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A general coefficient inequality [1]

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Abstract

In the present note, we obtain a general coefficient inequality regarding a multiplier transformation in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$. As special case to our main result, we obtain a coefficient inequality for n-starlikeness of analytic functions. Also certain known coefficient inequalities for starlikeness and convexity of analytic functions appear as particular cases of our main result.

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ p \in \mathbb{N} = \{1, 2, \cdots\},$$

which are analytic in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$.

Denote by $S^*(\alpha)$ and $K(\alpha)$, the classes of starlike functions of order α and convex functions of order α respectively, which are analytically defined as follows:

$$S^*(\alpha) = \left\{ f(z) \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{E} \right\}$$

and

$$K(\alpha) = \left\{ f(z) \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in \mathbb{E} \right\}$$

where α is a real number such that $0 \le \alpha < 1$.

We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n, \lambda) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k + \lambda}{p + \lambda} \right)^n a_k z^k, \ (\lambda \ge 0, n \in \mathbb{Z}).$$

$I_1(n, 0)$ is the well-known Sălăgean [2] derivative operator D^n, defined as

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and for $f \in \mathcal{A}$.

Denote by $S_n^*(\alpha)$, the class of n-starlike functions of order α, which is analytically defined as follows

$$S_n^*(\alpha) = \left\{ f(z) \in \mathcal{A} : \Re \, \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, z \in \mathbb{E} \right\}$$

where α is a real number such that $0 \leq \alpha < 1$.

In the present note, we obtain a general coefficient inequality regarding multiplier transformation $I_p(n, \lambda)$ in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$. As special case to our main result, we obtain a coefficient inequality for n-starlikeness of analytic functions. Also certain known coefficient inequalities for starlikeness and convexity of analytic functions appear as particular cases of our main result.

2 Main Result

Theorem 1 *If $f \in \mathcal{A}_p$ satisfies*

$$\sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda} \right)^n \left(\frac{k+\lambda}{p+\lambda} - 1 + p - \alpha \right) |a_k| \leq p - \alpha,$$

then

$$\left| \frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)} - 1 \right| < p - \alpha, \ 0 \leq \alpha < p, \ z \in \mathbb{E}.$$

Proof. To prove the required result, we prove that

$$|I_p(n+1, \lambda) f(z) - I_p(n, \lambda) f(z)| - (p - \alpha) \, |I_p(n, \lambda) f(z)| \leq 0$$

Indeed, we have

$$|I_p(n+1, \lambda) f(z) - I_p(n, \lambda) f(z)| - (p - \alpha) \, |I_p(n, \lambda) f(z)|$$

$$\begin{aligned}
&= \left|\sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^{n}\left(\frac{k+\lambda}{p+\lambda}-1\right)a_k z^k\right| \\
&\quad -(p-\alpha)\left|z^p+\sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^{n}a_k z^k\right| \\
&\le \sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^{n}\left(\frac{k+\lambda}{p+\lambda}-1\right)|a_k|\left|z^k\right| \\
&\quad -(p-\alpha)\left(|z^p|-\sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^{n}|a_k|\left|z^k\right|\right) \\
&= \sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^{n}\left(\frac{k+\lambda}{p+\lambda}-1+p-\alpha\right)|a_k|\left|z^k\right|-(p-\alpha)\left|z^p\right| \\
&< \sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^{n}\left(\frac{k+\lambda}{p+\lambda}-1+p-\alpha\right)|a_k|-(p-\alpha)\le 0.
\end{aligned}$$

Hence

$$\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}-1\right| < p-\alpha,\ 0\le\alpha<p,\ z\in\mathbb{E}.$$

3 Applications

For $\lambda = 0$ and $p = 1$ in Theorem 1, we obtain the following result.

Corollary 1 *If $f \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty} k^n(k-\alpha)|a_k| \le 1-\alpha,$$

then $f \in S_n^*(\alpha)$.

For $\lambda = 0, p = 1$ and $n = 0$ in Theorem 1, we obtain the following result.

Corollary 2 *If $f \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k| \leq 1-\alpha,$$

then $f \in S^*(\alpha)$.

For $\lambda = 0, p = 1$ and $n = 1$ in Theorem 1, we obtain the following result.

Corollary 3 *If $f \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}k(k-\alpha)|a_k| \leq 1-\alpha,$$

then $f \in K(\alpha)$.

Remark 1 *For $\alpha = 0$, Corollary 2 and Corollary 3 are the results of Clunie and Keogh [1] and Silverman [3].*

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