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Notes on radius problems of certain univalent functions

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Abstract

For analytic functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U}$, a class $\mathcal{P}(\beta_1, \beta_2; \lambda)$ of $f(z)$ defined by some conditions with some complex numbers β_1 and β_2 is introduced. The object of the present paper is to consider some radius problems of $\dfrac{1}{\alpha}f(\alpha z)$ for $f(z) \in \mathcal{S}$.

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1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$. For $f(z) \in \mathcal{A}$, we say that $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$ if $f(z)$ satisfies $\dfrac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$ and

$$\left| \beta_1 \left(\frac{z}{f(z)} \right)'' + \beta_2 \left(\frac{z}{f(z)} \right)''' \right| \leqq \lambda \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some complex numbers β_1 and β_2, and for some real $\lambda > 0$. Obradović and Ponnusamy [2] have studied the subclass $\mathcal{P}_2(\lambda)$ of $\mathcal{A}$ consisting of $f(z)$ satisfying $\dfrac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$ and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leqq \lambda \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some real $\lambda > 0$.

Let us consider a function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{\delta}} \qquad (\delta \geqq 0). \tag{1.4}$$

Then, we see that

$$\frac{f(z)}{z} = \frac{1}{(1-z)^{\delta}} \neq 0 \quad (z \in \mathbb{U}), \tag{1.5}$$

$$\left| \left(\frac{z}{f(z)} \right)'' \right| = \left| \delta(\delta-1)(1-z)^{\delta-2} \right| < \delta(\delta-1)2^{\delta-2} \qquad (\delta \geqq 2) \tag{1.6}$$

and

(1.7)
$$\left| \left(\frac{z}{f(z)} \right)''' \right| = \left| \delta(\delta-1)(\delta-2)(1-z)^{\delta-3} \right| < \delta(\delta-1)(\delta-2)2^{\delta-3} \qquad (\delta \geqq 3)\ .$$

Therefore, Koebe function $f(z) = \dfrac{z}{(1-z)^2}$ belongs to the class $\mathcal{P}(1,0;2)$ and $\mathcal{P}(0,1;\lambda)$ for any $\lambda > 0$.
If we consider the function $f(z)$ defined by

$$f(z) = \frac{z}{\sum\limits_{k=0}^{n} z^k},$$

then

$$\begin{aligned}\left|\beta_1\left(\frac{z}{f(z)}\right)'' + \beta_2\left(\frac{z}{f(z)}\right)'''\right| &< |\beta_1|\sum_{k=2}^{n} k(k-1) + |\beta_2|\sum_{k=3}^{n} k(k-1)(k-2) \\ &= \frac{n(n+1)(n-1)(4|\beta_1| + 3(n-2)|\beta_2|)}{12}.\end{aligned}$$

This means that $f(z) \in \mathcal{P}(\beta_1, \beta_2; \lambda)$ with

$$\lambda = \frac{n(n+1)(n-1)(4|\beta_1| + 3(n-2)|\beta_2|)}{12}.$$

2 Main results

To consider our problem for the class $\mathcal{P}(\beta_1, \beta_2; \lambda)$, we need the following lemma due to Goodman [1].

Lemma 1 *If* $f(z) \in \mathcal{S}$ *and*

$$\frac{z}{f(z)} = 1 + \sum_{n=2}^{\infty} b_n z^n, \tag{2.1}$$

then we have

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leqq 1. \tag{2.2}$$

Further, we need the following lemma.

Lemma 2 *Let* $f(z) \in \mathcal{A}$ *and* $\dfrac{z}{f(z)} = 1 + \sum\limits_{n=1}^{\infty} b_n z^n \neq 0 \quad (z\in\mathbb{U})$. *If* $f(z)$ *satisfies*

(2.3)
$$2|\beta_1||b_2| + \sum_{n=3}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n| \leqq \lambda,$$

for some complex numbers β_1 *and* β_2, *then* $f(z)\in\mathcal{P}(\beta_1, \beta_2; \lambda)$.

Proof. We note that

(2.4)
$$\left|\beta_1\left(\frac{z}{f(z)}\right)'' + \beta_2\left(\frac{z}{f(z)}\right)'''\right| < 2|\beta_1||b_2| + \sum_{n=3}^{\infty} n(n-1)(|\beta_1| + (n-2)|\beta_2|)|b_n|.$$

Thus, if $f(z)$ satisfies the inequality (2.3), then $f(z)\in\mathcal{P}(\beta_1, \beta_2; \lambda)$.

Now, we derive

Theorem 1 *Let* $f(z) \in \mathcal{S}$ *and* $\alpha \in \mathbb{C}$ $(|\alpha| < 1)$. *Then the function* $\dfrac{1}{\alpha} f(\alpha z)$ *belongs to the class* $\mathcal{P}(\beta_1, \beta_2; \lambda)$ *for* $0 < |\alpha| \leqq |\alpha_0(\lambda)|$, *where* $|\alpha_0| = |\alpha_0(\lambda)|$ *is the smallest root of the equation*

(2.5)

$$|\beta_1|\frac{|\alpha|^2\sqrt{2\left(|\alpha|^2+2\right)}}{\left(1-|\alpha|^2\right)^2}+|\beta_2|\frac{|\alpha|^3\sqrt{6\left(3|\alpha|^4+14|\alpha|^2+3\right)}}{\left(1-|\alpha|^2\right)^3}\left(1-|b_2|^2\right)^{\frac{1}{2}}=\lambda$$

in $0<|\alpha|<1$.

Proof. Since $\dfrac{z}{f(z)}\neq 0 \quad (z\in\mathbb{U})$ for $f(z)\in\mathcal{S}$, if we write

$$\frac{z}{f(z)}=1+\sum_{n=1}^{\infty}b_nz^n,$$

then

$$\frac{z}{\frac{1}{\alpha}f(\alpha z)}=1+\sum_{n=1}^{\infty}\alpha^nb_nz^n \tag{2.6}$$

for $0<|\alpha|<1$.

To show that $\frac{1}{\alpha}f(\alpha z)\in\mathcal{P}(\beta_1,\beta_2;\lambda)$, we have to prove that

$$|\beta_1|\sum_{n=2}^{\infty}\frac{n!}{(n-2)!}\left|\alpha^nb_n\right|+|\beta_2|\sum_{n=3}^{\infty}\frac{n!}{(n-3)!}\left|\alpha^nb_n\right|\leqq\lambda \tag{2.7}$$

which is equivalent to (2.3) by means of Lemma 2. Indeed, applying the Cauchy-Schwarz inequality for the left hand of (2.7), we obtain that

$$|\beta_1|\sum_{n=2}^{\infty}\frac{n!}{(n-2)!}\left|\alpha^nb_n\right|+|\beta_2|\sum_{n=3}^{\infty}\frac{n!}{(n-3)!}\left|\alpha^nb_n\right|=|\beta_1|\sum_{n=2}^{\infty}\left(n^2(n-1)|\alpha|^{2n}\right)^{\frac{1}{2}}\left((n-1)|b_n|^2\right)^{\frac{1}{2}}+|\beta_2|\sum_{n=3}^{\infty}\left(n^2(n-1)(n-2)^2|\alpha|^{2n}\right)^{\frac{1}{2}}\left((n-1)|b_n|^2\right)^{\frac{1}{2}} \tag{2.8}$$

$$\leqq |\beta_1| \left(\sum_{n=2}^{\infty} n^2(n-1)|\alpha|^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} (n-1)|b_n|^2 \right)^{\frac{1}{2}}$$

$$+|\beta_2| \left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|\alpha|^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=3}^{\infty} (n-1)|b_n|^2 \right)^{\frac{1}{2}}$$

$$\leqq |\beta_1| \left(\sum_{n=2}^{\infty} n^2(n-1)|\alpha|^{2n} \right)^{\frac{1}{2}} + |\beta_2| \left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|\alpha|^{2n} \right)^{\frac{1}{2}} \left(1 - |b_2|^2 \right)^{\frac{1}{2}}$$

$$= |\beta_1| \frac{|\alpha|^2 \sqrt{2\left(|\alpha|^2+2\right)}}{\left(1-|\alpha|^2\right)^2} + |\beta_2| \frac{|\alpha|^3 \sqrt{6\left(3|\alpha|^4+14|\alpha|^2+3\right)}}{\left(1-|\alpha|^2\right)^3} \left(1 - |b_2|^2 \right)^{\frac{1}{2}}.$$

Now, we consider the complex number α $\left(0 < |\alpha| < 1\right)$ such that

(2.9)
$$|\beta_1| \frac{|\alpha|^2 \sqrt{2\left(|\alpha|^2+2\right)}}{\left(1-|\alpha|^2\right)^2} + |\beta_2| \frac{|\alpha|^3 \sqrt{6\left(3|\alpha|^4+14|\alpha|^2+3\right)}}{\left(1-|\alpha|^2\right)^3} \left(1 - |b_2|^2 \right)^{\frac{1}{2}} = \lambda.$$

This give that

$$h(|\alpha|) = -\lambda|\alpha|^6 + \left(3\lambda + |\beta_1|\sqrt{2(|\alpha|^2+2)}\right)|\alpha|^4$$

$$-|\beta_2|\sqrt{6(3|\alpha|^4+14|\alpha|^2+3)(1-|b_2|^2)}|\alpha|^3 - \left(3\lambda + |\beta_1|\sqrt{2(|\alpha|^2+2)}\right)|\alpha|^2 + \lambda = 0.$$

Noting that $h(0) = \lambda > 0$ and $h(1) = -2\sqrt{30(1-|b_2|^2)}|\beta_2| < 0$, $h(|\alpha|) = 0$ has a root $|\alpha_0| = |\alpha_0(\lambda)|$ in $0 < |\alpha| < 1$. This completes the proof of the theorem.

Remark 1 In the proof of Theorem 1, we calculate

$$\left(\sum_{n=2}^{\infty} n^2(n-1)|\alpha|^{2n} \right)^{\frac{1}{2}} = \frac{|\alpha|^2 \sqrt{2\left(2+|\alpha|^2\right)}}{\left(1-|\alpha|^2\right)^2}$$

as follows. Note that

$$\sum_{n=2}^{\infty} n^2(n-1)t^n = t^2\left(\sum_{n=2}^{\infty} nt^n\right)'' = t^2\left(\frac{-t^3+2t^2}{(1-t)^2}\right)''$$

$$= t^2\left(\frac{t^3-3t^2+4t}{(1-t)^3}\right)' = \frac{2t^2(t+2)}{(1-t)^4} .$$

Letting $t=|\alpha|^2$, we have

$$\left(\sum_{n=2}^{\infty} n^2(n-1)|\alpha|^{2n}\right)^{\frac{1}{2}} = \frac{|\alpha|^2\sqrt{2\big(2+|\alpha|^2\big)}}{\big(1-|\alpha|^2\big)^2} .$$

Further, we prove

$$\left(\sum_{n=2}^{\infty} n^2(n-1)(n-2)^2|\alpha|^{2n}\right)^{\frac{1}{2}} = \frac{|\alpha|^3\sqrt{6\big(3|\alpha|^4+14|\alpha|^2+3\big)}}{\big(1-|\alpha|^2\big)^3}$$

as follows. Note that

$$\sum_{n=2}^{\infty} n^2(n-1)(n-2)^2t^n = t^3\left(\sum_{n=3}^{\infty} n(n-2)t^n\right)''' = t^3\left(\frac{-t^4+3t^3}{(1-t)^3}\right)'''$$

$$= t^3\left(\frac{t^4-4t^3+9t^2}{(1-t)^4}\right)'' = t^3\left(\frac{6t^2+18t}{(1-t)^5}\right)' = 6t^3\frac{3t^2+14t+3}{(1-t)^6} .$$

Letting $t=|\alpha|^2$, we have

$$\left(\sum_{n=2}^{\infty} n^2(n-1)(n-2)^2|\alpha|^{2n}\right)^{\frac{1}{2}} = \frac{|\alpha|^3\sqrt{6\big(3|\alpha|^4+14|\alpha|^2+3\big)}}{\big(1-|\alpha|^2\big)^3} .$$

Remark 2 If we take $\alpha = \frac{1}{2}e^{i\theta}$ in (2.5), then we have

$$\lambda = \frac{2\sqrt{2}}{3}|\beta_1| + \frac{2\sqrt{642}}{27}|\beta_2|\sqrt{1-|b_2|^2}.$$

If we put $\lambda = |\beta_1| = |\beta_2| = 1$ and $|b_2| = \dfrac{1}{2}$ in (2.5), then we have

$$(1-|\alpha|)^2|\alpha|^2\sqrt{2(|\alpha|^2+2)}+|\alpha|^3\sqrt{6(3|\alpha|^4+14|\alpha|^2+3)}\frac{\sqrt{3}}{2}-\left(1-|\alpha|^2\right)^3=0.$$

It is easy to see that the above equation has a root $|\alpha_0|$ such that $0.3999 < |\alpha_0| < 0.4002$.

References

[1] A. W. Goodman, Univalent Functions, Vol.I and II, Mariner, Tampa, Florida, 1983.

[2] M. Obradović and S. Ponnusamy, Radius properties for subclasses of univalent functions, Analysis, 25(2005), 183-188.

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