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Approximation of common fixed points for a finite family of Zamfirescu operators [1]

Shaini Pulickakunnel, Neeta Singh

Abstract

In this paper we introduce a new composite implicit iteration scheme with errors and a strong convergence theorem is established for a finite family of Zamfirescu operators in arbitrary normed spaces. As a corollary we observe that the iteration scheme introduced by Su and Li (18) converges to the common fixed point of a finite family of Zamfirescu operators.

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1 Introduction and preliminary definitions

In recent years, iterative techniques for approximating the common fixed points of a finite family of pseudocontractive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings or nonexpansive mappings in Hilbert spaces, uniformly convex Banach spaces or arbitrary Banach spaces have been considered by several authors. [eg., 4, 9, 12, 17, 19, 20, 21]. In 2001, Xu and Ori [22] introduced an implicit iteration process for a finite family of nonexpansive mappings as follows:

Let K be a nonempty closed convex subset of a normed space E. Let

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$\{T_1, T_2, ..., T_N\}$ be N nonexpansive self-maps of K. Then for an arbitrary point $x_0 \in K$, and $\{\alpha_n\} \subset (0,1)$, the sequence $\{x_n\}$ generated can be written in the compact form as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \qquad \forall n \geq 1, \tag{1}$$

where $T_n = T_{n(modN)}$ (the mod$N$ function takes values in $I = \{1, 2, 3, ..., N\}$). Xu and Ori proved the weak convergence of this process to a common fixed point of a finite family of nonexpansive mappings defined in a Hilbert space. In 2004, Osilike [12] extended the results of Xu and Ori from nonexpansive mappings to strictly pseudocontractive mappings.

Inspired by the above facts, in 2006 Su and Li [18] introduced a new two-step implicit iteration process which is defined as follows:

Let E be a real Banach space and K a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N$ be N strictly pseudocontractive self-maps of K. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n \\ y_n &= \beta_n x_{n-1} + (1-\beta_n)T_n x_n \end{aligned} \tag{2}$$

where $T_n = T_{n(modN)}$ and $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$.

Using this iteration they proved a convergence theorem for a finite family of strictly pseudocontractive maps. It is observed that the class of Zamfirescu operators is independent (see Rhoades [16]) of the class of strictly pseudocontractive operators.

Consideration of error terms in iterative processes is an important part of the theory. Several authors have introduced and studied one-step, two-step as well as multi-step iteration schemes with errors to approximate fixed points of various classes of mappings in Banach spaces [2, 5, 6, 7, 8, 10, 13, 14].

Let K be a nonempty closed convex subset of a normed space E. Motivated by the above facts, we introduce the following composite implicit iteration processes with errors for a finite family of Zamfirescu operators $\{T_i\}_{i=1}^N : K \to K$, and define the sequences $\{x_n\} \subset K$ as follows:

$$\begin{aligned} &x_0 \in K, \\ x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n + u_n, \\ y_n &= \beta_n x_{n-1} + (1-\beta_n)T_n x_n + v_n, \end{aligned}$$

where $T_n = T_{n(modN)}$ (the mod$N$ function takes values in $I = \{1, 2, 3, ..., N\}$), $\{u_n\}$ and $\{v_n\}$ are two summable sequences in E, ie., $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\sum_{n=0}^{\infty} \|v_n\| < \infty$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$, satisfying certain restrictions.

In particular if $u_n = 0, v_n = 0$ for all $n > 0$, then the iteration scheme obtained is the scheme introduced by Su and Li.

We recall the following definitions in a metric space (X, d), from Berinde [1, p.6, 50-51, 131] and Ciric [3, p.268].

A mapping $T : X \to X$ is called an a-contraction if

(z_1) $\quad d(Tx, Ty) \leq ad(x, y)$ for all $x, y \in X$, where $a \in [0, 1)$.

The map T is called a Kannan mapping if there exists $b \in [0, \frac{1}{2})$ such that

(z_2) $\quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$.

A similar definition is due to Chatterjea : there exists $c \in [0, \frac{1}{2})$ such that

(z_3) $\quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$.

It is known, see Rhoades [15] that $(z_1), (z_2),$ and (z_3) are independent contractive conditions. An operator T which satisfies at least one of the contractive conditions $(z_1), (z_2)$ and (z_3) is called a Zamfirescu operator or a Z-operator. Alternatively we say that T satisfies Condition Z.

The main purpose of this paper is to establish a strong convergence theorem to approximate common fixed points of a finite family of Zamfirescu operators in normed spaces using the new iteration scheme defined above.

We need the following lemma.

Lemma 1 [11]. *Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n)r_n + s_n t_n + k_n, \quad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $\lim_{n\to\infty} r_n = 0$.

2 Main result

Theorem 2 *Let K be a nonempty closed convex subset of a normed space E. Let $\{T_1, T_2, T_3 ..., T_N\} : K \to K$ be N, Zamfirescu operators with $F =$*

$\cap_{i=1}^{N} F(T_i) \neq \phi$ (F denotes the set of common fixed points of $\{T_1, T_2, T_3 ..., T_N\}$). Let $\{u_n\}$ and $\{v_n\}$ be two summable sequences in E, and $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0, 1]$ satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \beta_n(1 - \alpha_n) = \infty$;

(ii) $\|v_n\| = o(\beta_n)$.

For any $x_0 \in K$, let the sequence $\{x_n\} \subset K$ be defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n y_n + u_n \tag{3}$$

$$y_n = \beta_n x_{n-1} + (1 - \beta_n) T_n x_n + v_n$$

where $T_n = T_{n(modN)}$ (the modN function takes values in $I = \{1, 2, 3, ..., N\}$). Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, T_3 ..., T_N\}$.

Proof. It follows from the assumption $F = \cap_{i=1}^{N} F(T_i) \neq \phi$, that the operators $\{T_1, T_2, T_3 ..., T_N\}$ have a common fixed point in K, say p. Consider $x, y \in K$. Since each T_i is a Zamfirescu operator, each T_i satisfies at least one of the conditions $(z_1), (z_2)$ and (z_3).

If (z_2) holds, then for any $x, y \in K$

$$\begin{aligned} \|T_i x - T_i y\| &\leq b[\|x - T_i x\| + \|y - T_i y\|] \\ &\leq b[\|x - T_i x\| + \|y - x\| + \|x - T_i x\| + \|T_i x - T_i y\|], \end{aligned}$$

which implies

$$(1 - b) \|T_i x - T_i y\| \leq b \|x - y\| + 2b \|x - T_i x\|,$$

since $0 \leq b < \frac{1}{2}$ we get

$$\|T_i x - T_i y\| \leq \frac{b}{1 - b} \|x - y\| + \frac{2b}{1 - b} \|x - T_i x\|. \tag{4}$$

Similarly, if (z_3) holds, then we have for any $x, y \in K$

$$\begin{aligned} \|T_i x - T_i y\| &\leq c[\|x - T_i y\| + \|y - T_i x\|] \\ &\leq c[\|x - T_i x\| + \|T_i x - T_i y\| + \|y - x\| + \|x - T_i x\|] \end{aligned}$$

which implies

$$(1 - c) \|T_i x - T_i y\| \leq c \|x - y\| + 2c \|x - T_i x\|,$$

since $0 \leq c < \dfrac{1}{2}$ we get

$$\|T_i x - T_i y\| \leq \frac{c}{1-c}\|x-y\| + \frac{2c}{1-c}\|x - T_i x\|. \tag{5}$$

Denote

$$\delta = max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{6}$$

Then we have $0 \leq \delta < 1$ and, in view of $(z_1), (4), (5)$ and (6), it results that the inequality

$$\|T_i x - T_i y\| \leq \delta\|x-y\| + 2\delta\|x - T_i x\| \tag{7}$$

holds for all $x, y \in K$ and for every $i \in \{1, 2, 3, ..., N\}$.

Now, since $T_i p = p$, $T_n = T_{n(modN)}$ and the modN function takes values in $\{1, 2, 3, ..., N\}$, for $y = x_n$ and $x = p$, the above inequality (7) gives the following result

$$\|T_n x_n - p\| \leq \delta\|x_n - p\| \tag{8}$$

Again, with $y = y_n$ and $x = p$, in (7) we get

$$\|T_n y_n - p\| \leq \delta\|y_n - p\|. \tag{9}$$

Now, let $\{x_n\}$ be the implicit iteration process with errors defined by (3) and $x_0 \in K$ be arbitrary.

Then

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n x_{n-1} + (1-\alpha_n)T_n y_n + u_n - p\| \\
&= \|\alpha_n x_{n-1} + (1-\alpha_n)T_n y_n + u_n - (\alpha_n + 1 - \alpha_n)p\| \\
&= \|\alpha_n(x_{n-1} - p) + (1-\alpha_n)(T_n y_n - p) + u_n\| \\
&\leq \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)\|T_n y_n - p\| + \|u_n\|.
\end{aligned}$$

Using (9) in the above inequality we obtain that

$$\|x_n - p\| \leq \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)\delta\|y_n - p\| + \|u_n\|$$

Substitute for y_n from (3) we get

$$\begin{aligned}\|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\delta\|\beta_n x_{n-1} \\ &\quad + (1-\beta_n)T_n x_n + v_n - p\| + \|u_n\| \\ &= \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\delta\|\beta_n x_{n-1} \\ &\quad + (1-\beta_n)T_n x_n + v_n - (\beta_n + 1 - \beta_n)p\| + \|u_n\| \\ &= \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\delta\|\beta_n(x_{n-1} - p) \\ &\quad + (1-\beta_n)(T_n x_n - p) + v_n\| + \|u_n\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\delta\Big\{\beta_n \|x_{n-1} - p\| \\ &\quad + (1-\beta_n)\|T_n x_n - p\| + \|v_n\|\Big\} + \|u_n\|.\end{aligned}$$

Using (8) in the above inequality we get that

$$\begin{aligned}\|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\delta\Big\{\beta_n \|x_{n-1} - p\| \\ &\quad + (1-\beta_n)\delta\|x_n - p\| + \|v_n\|\Big\} + \|u_n\| \\ &= \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\delta\beta_n \|x_{n-1} - p\| \\ &\quad + (1-\alpha_n)(1-\beta_n)\delta^2 \|x_n - p\| + (1-\alpha_n)\delta\|v_n\| + \|u_n\|\end{aligned}$$

that is

$$\begin{aligned}(1-(1-\alpha_n)(1-\beta_n)\delta^2)\|x_n - p\| &\leq [\alpha_n + (1-\alpha_n)\beta_n\delta]\|x_{n-1} - p\| \\ &\quad + (1-\alpha_n)\delta\|v_n\| + \|u_n\|\end{aligned}$$

since $0 \leq (1-\alpha_n)(1-\beta_n)\delta^2 < 1$, we have

$$\begin{aligned}\text{(10)}\quad \|x_n - p\| &\leq \frac{[\alpha_n+(1-\alpha_n)\beta_n\delta]\|x_{n-1}-p\|+(1-\alpha_n)\delta\|v_n\|+\|u_n\|}{1-(1-\alpha_n)(1-\beta_n)\delta^2} \\ &= \frac{\alpha_n+(1-\alpha_n)\beta_n\delta}{1-(1-\alpha_n)(1-\beta_n)\delta^2}\|x_{n-1}-p\| + \frac{(1-\alpha_n)\delta\|v_n\|+\|u_n\|}{1-(1-\alpha_n)(1-\beta_n)\delta^2}.\end{aligned}$$

Let

$$A_n = \alpha_n + (1-\alpha_n)\beta_n\delta$$
$$B_n = 1 - (1-\alpha_n)(1-\beta_n)\delta^2.$$

Consider

$$\begin{aligned}1 - \frac{A_n}{B_n} &= 1 - \frac{\alpha_n + (1-\alpha_n)\beta_n\delta}{1-(1-\alpha_n)(1-\beta_n)\delta^2} \\ &= \frac{1-(1-\alpha_n)(1-\beta_n)\delta^2 - [\alpha_n + (1-\alpha_n)\beta_n\delta]}{1-(1-\alpha_n)(1-\beta_n)\delta^2}\end{aligned}$$

$$= \frac{1-[(1-\alpha_n)(1-\beta_n)\delta^2+\alpha_n+(1-\alpha_n)\beta_n\delta]}{1-(1-\alpha_n)(1-\beta_n)\delta^2}. \tag{11}$$

Since $1-(1-\alpha_n)(1-\beta_n)\delta^2 \leq 1$, from (11) we have

$$1-\frac{A_n}{B_n} \geq 1-[(1-\alpha_n)(1-\beta_n)\delta^2+\alpha_n+(1-\alpha_n)\beta_n\delta]$$

that is

$$\frac{A_n}{B_n} \leq (1-\alpha_n)(1-\beta_n)\delta^2+\alpha_n+(1-\alpha_n)\beta_n\delta.$$

Using the facts that $\{\alpha_n\},\{\beta_n\} \subset [0,1]$ and $\delta < 1$, we get

$$\begin{aligned} \frac{A_n}{B_n} &\leq (1-\alpha_n)(1-\beta_n)+\alpha_n+(1-\alpha_n)\beta_n\delta \\ &= 1-\alpha_n-\beta_n+\alpha_n\beta_n+\alpha_n+(1-\alpha_n)\beta_n\delta \\ &= 1-\beta_n(1-\alpha_n)+(1-\alpha_n)\beta_n\delta = 1-\beta_n(1-\alpha_n)(1-\delta). \end{aligned} \tag{12}$$

Hence from (10) and (12) we have

$$\begin{aligned} \|x_n-p\| &\leq [1-(1-\delta)\beta_n(1-\alpha_n)]\|x_{n-1}-p\| \\ &\quad + \frac{(1-\alpha_n)\delta\|v_n\|}{1-(1-\alpha_n)(1-\beta_n)\delta^2} + \frac{\|u_n\|}{1-(1-\alpha_n)(1-\beta_n)\delta^2} \end{aligned}$$

which, by the inequality

$$1-\delta \leq 1-(1-\alpha_n)(1-\beta_n)\delta^2,$$

implies that

$$\|x_n-p\| \leq [1-(1-\delta)\beta_n(1-\alpha_n)]\|x_{n-1}-p\| + \frac{(1-\alpha_n)\delta}{1-\delta}\|v_n\| + \frac{1}{1-\delta}\|u_n\|.$$

Since $\|v_n\| = o(\beta_n)$ by assumption, let $\|v_n\| = d_n\beta_n$ and $d_n \to 0$. Therefore from the above inequality we obtain that

$$\begin{aligned} \|x_n-p\| &\leq [1-(1-\delta)\beta_n(1-\alpha_n)]\|x_{n-1}-p\| \\ &\quad + \frac{(1-\delta)(1-\alpha_n)\delta d_n\beta_n}{(1-\delta)^2} + \frac{1}{1-\delta}\|u_n\|. \\ &= [1-(1-\delta)\beta_n(1-\alpha_n)]\|x_{n-1}-p\| \\ &\quad + \frac{(1-\delta)\beta_n(1-\alpha_n)\delta d_n}{(1-\delta)^2} + \frac{1}{1-\delta}\|u_n\|. \end{aligned}$$

Setting $r_n = \|x_{n-1} - p\|$, $s_n = (1-\delta)\beta_n(1-\alpha_n)$, $t_n = \dfrac{\delta}{(1-\delta)^2} d_n$, $k_n = \frac{1}{1-\delta}\|u_n\|$, and using the facts that $0 \le \delta < 1$, $0 \le \alpha_n \le 1$, $0 \le \beta_n \le 1$, $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty$, $d_n \to 0$ and $\sum_{n=1}^{\infty} \|u_n\| < \infty$, it follows from Lemma 1 that

$$\lim_{n\to\infty} \|x_n - p\| = 0$$

which implies that $x_n \to p \in F$. Hence the proof.

Corollary 3 *Let K be a nonempty closed convex subset of a normed space E, and let $\{T_1, T_2, T_3 ..., T_N\} : K \to K$ be N, Zamfirescu operators with $F = \cap_{i=1}^{N} F(T_i) \neq \phi$ ($F(T_i)$ denotes the set of fixed points of T_i). Let $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ be two real sequences satisfying the condition $\sum_{n=1}^{\infty}(1-\alpha_n)\beta_n = \infty$. For $x_0 \in K$, let the sequence $\{x_n\}$ be defined by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n y_n$$

$$y_n = \beta_n x_{n-1} + (1-\beta_n) T_n x_n$$

where $T_n = T_{n(modN)}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, T_3 ..., T_N\}$.

Remark 4 *Chatterjea's and Kannan's contractive conditions (z_2) and (z_3) are both included in the class of Zamfirescu operators and so their convergence theorems for the implicit iteration process with errors defined by (3) are obtained in Theorem 2.*

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Shaini Pulickakunnel and Neeta Singh
Department of Mathematics
University of Allahabad
Allahabad-211002, India
e-mail: shainipv@gmail.com