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Some analytic and multivalent functions defined by subordination property [1]

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Abstract

In this paper we introduce some functions which are multivalently analytic defined by the subordination property and the Dziok-Srivastava linear operator. We obtain characterizing property, growth and distortion inequalities, closure theorem, extreme points, radius of starlikeness, convexity, and close-to-convexity for the functions in the class. We also discuss inclusion and neighbourhood properties, region of p-valency and a class preserving linear operator for these functions. Interesting consequences of the results obtained are also indicated.

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1 Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0, \;\; p, n \in I\!N) \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. If $f(z) \in A(p)$ is given by (1) and $g(z) \in A(p)$ is given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k \quad (b_k \geq 0, \;\; p, n \in I\!N_0) \tag{2}$$

the convolution $(f * g)(z)$ of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k := (g * f)(z). \tag{3}$$

A function $f \in A(p)$ is said to be p-valently starlike of order ρ $(0 \leq \rho < p)$ in U if and only if,

$$Re\left\{z\frac{f'(z)}{f(z)}\right\} > \rho. \tag{4}$$

Similarly, a function $f(z)$ is p-valently convex of order ρ $(0 \leq \rho < p)$ in U if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho. \tag{5}$$

It follows from expression (4) and (5) that f is convex if and only if, zf' is starlike.A function $f(z) \in A(p)$ is close-to-convex of order ρ if

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \rho \;\; (0 \leq \rho < p). \tag{6}$$

For the two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f \prec g$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

The operator

$$
\begin{aligned}
(H_s^q[a_1]f)(z) &:= H_s^q(a_1, \cdots, a_q; b_1, \cdots, b_s)f(z) \\
&= z^p \ {}_qF_s(a_1, \cdots, a_q; b_1, \cdots, b_s; z) * f(z) \\
&= z^p + \sum_{k=p+n}^{\infty} \frac{(a_1)_{k-p} \cdots (a_q)_{k-p} a_k}{(b_1)_{k-p} \cdots (b_s)_{k-p}(k-p)!} z^k \qquad (7) \\
&= z^p + \sum_{k=p+n}^{\infty} h(k) a_k z^k
\end{aligned}
$$

where

$$
h(k) = \frac{(a_1)_{k-p} \cdots (a_q)_{k-p}}{(b_1)_{k-p} \cdots (b_s)_{k-p}(k-p)!} \qquad (8)
$$

Here ${}_qF_s(z)$ is the generalized hypergeometric function for $a_j \in \mathbb{C}$ $(j = 1, 2, \cdots, q)$ and $b_j \in \mathbb{C}$ $(j = 1, 2, \cdots, s)$ such that $b_j \neq 0, -1, -2, \cdots$ $(j = 1, 2, \cdots, s)$ defined by

$$
\begin{aligned}
{}_qF_s(z) &= {}_qF_s(a_1, \cdots, a_q; b_1, \cdots, b_s; z) \qquad (9) \\
&= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_s)_k k!} z^k \quad (q \leq s+1, q, s \in I\!N_0, \ z \in U)
\end{aligned}
$$

where

$$
(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0) \\ \lambda(\lambda+1) \cdots (\lambda + k - 1) & (k \in I\!N) \end{cases}
$$

The series ${}_qF_s(z)$ in (9) converges absolutely for $|z| < \infty$ if $q < s+1$ and for $|z| = 1$ if $q = s+1$.The linear operator defined in (7) is the Dziok-Srivastava operator (for details see [2], [3]) which contains the well-known operators like the Hohlov linear operator [6], the Carlson-Shafer operator [1], the Ruscheweyh derivative operator [11], the Srivastava-Owa fractional derivative operator [9], the Saitoh generalized linear operator, the Bernardi-Libera-Livingston operator and many others. One may refer [9] for further details and references for these operators.

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions f of the form

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0, p, n \in I\!N) \tag{10}$$

which are analytic and p-valent in U.

By applying the subordination definition we introduce a new class $K(\lambda, \mu, A, B)$ of functions belonging to $T(p)$ and satisfying

$$L(z) = \frac{z(H_s^q[a_1]f)' + \lambda z^2 (H_s^q[a_1]f)''}{(1-\mu)(H_s^q[a_1]f) + \mu z (H_s^q[a_1]f)'} \prec p\frac{1+Az}{1+Bz} \tag{11}$$

$(0 \leq \mu \leq \lambda \leq 1, -1 \leq A < B \leq 1, a_j \in \mathbb{C}\ (j = 1, 2, \cdots, q), b_j \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}\ (j = 1, 2, \cdots, s), q \leq s+1, q, s \in I\!N, z \in U)$.

Following the work of Goodman [5] and Ruscheweyh [11], we define the (n, δ)-neighbourhood of a function $f \in T(p)$ by

$$N_{n,\delta}(f) := \left\{ g \in T(p) : g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \text{ and } \sum_{k=p+n}^{\infty} k|a_k - b_k| \leq \delta \right\}. \tag{12}$$

In particular, for the function $e(z) = z^p \quad (p \in I\!N)$

(13)

$$N_{n,\delta}(e) := \left\{ g \in T(p) : g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \text{ and } \sum_{k=p+n}^{\infty} k|b_k| \leq \delta \right\}.$$

A function $f(z) \in T(p)$ defined by (10) is said to be in the class $K^{\alpha}(\lambda, \mu, A, B)$ if there exists a function $g(z) \in K(\lambda, \mu, A, B)$ such that

(14) $$\left| \frac{f(z)}{g(z)} - 1 \right| < p - \alpha \quad (z \in U, 0 \leq \alpha < p)$$

2 Main results and properties of the Class $K(\lambda, \mu, A, B)$

Theorem 1. *Let the function $f(z)$ be defined by (10). Then the function $f(z)$ belongs to the class $K(\lambda, \mu, A, B)$ if and only if*

(15) $$\sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) a_k \leq 1$$

where

(16)

$$M(\lambda, \mu, A, B, k) = \frac{[k(1 + B)(1 + \lambda(k - 1)) - p(1 + A)(1 + \mu(k - 1))]h(k)}{p[(B - 1)(1 + \lambda(p - 1)) - (A - 1)(1 + \mu(p - 1))]}$$

for

$$h(k) = \frac{(a_1)_{k-p} \cdots (a_q)_{k-p}}{(b_1)_{k-p} \cdots (b_s)_{k-p}(k - p)!}$$

$(0 \leq \mu \leq \lambda \leq 1, -1 \leq A < B \leq 1, a_j \in \mathbb{C}\ (j = 1, 2, \cdots, q), b_j \in \mathbb{C} \setminus \{0, -1, -2, \cdots\} \quad (j = 1, 2, \cdots, s))$. The result is sharp with the extremal function $f(z)$ given by

(17) $$f(z) = z^p - \frac{1}{M(\lambda, \mu, A, B, k)} z^{p+n}\ (n \in I\!N).$$

Proof. We suppose that $f(z) \in K(\lambda,\mu,A,B)$. Then by recalling the condition (11), we have

(18)
$$\left|\frac{p[1+\lambda(p-1)-(1+\mu(p-1))]-\sum_{k=p+n}^{\infty}[k(1+\lambda(k-1))-p(1+\mu(k-1))]h(k)a_k z^{k-p}}{p[B(1+\lambda(p-1))-A(1+\mu(p-1))]-\sum_{k=p+n}^{\infty}[kB(1+\lambda(k-1))-Ap(1+\mu(k-1))]h(k)a_k z^{k-p}}\right| \le 1 \quad (z\in U).$$

Now choosing values of z on the real axis and allowing $z \to 1$ from the left through real values, the inequality (18) immediately yields the desired condition in (15).

Conversely, by assuming the hypothesis (15) and $|z| = 1$, we note the following by the subordination property:

$$\left|\frac{L(z)-p}{L(z)B-pA}\right|$$
$$\le \left|\frac{p[1+\lambda(p-1)-(1+\mu(p-1))]-\sum_{k=p+n}^{\infty}[k(1+\lambda(k-1))-p(1+\mu(k-1))]h(k)a_k z^{k-p}}{p[B(1+\lambda(p-1))-A(1+\mu(p-1))]-\sum_{k=p+n}^{\infty}[kB(1+\lambda(k-1))-Ap(1+\mu(k-1))]h(k)a_k z^{k-p}}\right|$$
$$\le \frac{p[1+\lambda(p-1)-(1+\mu(p-1))]+\sum_{k=p+n}^{\infty}[k(1+\lambda(k-1)-p(1+\mu(k-1))]h(k)a_k}{p[B(1+\lambda(p-1))-A(1+\mu(p-1))]-\sum_{k=p+n}^{\infty}[k(B(1+\lambda(k-1))-Ap(1+\mu(k-1))]h(k)a_k}.$$

Hence by maximum modulus theorem $f(z) \in K(\lambda,\mu,A,B)$.Finally, it is observed that the result is sharp and the extremal function is given by (17). Theorem 1 immediately yields the following result.

Corollary 1. *If a function $f(z)$ of the form (10) is in $T(p)$ and belongs to the class $K(\lambda,\mu,A,B)$, then*

(19)
$$a_k \le \frac{p[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]}{[k(1+B)(1+\lambda(k-1))-p(1+A)(1+\mu(k-1))]h(k)} \quad (k \ge p+n, n \in I\!N)$$

where the equality holds true for the function (17).

Proof. The result (19) follows from the fact that the series in (15) converges.

Next we give some more interesting properties of the class $K(\lambda, \mu, A, B)$.

Theorem 2. *Let* $0 \le \lambda \le \mu \le 1, -1 \le A < B \le 1, -1 \le A' < B' \le 1$. *Then*

$$K(\lambda, \mu, A, B) = K(\lambda, \mu, A', B') \tag{20}$$

if and only if

$$M(\lambda, \mu, A, B, k) = M(\lambda, \mu, A', B', k). \tag{21}$$

Proof. Let $f(z) \in K(\lambda, \mu, A, B)$ and (21) hold true. Then by Theorem 1, we have

$$\sum_{k=p+n}^{\infty} M(\lambda, \mu, A', B', k) a_k = \sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) a_k \le 1.$$

This implies $f(z) \in K(\lambda, \mu, A', B')$. Similarly it can be shown that $f(z) \in K(\lambda, \mu, A', B')$ implies $f(z) \in K(\lambda, \mu, A, B)$. Hence (21) implies $K(\lambda, \mu, A, B) = K(\lambda, \mu, A', B')$. Conversely, suppose (20) holds true. Notice that a function defined by (10) belonging to $K(\lambda, \mu, A, B)$ will belong to $K(\lambda, \mu, A', B')$ only if

$$\sum_{k=p+n}^{\infty} M(\lambda, \mu, A', B', k) a_k \le \sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) a_k$$

that is if

$$M(\lambda, \mu, A', B', k) \le M(\lambda, \mu, A, B, k). \tag{22}$$

Similarly, we can show that

$$M(\lambda, \mu, A, B, k) \le M(\lambda, \mu, A', B', k). \tag{23}$$

(22) and (23) together imply (21). Hence the result.

We state some more interesting deductions which follow using Theorem 1 and Theorem 2.

Theorem 3. *Let* $0 \le \lambda \le \mu \le 1, -1 \le A < B_1 \le B_2 \le 1$. *Then*

$$K(\lambda, \mu, A, B_1) \supseteq K(\lambda, \mu, A, B_2).$$

Proof. Notice that

$$M(\lambda, \mu, A, B_1, k) \le M(\lambda, \mu, A, B_2, k) \text{ for } B_1 \le B_2. \tag{24}$$

If $f(z) \in K(\lambda, \mu, A, B_2)$ we have

$$\sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B_1, k)a_k \le \sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B_2, k)a_k \le 1$$

Thus by Theorem 1 it follows that $f(z) \in K(\lambda, \mu, A, B_1)$. Hence the theorem is proved.

Theorem 4. *Let* $0 \le \lambda \le \mu \le 1, -1 \le A_1 \le A_2 < B \le 1$. *Then*

$$K(\lambda, \mu, A_1, B) \subseteq K(\lambda, \mu, A_2, B).$$

Proof. The proof of the theorem is on the lines of Theorem 3 above.

Next we give a result which follows from Theorem 3 and Theorem 4.

Corollary 2. *Let* $0 \le \lambda \le \mu \le 1, -1 \le A_1 \le A_2 < B_1 \le B_2 \le 1$. *Then*

$$K(\lambda, \mu, A_1, B_2) \subseteq K(\lambda, \mu, A_2, B_2) \subseteq K(\lambda, \mu, A_2, B_1).$$

3 Growth and Distortion Theorem

Let us recall again the function $h(k)$ given by (8)

$$h(k) = \frac{(a_1)_{k-p} \cdots (a_q)_{k-p}}{(b_1)_{k-p} \cdots (b_s)_{k-p}(k-p)!}.$$

We note that $h(k)$ is a non-decreasing function of k for $k \geq p+n, n \in I\!N$. Thus

$$h(k) \geq h(p+1) = \frac{a_1 \cdots a_q}{b_1 \cdots b_s} \geq 0. \tag{25}$$

We now state the following growth and distortion inequalities for the class $K(\lambda, \mu, A, B)$.

Theorem 5. *If the function $f(z)$ defined by (10) is in the class $K(\lambda, \mu, A, B)$, then*

$$||f(z)|-|z|^p| \leq \frac{p[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1))-p(1+A)(1+\mu(p+n-1))]h(p+n)}|z|^{p+n}, \ (n \in I\!N) \tag{26}$$

and

$$||f'(z)|-p\,|z|^{p-1}| \leq \frac{p(p+n)[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1))-p(1+A)(1+\mu(p+n-1))]h(p+n)}|z|^{p+n-1}, \ (n \in I\!N). \tag{27}$$

The result in (26) and (27) are sharp with the extremal function

$$f(z) = z^p - \frac{p[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1))-p(1+A)(1+\mu(p+n-1))]h(p+n)}z^{p+n}, \ \ (n \in I\!N).$$

Proof. We have

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad \text{therefore}$$

$$\begin{aligned} |f(z)| &\leq |z|^p + \sum_{k=p+n}^{\infty} a_k |z|^k \leq |z|^p + |z|^{p+n} \sum_{k=p+n}^{\infty} a_k \\ &\leq |z|^p + \frac{p[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1))-p(1+A)(1+\mu(p+n-1))]h(p+n)}|z|^{p+n}. \end{aligned} \tag{28}$$

Similarly

$$\begin{aligned} |f(z)| &\geq |z|^p - \sum_{k=p+n}^{\infty} a_k |z|^k \geq |z|^p - |z|^{p+n} \sum_{k=p+n}^{\infty} a_k \\ &\geq \frac{p(p+n)[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1))-p(1+A)(1+\mu(p+n-1))]h(p+n)}|z|^{p+n}. \end{aligned} \tag{29}$$

Combining (28) and (29) we get the result (26). The next result in (27) can be derived similarly.

Remark. *Let the function $f(z)$ defined by (10) be in the class $K(\lambda, \mu, A, B)$. Then $f(z)$ is included in a disc with centre at the origin and radius*

$$R_1 = 1 + \frac{p[(B-1)(1+\lambda(p-1)) - (A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1)) - p(1+A)(1+\mu(p+n-1))]h(p+n)}, \quad (n \in I\!N).$$

and $f'(z)$ is included in a disc with centre at origin and radius

$$R_2 = p + \frac{p(p+n)[(B-1)(1+\lambda(p-1)) - (A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1)) - p(1+A)(1+\mu(p+n-1))]h(p+n)}, \quad (n \in I\!N).$$

Now we state a theorem of convex linear combinations of the functions in the class $K(\lambda, \mu, A, B)$.

Theorem 6. *Let the function*

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0, \quad j = 1, 2, \cdots, l)$$

be in the class $K(\lambda, \mu, A, B)$. Then

$$h(z) = \sum_{j=1}^{l} c_j f_j(z) \in K(\lambda, \mu, A, B)$$

where $\sum\limits_{j=1}^{l} c_j = 1$ and $c_j \geq 0$ $(j = 1, 2, \cdots, l)$. Thus, we note that $K(\lambda, \mu, A, B)$ is a convex set.

Proof. We have

$$h(z) = \sum_{j=1}^{\ell} c_j \left(z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \right) \tag{30}$$
$$= z^p \sum_{j=1}^{\ell} c_j - \sum_{j=1}^{\ell} \sum_{k=p+n}^{\infty} c_j a_{k,j} z^k$$
$$= z^p - \sum_{k=p+n}^{\infty} \left(\sum_{j=1}^{\ell} a_{k,j} c_j \right) z^k$$
$$= z^p - \sum_{k=p+n}^{\infty} e_k z^k$$

where $e_k = \sum_{j=1}^{\ell} a_{k,j} c_j$.

Since $f_j \in K(\lambda, \mu, A, B)$ by (15), we have

$$\sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) a_{k,j} \leq 1. \tag{31}$$

In view of (30) $h(z) \in K(\lambda, \mu, A, B)$ if

$$\sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) e_k \leq 1.$$

Now, we have

$$\begin{aligned} \sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) e_k &= \sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) \sum_{j=1}^{\ell} a_{k,j} c_j \\ &= \sum_{j=1}^{\ell} c_j \sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) a_{k,j} \\ &\leq \sum_{j=1}^{\ell} c_j = 1. \end{aligned}$$

Thus $h(z) \in K(\lambda, \mu, A, B)$.

4 Extreme Points

Theorem 7. *Let $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{p[(B-1)(1+\lambda(p-1)) - (A-1)(1+\mu(p-1))]}{[k(1+B)(1+\lambda(k-1)) - p(1+A)(1+\mu(k-1))]h(k)} z^k$$

$(k \geq p+n, n \in I\!N)$. Then $f(z) \in K(\lambda, \mu, A, B)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{k=p+n}^{\infty} d_k f_k(z) \tag{32}$$

where $d_k \geq 0$ *and* $\sum_{k=p+n}^{\infty} d_k = 1, n \in I\!N_0$.

Proof. Let $f(z)$ be expressible in the form

$$\begin{aligned} f(z) &= \sum_{k=p+n}^{\infty} \lambda_k f_k(z) \quad (n \in I\!N_0) \\ &= z^p - \sum_{k=p+n}^{\infty} \frac{1}{M(\lambda, \mu, A, B, k)} \; d_k z^k \quad (n \in I\!N). \end{aligned}$$

Now

$$\sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) \frac{1}{M(\lambda, \mu, A, B, k)} d_k = \sum_{k=p+n}^{\infty} d_k = 1 - d_p \leq 1 \quad (n \in I\!N).$$

Therefore, $f(z) \in K(\lambda, \mu, A, B)$. Conversely, suppose that $f(z) \in K(\lambda, \mu, A, B)$. Then setting

$$d_k = \frac{1}{M(\lambda, \mu, A, B, k)} a_k \quad \text{and} \quad d_p = 1 - \sum_{k=p+n}^{\infty} d_k \quad (n \in I\!N)$$

we notice that $f(z)$ can be expressed in the form (32).

Remark. *The extreme points of the class* $K(\lambda, \mu, A, B)$ *are* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{p[(B-1)(1+\lambda(p-1)) - (A-1)(1+\mu(p-1))]}{[k(1+B)(1+\lambda(k-1)) - p(1+A)(1+\mu(k-1))h(k)} z^k, \quad (k \geq p+n, n \in I\!N).$$

5 Inclusion Property

We now obtain an inclusion relation for the functions in the class $K(\lambda, \mu, A, B)$.

Theorem 8. *If* $h(k) \geq h(p+n)$ *for* $k \geq p+n, n \in I\!N$ *and*

(33)

$$\delta := \frac{p(p+n)[(B-1)(1+\lambda(p-1)) - (A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1)) - p(1+A)(1+\mu(p+n-1))]h(p+n)}$$

then

$$K(\lambda,\mu,A,B)\subseteq N_{n,\delta}(e). \tag{34}$$

Proof. Let $f(z)\in K(\lambda,\mu,A,B)$. Then in view of assertion (15) of Theorem 1 and the condition $h(k)\geq h(p+n)$ for $k\geq p+n, n\in I\!N$, we get

$$h(p+n)[(p+n)(1+B)(1+\lambda(p+n-1))-p(1+A)(1+\mu(p+n-1))]\sum_{k=p+n}^{\infty}a_k$$
$$\leq \sum_{k=p+n}^{\infty}[k(1+B)(1+\lambda(k-1))-p(1+A)(1+\mu(k-1))]h(k)a_k$$
$$\leq p[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))] \tag{35}$$

which implies

$$\sum_{k=p+n}^{\infty}a_k\leq \frac{p[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]}{[(p+n)(1+B)(1+\lambda(p+n-1))-p(1+A)(1+\mu(p+n-1))]h(p+n)}. \tag{36}$$

Applying the assertion (15) of Theorem 1 in conjunction with (36), we obtain

$$(1+B)(1+\lambda(p+n-1))h(p+n)\sum_{k=p+n}^{\infty}ka_k$$
$$\leq p[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]+p(1+A)(1+\mu(p+n-1))h(p+n)\sum_{k=p+n}^{\infty}a_k$$
$$\leq \frac{p(p+n)[(B-1)(1+\lambda(p-1))-(A-1)(1+\mu(p-1))]}{(p+n)(1+B)(1+\lambda(p+n-1))-p(1+A)(1+\mu(p+n-1))]h(p+n)}:=\delta$$

which by virtue of (12) establishes the inclusion relation (34).

6 Neighbourhood Property

In this section we determine the neighbourhood property for the class $K^{\alpha}(\lambda,\mu,A,B)$.

Theorem 9. *If* $g(z)\in K(\lambda,\mu,A,B)$ *and*

$$\alpha=p-\frac{\delta}{p+n}\frac{M(\lambda,\mu,A,B,p+n)}{M(\lambda,\mu,A,B,p+n)-1} \tag{37}$$

then

$$N_{n,\delta}(g) \subset K^{\alpha}(\lambda, \mu, A, B).$$

Proof. Suppose that $f(z) \in N_{n,\delta}(g)$. We then find from (12) that

$$\sum_{k=p+n}^{\infty} k|a_k - b_k| \leq \delta$$

which readily implies the following coefficient inequality

$$\sum_{k=p+n}^{\infty} |a_k - b_k| \leq \frac{\delta}{p+n} \quad (n \in \mathbb{N}). \tag{38}$$

Next, since $g(z) \in K(\lambda, \mu, A, B)$, in view of (36), we have

$$\sum_{k=p+n}^{\infty} b_k \leq \frac{1}{M(\lambda, \mu, A, B, p+n)} \tag{39}$$

Using (38) and (39), we get

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum\limits_{k=p+n}^{\infty} |a_k - b_k|}{1 - \sum\limits_{k=p+n}^{\infty} b_k} \leq \frac{\delta}{p+n} \frac{M(\lambda, \mu, A, B, p+n)}{M(\lambda, \mu, A, B, p+n) - 1}$$

provided that α is given by (37). Thus by condition (14) $f(z) \in K^{\alpha}(\lambda, \mu, A, B)$ where α is given by (37).

7 Radius of Starlikeness, Convexity and Close-to-convexity

Using the inequalities (4), (5) and (6) and Theorem 1 we can compute the radius of starlikeness, convexity and close-to-convexity.

Theorem 10. *Let a function $f(z) \in K(\lambda, \mu, A, B)$. Then $f(z)$ is p-valently starlike of order ρ $(0 \leq \rho < p)$ in the disc $|z| < R_3$ where*

$$R_3 = \inf_k \left\{ \frac{(p-\rho)}{(k-\rho)} M(\lambda, \mu, A, B, k) \right\}^{\frac{1}{k-p}} \quad (k \geq p+n, \quad n \in I\!N)$$

for $M(\lambda, \mu, A, B, k)$ given by (16).

Proof : It is sufficient to show that

$$\left| \frac{zf'}{f} - p \right| \leq p - \rho \ \text{ for } \ 0 \leq \rho < p \ \text{ and } \ |z| < R_3$$

(40)

$$\left| \frac{zf'}{f} - p \right| = \left| \frac{-\sum\limits_{k=p+n}^{\infty} (k-p) a_k z^{k-p}}{1 - \sum\limits_{k=p+n}^{\infty} a_k z^{k-p}} \right|$$

(40) is bounded above by $p - \rho$ if

$$\sum_{k=p+n}^{\infty} \frac{(k-\rho)}{(p-\rho)} a_k |z|^{k-p} \leq 1. \tag{41}$$

Also from Theorem 1, if $f(z) \in K(\lambda, \mu, A, B)$ then

$$\sum_{k=n+p}^{\infty} M(\lambda, \mu, A, B, k) a_k \leq 1. \tag{42}$$

In view of (42) we notice that (41) holds true if

$$\frac{(k-\rho)}{(p-\rho)} |z|^{k-p} \leq M(\lambda, \mu, A, B, k).$$

That is if

$$|z| \leq \left\{ \frac{(p-\rho) M(\lambda, \mu, A, B, k)}{(k-\rho)} \right\}^{\frac{1}{k-p}}.$$

Setting $|z| = R_3$ we get the desired result.

Theorem 11. *Let a function $f(z) \in K(\lambda, \mu, A, B)$. Then $f(z)$ is p-valently convex of order ρ $(0 \leq \rho < p)$ in the disc $|z| < R_4$ where*

$$R_4 = \inf_k \left\{ \frac{p(p-\rho)}{k(k-\rho)} M(\lambda, \mu, A, B, k) \right\}^{\frac{1}{k-p}} \quad (k \geq p+n, \quad n \in I\!N)$$

for $M(\lambda, \mu, A, B, k)$ given by (16).

Proof : It is sufficient to show that

$$\left| \frac{zf''}{f} + 1 - p \right| \leq p - \rho \text{ for } 0 \leq \rho < p \text{ and } |z| < R_4.$$

Using arguments similar to the proof of Theorem 10, we get the result.

Theorem 12. *Let a function $f(z) \in K(\lambda, \mu, A, B)$. Then $f(z)$ is p-valently close-to-convex of order ρ $(0 \leq \rho < p)$ in the disc $|z| < R_5$ where*

$$R_5 = \inf_k \left\{ \frac{(p-\rho)}{k} \quad M(\lambda, \mu, A, B, k) \right\}^{\frac{1}{k-p}} \quad (k \geq p+n, \ n \in I\!N)$$

for $M(\lambda, \mu, A, B, k)$ given by (16).

Proof : It is sufficient to show that

$$\left| \frac{f'}{z^{p-1}} - p \right| \leq p - \rho \text{ for } 0 \leq \rho < p \text{ and } |z| < R_5.$$

The result follows by application of arguments similar to the proof of Theorem 10.

8 Application of Class Preserving Integral Operator

In this Section we give a class preserving integral operator due to Jung-Kim-Srivastava, please refer [8].

(43)
$$I(z) = Q^{\alpha}_{\beta,p} f(z) = \begin{pmatrix} \alpha+\beta+p-1 \\ \beta+p-1 \end{pmatrix} \frac{\alpha}{z^{\beta}} \int_0^z t^{\beta-1}(1-\frac{t}{z})^{\alpha-1} f(t)dt$$

$(\alpha > 0, \beta > -p, z \in U)$.

It can be easily verified that

(44)
$$I(z) = Q^{\alpha}_{\beta,p} f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{\Gamma(\beta+k)\Gamma(\alpha+\beta+k)}{\Gamma(\alpha+\beta+k)\Gamma(\beta+p)} a_k z^k.$$

A function $I(z)$ is said to be close-to-convex and p-valent in the disc $|z| < R_6$ if

(45)
$$\left| \frac{I'(z)}{z^{p-1}} - p \right| \leq p \text{ in } |z| < R_6$$

and $(\alpha > 0, \beta > -p, z \in U)$.

Theorem 13. *Let $\alpha > 0, \beta > -p$ and $f(z)$ belong to the class $K(\lambda, \mu, A, B)$. Then the function $I(z)$ defined by (43) is close-to-convex and p-valent in the disc $|z| < R_6$, where*

(46)
$$R_6 = \inf_k \left\{ \frac{p\Gamma(\alpha+\beta+k)\Gamma(\beta+p)M(\lambda,\mu,A,B,k)}{k\Gamma(\beta+k)\Gamma(\alpha+\beta+k)} \right\}^{\frac{1}{k-p}}.$$

Proof : We show that

(47)
$$\left| \frac{I'(z)}{z^{p-1}} - p \right| \leq p \text{ in } |z| < R_6$$

R_6 is given by (46).

In view of (44), we have

$$\left|\frac{I'(z)}{z^{p-1}} - p\right| = \left|-\sum_{k=p+n}^{\infty} \frac{k\Gamma(\beta+k)\Gamma(\alpha+\beta+k)}{\Gamma(\alpha+\beta+k)\Gamma(\beta+p)} a_k z^{k-p}\right|$$
$$\leq \sum_{k=p+n}^{\infty} \frac{k\Gamma(\beta+k)\Gamma(\alpha+\beta+k)}{\Gamma(\alpha+\beta+k)\Gamma(\beta+p)} a_k |z|^{k-p}.$$

The last inequality is bounded above by p if

$$\sum_{k=p+n}^{\infty} \frac{k\Gamma(\beta+k)\Gamma(\alpha+\beta+k)}{p\Gamma(\alpha+\beta+k)\Gamma(\beta+p)}\ a_k |z|^{k-p} \leq 1. \tag{48}$$

Also, since $f(z) \in K(\lambda, \mu, A, B)$ by Theorem 1, we have

$$\sum_{k=p+n}^{\infty} M(\lambda, \mu, A, B, k) a_k \leq 1 \tag{49}$$

where $M(\lambda, \mu, A, B, k)$ is given in (19). Thus (48) and consequently (47) will hold if

$$\frac{k\Gamma(\beta+k)\Gamma(\alpha+\beta+k)}{p\Gamma(\alpha+\beta+k)\Gamma(\beta+p)} a_k |z|^{k-p} \leq M(\lambda, \mu, A, B, k) a_k.$$

That is, if

$$|z| \leq \left\{\frac{p\Gamma(\alpha+\beta+k)\Gamma(\beta+p)M(\lambda, \mu, A, B, k)}{k\Gamma(\beta+k)\Gamma(\alpha+\beta+k)}\right\}^{\frac{1}{k-p}}$$

for $k \geq p+n, n \in I\!N$. The result follows by setting $|z| = R_6$.

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