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On some integral operators on analytic functions

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Abstract

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1 Introduction and Preliminaries

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

We denote with T the subset of the functions $f \in S$, which have the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \,,\; a_j \geq 0 \,,\; j \geq 2 \,,\; z \in U \tag{1}$$

and with $T^* = T \bigcap S^*$, $T^*(\alpha) = T \bigcap S^*(\alpha)$, $T^c = T \bigcap S^c$ and $T^c(\alpha) = T \bigcap S^c(\alpha)$, where $0 \leq \alpha < 1$.

Theorem 1. *[5] For a function f having the form (1) the following assertions are equivalents:*

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$;

(ii) $f \in T$;

(iii) $f \in T^*$.

Regarding the classes $T^*(\alpha)$ and $T^c(\alpha)$ with $0 \leq \alpha < 1$, we recall here the following result:

Theorem 2. *[5] A function f having the form (1) is in the class $T^*(\alpha)$ if and only if:*

$$\sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} a_j \leq 1 \,, \tag{2}$$

and is in the class $T^c(\alpha)$ if and only if:

$$\sum_{j=2}^{\infty} \frac{j(j-\alpha)}{1-\alpha} a_j \leq 1 \,. \tag{3}$$

Definition 1. *[1] Let $S^*(\alpha, \beta)$ denote the class of functions having the form (1) which are starlike and satisfy*

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + (1-2\alpha)} \right| < \beta \tag{4}$$

for $0 \leq \alpha < 1$ *and* $0 < \beta \leq 1$. *And let* $C^*(\alpha, \beta)$ *denote the class of functions such that* $zf'(z)$ *is in the class* $S^*(\alpha, \beta)$.

Theorem 3. *[1] A function* f *having the form (1) is in the class* $S^*(\alpha, \beta)$ *if and only if:*

$$\sum_{j=2}^{\infty} \{(j-1) + \beta(j+1-2\alpha)\} a_j \leq 2\beta(1-\alpha)\,, \tag{5}$$

and is in the class $C^*(\alpha, \beta)$ *if and only if:*

$$\sum_{j=2}^{\infty} j\,\{(j-1) + \beta(j+1-2\alpha)\} a_j \leq 2\beta(1-\alpha)\,. \tag{6}$$

Let D^n be the Sălăgean differential operator (see [2]) defined as:

$$D^n : A \to A\,, \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)\,, \quad D^n f(z) = D(D^{n-1} f(z)).$$

In [3] the author define the class $T_n(\alpha, \beta)$, from which by choosing different values for the parameters we obtain variously subclasses of analytic functions with negative coefficients (for example $T_n(\alpha, 1) = T_n(\alpha)$ which is the class of n-starlike of order α functions with negative coefficients and $T_0(\alpha, \beta) = S^*(\alpha, \beta) \bigcap T$, where $S^*(\alpha, \beta)$ is the class defined by (4)).

Definition 2. *[3] Let* $\alpha \in [0, 1)$, $\beta \in (0, 1]$ *and* $n \in \mathbb{N}$. *We define the class* $S_n(\alpha, \beta)$ *of the* n*-starlike of order* α *and type* β *through*

$$S_n(\alpha, \beta) = \{f \in A\,;\, |J(f, n, \alpha; z)| < \beta\}$$

where $J(f,n,\alpha;z) = \dfrac{D^{n+1}f(z) - D^n f(z)}{D^{n+1}f(z) + (1-2\alpha)D^n f(z)}$, $z \in U$. *Consequently* $T_n(\alpha,\beta) = S_n(\alpha,\beta)\bigcap T$.

Theorem 4. *[3] Let f be a function having the form (1). Then $f \in T_n(\alpha,\beta)$ if and only if*

$$\sum_{j=2}^{\infty} j^n\left[j-1+\beta(j+1-2\alpha)\right]a_j \leq 2\beta(1-\alpha)\,. \tag{7}$$

2 Main results

From [4] we have the following definitions:

Let $f(z) \in T$, $f(z) = z - \sum\limits_{n=2}^{\infty} a_n z^n$, satisfies $V_\mu(f)(z) = \displaystyle\int_0^1 \mu(t)\frac{f(tz)}{t}dt$, where μ is a real-valued, non-negative weight function normalized so that $\displaystyle\int_0^1 \mu(t)dt = 1$.

If $\mu(t) = \dfrac{(c+1)^\delta}{\mu(\delta)}t^c\left(log\dfrac{1}{t}\right)^{\delta-1}$ $(c > -1\,;\, \delta > 0)$, which gives the Komatu operator. Then we have

$$V_\mu(f)(z) = z - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^n\,. \tag{8}$$

Remark 1. *We notice that* $0 < \left(\dfrac{c+1}{c+n}\right)^{\delta} < 1$, *where* $c > -1$, $\delta > 0$ *and* $j \geq 2$.

Remark 2. *It is easy to prove, by using Theorem 1 and Remark 1, that for $F(z) \in T$ and $f(z) = V_\mu(F)(z)$, we have $f(z) \in T$, where V_μ is the integral operator defined by (8).*

Theorem 5. *Let $F(z)$ be in the class $T^*(\alpha)$, $\alpha \in [0,1)$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = V_\mu(F)(z) \in T^*(\alpha)$, where V_μ is the integral operator defined by (8).*

Proof. From Remark 2 we obtain $f(z) = V_\mu(F)(z) \in T$.

We have $f(z) = z - \sum_{j=2}^{\infty} b_j z^j$, where $b_j = \left(\frac{c+1}{c+j}\right)^{\delta} a_j z^j$. By using Remark 1 we obtain $\frac{j-\alpha}{1-\alpha} b_j < \frac{j-\alpha}{1-\alpha} a_j$, for $j = 2, 3, \ldots$, $0 \leq \alpha < 1$,

and thus $\sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} b_j \leq \sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} a_j \leq 1$. This mean (see Theorem 2) that $f(z) = V_\mu(F)(z) \in T^*(\alpha)$.

Similarly (by using Remark 2 and the Theorems 2, 3 and 4) we obtain:

Theorem 6. *Let $F(z)$ be in the class $T^c(\alpha)$, $\alpha \in [0,1)$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = V_\mu(F)(z) \in T^c(\alpha)$, where V_μ is the integral operator defined by (8).*

Theorem 7. *Let $F(z)$ be in the class $C^*(\alpha, \beta)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = V_\mu(F)(z) \in C^*(\alpha, \beta)$, where V_μ is the integral operator defined by (8).*

Theorem 8. *Let $F(z)$ be in the class $T_n(\alpha, \beta)$, $\alpha \in [0,1)$, $\beta \in (0,1]$, $n \in \mathbb{N}$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. Then $f(z) = V_\mu(F)(z) \in T_n(\alpha, \beta)$, where V_μ is the integral operator defined by (8).*

Remark 3. *By choosing $\beta = 1$, respectively $n = 0$, in the above theorem, we obtain the similarly results for the classes $T_n(\alpha)$ and $S^*(\alpha, \beta)$.*

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