N}$ the biggest, and the smallest integer for which $a_j = a_k = a_r$. Then, the equation $(\lambda I - R_a)f = 0$ has a solution $f \in s$, $f \neq 0$. Remains to study if the property is true in $l^p$.

Let $I = \{n \geq r | a_{n+1} = 0\}$. card $\{n \geq r | a_{n+1}\}$ is finite, because in the principal diagonal of matrix $R_a$ isn't an infinite number of entries equal between them.

Then,

$$\sum_{n=0}^{\infty} |f(n)|^p = \sum_{n \geq j, n \notin I} |f(n)|^p + \sum_{n \geq j, n \in I} |f(n)|^p,$$

and

$$\lim_{n \to \infty} \left| \frac{a_j}{a_{n+1}} - 1 \right|^p \neq 0,$$

such that $f \notin l^p$.

So, the operator $\lambda I - R_a$ is injective.

b) Obviously $\lambda I - R_a^*$ isn't injective.

c) The operator $a_j I - R_a^*$ isn't surjective. For $n \geq j$, $f = Bg$, where $B$ are the entries

$$b_{nk} = 0 \qquad k > n \qquad n \geq j+1$$

$$b_{nn} = \frac{1}{a_j} \qquad n \geq j+1$$

$$b_{n,n-1} = -a_{n-1}\lambda^{-2} \qquad n \geq j+2$$

$$b_{n,n-k} = -a_j^{-1} a_{n-k} \left[ 1 + \sum_{r=1}^{k-1} (-1)^r \left( \sum_{n-k<i_1<...<i_r<n} a_{i1} a_{i2} ... a_{ir} \right) \lambda^{-(r+1)} \right], \ n \geq j+3$$

$$b_{n,0} = -a_j^{-1} \left[ 1 + \sum_{r=1}^{n-2} (-1)^r \left( \sum_{2 \leq i_1 < ... < i_r \leq n-1} a_{i1} a_{i2} ... a_{ir} \right) a_j^{-r} \right], \ n \geq j+1.$$

(5)

If $j = 0$ and $|m_1| \leq |a_2| \leq |m_2| \ \ \forall n \in \mathbb{N}$,

$$\sum_{n=j+1}^{\infty} |b_{n_j}| \geq \sum_{n=j+1}^{\infty} \left| \frac{1}{a_j} \right| \left| 1 + \sum_{\substack{r=1 \\ r \text{ impar}}}^{n-2} (-1)^r \left( \sum_{2 \leq i_1 < ... < i_r \leq n-1} a_{i1} a_{i2} ... a_{ir} \right) a_j^{-r} \right| \geq$$

$$\geq \sum_{n=j+1}^{\infty} \frac{1}{|m_2|} \left| 1 + \sum_{\substack{r=1 \\ r \text{ impar}}}^{n-2} (-1)^r \sum_{2 \leq i_1 < ... < i_r \leq n-1} \frac{|m_1|^r}{|m_2|^r} \right| \geq$$

$$\geq \frac{1}{|m_2|} \left| 1 + \sum_{\substack{r=1 \\ r \text{ impar}}}^{j-1} (-1)^r \frac{|m_1|^r}{|m_2|^r} (j-1) \right|$$

$\rightarrow \infty (j \rightarrow \infty)$. So, the operator $a_j I - R_a^*$ isn't surjective.

## References

[1] Boyd, D. W., *The spectrum of the Cesàro operator*, Acta Soc. Math., 29(1980).

[2] Leibowitz, G., *Rhaly matrices*, J. math. Analysis Appl., 128, 272-386(1987).

[3] Rhaly, H. C. Jr., *Posinormal operators*, J. Math. Soc. Japan, vol. 46, no. 4, 425-428(1999).

[4] Rhaly, H. C. Jr., *Hyponormal terraced matrices*, For East J. Math. Soc. 5, (1997).

[5] Rhaly, H. C. Jr., *Terraced matrices*, Bull. London Math. Soc., 21, 399-406(1989).

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