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Inclusion and neighborhoods of certain classes of analytic functions of complex order [1]

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Abstract

By means of a certain extended derivative operator, the authors introduce and investigate two new subclasses of analytic functions of complex order.The various results obtained here for each of these function classes include coefficient inequalities and the consequent inclusion relationships involving the neighborhoods of the analytic functions.

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1 Introduction

Let $T(m)$ denote the class of functions of the form:

$$(1) \qquad f(z) = z - \sum_{k=m+1}^{\infty} a_k z^k \quad (a_k \geq 0; k \geq m+1; m \in N = \{1, 2, ...\}),$$

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which are analytic and univalent in the unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$.

We say that a function $f(z) \in T(m)$ is starlike of complex ordor b ($b \in C^* = C\backslash\{0\}$), that is, $f \in S^*_m(b)$, if it satisfies the following inequality:

$$Re\{1 + \frac{1}{b}(\frac{zf'(z)}{f(z)} - 1)\} > 0 \qquad (z \in U; b \in C^*). \tag{2}$$

Furthermore, a function $f(z) \in T(m)$ is said to be convex of complex order b ($b \in C^*$), that is, $f \in C_m(b)$, if it also satisfies the following inequality:

$$Re\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\} > 0 \qquad (z \in U; b \in C^*). \tag{3}$$

The classes $S^*_m(b)$ and $C_m(b)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf ([11] and [12]) and Wiatrowski [18], respectively (see also [5] and [7]).

For the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (j = 1, 2), \tag{4}$$

let $(f_1 * f_2)(z)$ denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1}.a_{k,2} z^k = (f_2 * f_1)(z). \tag{5}$$

Let

$$D^0_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \;\; \lambda \geq 0,$$

$$D^1_\lambda f(z) = (1-\lambda)zf'(z) + \lambda z(zf'(z))',$$

$$D^n_\lambda f(z) = \frac{z}{n!}(z^{n-1}D_\lambda f(z))^{(n)} \qquad (n \in N_0 = N \cup \{0\}).$$

Note that if $f(z)$ is given by (1), then we can write

$$(6)\ D_\lambda^n f(z) = z - \sum_{k=m+1}^{\infty} [1+\lambda(k-1)]C(n,k)a_k z^k \quad (m \in N; n \in N_0; \lambda \geq 0),$$

where

$$(7) \qquad C(n,k) = \binom{k+n-1}{n} = \frac{\prod_{j=1}^{k-1}(j+n)}{(k-1)!}.$$

The operator $D_\lambda^n \lambda(z)$ was introduced by Al-Shaqsi and Darus [1].

In terms of this linear operator D_λ^n $(n \in N_0, \lambda \geq 0)$ defined by (6) above, let $S_m^\lambda(n,b,\beta)$ denote the subclass of $T(m)$ consisting of functions $f(z)$ which satisfy the following inequality:

$$(8) \qquad \left| \frac{1}{b}\left(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1\right) \right| < \beta$$

$$(z \in U; b \in C^*; n \in N_0; \lambda \geq 0; 0 < \beta \leq 1).$$

Also let $R_m^\lambda(n,b,\beta,\mu)$ denote the subclass of $T(m)$ consisting of functions $f(z)$ which satisfy the following inequality:

$$(9) \qquad \left| \frac{1}{b}\{(1-\mu)\frac{D_\lambda^n f(z)}{z} + \mu(D_\lambda^n f(z))' - 1\} \right| < \beta$$

$$(z \in U; b \in C^*; n \in N_0; \lambda \geq 0; 0 < \beta \leq 1; 0 \leq \mu \leq 1).$$

We note that:

(i) $S_m^0(\theta,b,\beta) = S_m(\theta,b,\beta)$ $(m \in N; b \in C^*; \theta > -1; 0 < \beta \leq 1)$ (Murugusundaramoorthy and Srivastava [10]);

(ii) $S_m^\lambda(0,b,\beta) = S_m(b,\lambda,\beta)$ (Altintas et al. [4]);

(iii) $R_m^0(\theta,b,\beta,\mu)$ $(m \in N; b \in C^*; \theta > -1; 0 < \beta \leq 1; 0 \leq \mu \leq 1)$ (Murugusundaramoorthy and Srivastava [10]);

(iv) $R_m^\lambda(0,b,\beta,1) = R_m(b,\lambda,\beta)$ (Altintas et al. [4]).

Now, following the earlier investigations by Goodman [9], Ruscheweyh [14], and others including Altintas and Owa [3], Altintas et al.([4] and [6]), Murugusundaramoorthy and Srivastava [10], Raina and Srivastava [13], Aouf [8] and Srivastava and Orhan [16] (see also [2], [15] and [17]), we define the (m,δ)-neighborhood of a function $f(z)\in T(m)$ by (see, for example [6,p.1668])

(10)
$$N_{m,\delta}(f)=\{g:g\in T(m),g(z)=z-\sum_{k=m+1}^{\infty}b_kz^k \text{ and } \sum_{k=m+1}^{\infty}k\left|a_k-b_k\right|\leq\delta\}.$$

In particular, if

$$e(z)=z, \tag{11}$$

we immediately have

$$N_{m,\delta}(e)=\{g:g\in T(m),g(z)=z-\sum_{k=m+1}^{\infty}b_kz^k \text{ and } \sum_{k=m+1}^{\infty}k\left|b_k\right|\leq\delta\}. \tag{12}$$

2 Neighborhoods for the classes $S_m^{\lambda}(n,b,\beta)$ and $R_m^{\lambda}(n,b,\beta,\mu)$

In our investigation of the inclusion relations involving $N_{m,\delta}(e)$, we shall require Lemmas 1 and 2 below.

Lemma 1 *Let the function $f(z)\in T(m)$ be defiend by (1).Then $f(z)$ is in the class $S_m^{\lambda}(n,b,\beta)$ if and only if*

$$\sum_{k=m+1}^{\infty}[1+\lambda(k-1)](\beta\left|b\right|+k-1)C(n,k)a_k\leq\beta\left|b\right|, \tag{13}$$

where $C(n,k)$ is defined by (7).

Proof. Let a function $f(z)$ of the form (1) belong to the class $S_m^\lambda(n,b,\beta)$.Then, in view of (6) and (8), we obtain the following inequality

$$Re(\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1) > -\beta\,|b| \qquad (z \in U), \tag{14}$$

or, equivalently,

$$Re\{\frac{-\sum\limits_{k=m+1}^{\infty}[1+\lambda(k-1)](k-1)C(n,k)a_k z^{k-1}}{1-\sum\limits_{k=m+1}^{\infty}[1+\lambda(k-1)]C(n,k)a_k z^{k-1}}\} > -\beta\,|b| \quad (z \in U). \tag{15}$$

Setting $z = r$ $(0 \le r < 1)$ in (15), we observe that the expression in the denominator of the left-hand side of (15) is positive for $r = 0$ and also for all r $(0 < r < 1)$.Thus, by letting $r \to 1^-$ through real values, (15) leads us to the desire assertion (13) of Lemma 1.

Conversely, by applying the hypothesis (13) and letting $|z| = 1$, we find from (8) that

$$\begin{aligned}
\left|\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1\right| &= \left|\frac{\sum\limits_{k=m+1}^{\infty}[1+\lambda(k-1)](k-1)C(n,k)a_k z^{k-1}}{1-\sum\limits_{k=m+1}^{\infty}[1+\lambda(k-1)]C(n,k)a_k z^{k-1}}\right| \\
&\le \frac{\sum\limits_{k=m+1}^{\infty}[1+\lambda(k-1)](k-1)C(n,k)a_k}{1-\sum\limits_{k=m+1}^{\infty}[1+\lambda(k-1)]C(n,k)a_k} \\
&\le \frac{\beta\,|b|\,(1-\sum\limits_{k=m+1}^{\infty}[1+\lambda(k-1)]C(n,k)a_k)}{1-\sum\limits_{k=m+1}^{\infty}[1+\lambda(k-1)]C(n,k)a_k} \\
&= \beta\,|b|\,.
\end{aligned} \tag{16}$$

Hence, by the maximum modulus theorem, we have $f(z) \in S_m^\lambda(n,b,\beta)$, which evidently complete the proof of Lemma 1.

Similarly, we can prove the following lemma.

Lemma 2 *Let the function* $f(z) \in T(m)$ *be defined by* (1).*Then* $f(z) \in R_m^\lambda(n,b,\beta,\mu)$ *if and only if*

$$\sum_{k=m+1}^{\infty}[1+\lambda(k-1)][1+\mu(k-1)]C(n,k)a_k \le \beta\,|b|\,. \tag{17}$$

Our first inclusion relation involving $N_{m,\delta}(e)$ is given in the following theorem.

Theorem 1 *Let*

$$\delta = \frac{\beta\,|b|\,(m+1)}{(1+\lambda m)(\beta\,|b|+m)C(n,m+1)}, \quad (|b|<1), \tag{18}$$

Then

$$S_m^\lambda(n,b,\beta) \subset N_{m,\delta}(e). \tag{19}$$

Proof. Let $f(z) \in S_m^\lambda(n,b,\beta)$.Then, in view of the assertion (13) of Lemma 1, we have

$$(1+\lambda m)(\beta\,|b|+m)C(n,m+1)\sum_{k=m+1}^{\infty}a_k$$

$$\le \sum_{k=m+1}^{\infty}[1+\lambda(k-1)](\beta\,|b|+k-1)C(n,k)a_k \le \beta\,|b|\,, \tag{20}$$

which readily yields

$$\sum_{k=m+1}^{\infty}a_k \le \frac{\beta\,|b|}{(1+\lambda m)(\beta\,|b|+m)C(n,m+1)}\,. \tag{21}$$

Making use of (13) again, in conjunction with (21), we get

$$\begin{aligned}
&(1+\lambda m)C(n,m+1)\sum_{k=m+1}^{\infty}ka_k \\
\le\ & \beta\,|b| + (1-\beta\,|b|)(1+\lambda m)C(n,m+1)\sum_{k=m+1}^{\infty}a_k \\
\le\ & \beta\,|b| + \frac{\beta\,|b|\,(1-\beta\,|b|)}{(\beta\,|b|+m)} = \frac{\beta\,|b|\,(1+m)}{(\beta\,|b|+m)}\,.
\end{aligned}$$

Hence

$$\sum_{k=m+1}^{\infty} ka_k \leq \frac{\beta\,|b|\,(1+m)}{(1+\lambda m)(\beta\,|b|+m)C(n,m+1)} = \delta \;\; (|b|<1), \tag{22}$$

which, by means of the definition (12), establishes the inclusion (19) asserted by Theorem 1.

Remark 1 *.(i) Putting $\lambda = 0$ and taking $n = \theta > -1$ in Theorem 1, we obtain the result of Murugusundaramoorthy and Srivastava [10, Theorem 1];*

(ii) Putting $n = 0$ in Theorem 1, we obtain the result obtained by Altintas et al.[4,Theorem 1];

(iii) Putting $\lambda = n = 0, \beta = 1$ and $b = 1-\alpha,\ 0 \leq \alpha < 1$ in Theorem 1, we obtain the result of Altintas and Owa [3, Theorem 2.1].

In a similar manner, by applying the assertion (17) of Lemma 2 instead of the assertion (13) of Lemma 1 to functions in the class $R_m^{\lambda}(n,b,\beta,\mu)$, we can prove the following inclusion relationship.

Theorem 2 *If*

$$\delta = \frac{\beta\,|b|\,(1+m)}{(1+\lambda m)(1+\mu m)C(n,m+1)}\ , \tag{23}$$

then

$$R_m^{\lambda}(n,b,\beta,\mu) \subset N_{m,\delta}(e). \tag{24}$$

Remark 2 *. (i) Putting $n = 0$ and $\mu = 1$ in Theorem 2, we obtain the result obtained by Altintas et al. [4,Theorem 2];*

(ii) Putting $\lambda = n = 0,\ \mu = \beta = 1$ and $b = 1-\alpha,\ 0 \leq \alpha < 1,$ in Theorem 2, we obtain the result obtained by Altintas and Owa [3, Theorem 3.1].

3 Neighborhoods for the classes $S_m^{\lambda,(\alpha)}(n,b,\beta)$ and $R_m^{\lambda,(\alpha)}(n,b,\beta,\mu)$

In this section, we determine the neighborhood for each of the classes $S_m^{\lambda,(\alpha)}(n,b,\beta)$ and $R_m^{\lambda,(\alpha)}(n,b,\beta,\mu)$, which we define as follows. A function $f(z) \in T(m)$ is said to be in the class $S_m^{\lambda,(\alpha)}(n,b,\beta)$ if there exists a function $g(z) \in S_m^{\lambda}(n,b,\beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha \qquad (z \in U; 0 \le \alpha < 1). \tag{25}$$

Analogously, a function $f(z) \in T(m)$ is said to be in the class $R_m^{\lambda,(\alpha)}(n,b,\beta,\mu)$ if there exists a function $g(z) \in R_m^{\lambda}(n,b,\beta,\mu)$ such that the inequality (25) holds true.

Theorem 3 *If* $g(z) \in S_m^{\lambda}(n,b,\beta)$ *and*

$$\alpha = 1 - \frac{\delta(1+\lambda m)(\beta\,|b|+m)C(n,m+1)}{(m+1)[(1+\lambda m)(\beta\,|b|+m)C(n,m+1)-\beta\,|b|]}, \tag{26}$$

then

$$N_{m,\delta}(g) \subset S_m^{\lambda,(\alpha)}(n,b,\beta), \tag{27}$$

where

$$\delta \le (m+1)\{1-\beta\,|b|\,[(1+\lambda m)(\beta\,|b|+m)C(n,m+1)]^{-1}\}. \tag{28}$$

Proof. Suppose that $f(z) \in N_{m,\delta}(g)$. We find from (10) that

$$\sum_{k=m+1}^{\infty} k\,|a_k - b_k| \le \delta, \tag{29}$$

which readily implies that

$$\sum_{k=m+1}^{\infty} |a_k - b_k| \le \frac{\delta}{m+1} \quad (m \in N). \tag{30}$$

Next, since $g(z) \in S_m^{\lambda}(n,b,\beta)$, we have [cf. equation (21)]

$$\sum_{k=m+1}^{\infty} b_k \leq \frac{\beta\,|b|}{(1+\lambda m)(\beta\,|b|+m)C(n,m+1)}, \tag{31}$$

so that

$$\begin{aligned} \left|\frac{f(z)}{g(z)}-1\right| &\leq \frac{\sum\limits_{k=m+1}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=m+1}^{\infty}b_k} \\ &\leq \frac{\delta}{m+1}.\frac{(1+\lambda m)(\beta\,|b|+m)C(n,m+1)}{(1+\lambda m)(\beta\,|b|+m)C(n,m+1)-\beta\,|b|} \\ &= 1-\alpha, \end{aligned} \tag{32}$$

provided that α is given by (26). Thus, by the above definition, $f(z) \in S_m^{\lambda,(\alpha)}(n,b,\beta)$ for α given by (26). This evidently proves Theorem 3.

Remark 3 *(i) Taking $\lambda = 0$ and $n = \theta > -1$ in Theorem 3, we obtain the result obtained by Murugusundaramoorthy and Srivastava [10, Theorem 3];*

(ii) Taking $n = 0$ in Theorem 3, we obtain the result obtained by Altintas et al .[4, Theorem 3].

The proof of Theorem 4 below is similar to that the proof of Theorem 3 above.

Theorem 4 *If $g(z) \in R_m^{\lambda}(n,b,\beta,\mu)$ and*

$$\alpha = 1 - \frac{\delta(1+\lambda m)(1+\mu m)C(n,m+1)}{(m+1)[(1+\lambda m)(1+\mu m)C(n,m+1)-\beta\,|b|]}, \tag{33}$$

then

$$N_{m,\delta}(g) \subset R_m^{\lambda,(\alpha)}(n,b,\beta,\mu), \tag{34}$$

where

$$\delta \leq (m+1)\{1-\beta\,|b|\,[(1+\lambda m)(1+\mu m)C(n,m+1)]^{-1}\}. \tag{35}$$

Remark 4 . *(i) Putting $\lambda = 0$ in Theorem 4, we obtain the result obtained by Murugusundaramoorthy and Srivastava [10, Theorem 4];*

(ii) Putting $n = 0$ and $\mu = 1$ in Theorem 4, we obtain the result obtained by Altintas et al. [4, Theorem 4].

References

[1] K. Al-Shaqsi, M. Darus, *On certain subclass of analytic functions with negative coefficients*, Appl. Math. Sci.1, no. 23, 2007, 1121-1128.

[2] O. P. Ahuja, M. Nunokawa, *Neighborhoods of analytic functions defined by Ruscheweyh derivatives*, Math. Japon. 51, 2003, 487-492.

[3] O. Altintas, S. Owa, *Neighborhoods of certain analytic functions with negative coefficients*, Internat. J. Math. Math. Sci. 19, 1996, 797-800.

[4] O. Altintas, O. Ozkan, H. M. Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Letters 13 (3), 2000, 63-67.

[5] O. Altintas, O. Ozkan, H. M. Srivastava, *Majorization by starlike function of complex order*, Complex Variables Theory Appl.46, 2001, 207-218.

[6] O. Altintas, O. Ozkan, H. M. Srivastava, *Neighborhoods of certain family of multivalent functions with negative coefficients*, Comput. Math. Appl. 47, 2004, 1667-1672.

[7] O. Altintas, H. M. Srivastava, *Some majorization problems associated with p-valently starlike and convex functions of complex order*, East Asian Math. J. 17, 2001,175-183.

[8] M. K. Aouf, *Neighborhoods of certain classes of analytic functions with negative coefficients*, Internat. J. Math. Math. Sci., Art. ID 38258, 2006, 1-6.

[9] A. W. Goodman, *Univalent functions and non-analytic curves*, Proc. Amer. Math. Soc. 8, 1957, 598-601.

[10] G. Murugusundaramoorthy, H. M. Srivastava, *Neighborhoods of certain classes of analytic functions of complex order*, J. Inequal. Pure Appl. Math. 5 (2), Art. 24, 2004, 1-8.

[11] M. A. Nasr, M. K. Aouf, *Starlike function of complex order*, J. Natur. Sci. Math. 25, 1985, 1-12.

[12] M. A. Nasr, M. K. Aouf, *On convex function of complex order*, Mansoura Bull. Sci. Egypt 9, 1982, 565-582.

[13] R. K. Raina, H. M. Srivastava, *Inclusion and neighborhood properties of some analytic and multivalent functions*, J. Inequal. Pure Appl. Math. 7 (1), 2006, Art. 5, 1-6.

[14] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. 81, 1981, 521-527.

[15] H. Silverman, *Neighborhoods of classes of analytic function*, Far East J. Math. Sci. 3, 1995, 165-169.

[16] H. M. Srivastava, H.Orhan, *Coefficient inqualities and inclusion relations for some families of analytic and multivalent functions*, Appl. Math. Letters 20, 2007, 686-691.

[17] H. M. Srivastava, S. Owa, *Current Topics in Analytic Function Theory*, World. Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[18] P. Wiatrowski, *On the coefficients of some family of holomorphic functions*, Zeszyty Nauk. Uniw. Lodz. Mat. Przyrod. (2) 39, 1970, 75-85.

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