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On some subclasses of starlike and convex functions[1]

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Abstract

Throughout this paper, in the second section, we prove that if $f \in A$, $\alpha \geq 0$ and $F(z) = zf'(z)\left(\alpha + \frac{zf'(z)}{f(z)}\right)$ is starlike then f is a starlike function and, in the third section, we prove that if $\alpha \in [0,1)$, $f \in A$ and $F(z) = zf'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right)$ is starlike of order α then f is a convex function of order α.

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1 Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and $H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}$.
We will also use the following notations:
$H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$ for $a \in \mathbb{C},\ n \in \mathbb{N}^*$,

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$A_n = \{f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots\}$, $n \in \mathbb{N}^*$, and for $n = 1$ we denote A_1 by A and this set is called **the class of analytic functions normalized in the origin**.

Let S be the class of holomorphic and univalent functions on the unit disc which are normalized with the conditions $f(0) = 0$, $f'(0) = 1$, so

$$S = \{f \in A : f \text{ is univalent in} \, U\}.$$

Definition 1.1. ([3]) *Let $f : U \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$. We say that f* **is starlike in U with respect to zero***(or, in brief, starlike)if the function f is univalent in U and $f(U)$ is a starlike domain with respect to zero, meaning that for each $z \in U$ the segment between the origin and $f(z)$ lies in $f(U)$.*

Theorem 1.1. ([3]) **(the theorem of analytical characterization of starlikeness)***Let $f \in H(U)$ be a function with $f(0) = 0$. Then f is starlike if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U.$$

Let S^* be the class of normalized starlike functions on the unit disc U, so

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U \right\}.$$

Definition 1.2. ([3]) *Let $f : U \to \mathbb{C}$ be a holomorphic function. We say that f is* **convex on** *U (or, in brief, convex) if f is univalent in U and $f(U)$ is a convex domain.*

Theorem 1.2. ([3]) **(the theorem of analytical characterization of convexity)***Let $f \in H(U)$. Then f is convex if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \; z \in U.$$

Let K be the class of normalized convex functions on the unit disc U and $K(\alpha)$ be the class of normalized convex functions of order α, i.e.

$$K(\alpha)=\left\{f\in A:\operatorname{Re}\frac{zf''(z)}{f'(z)}+1>\alpha,\, z\in U\right\}.$$

Lemma 1.1. ([2]) *Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ be a function that satisfies the condition*

$$\operatorname{Re}\psi(\rho i,\sigma,\mu+i\nu;z)\le 0\,,$$

when $\rho,\sigma,\mu,\nu\in\mathbb{R},\sigma\le-\frac{n}{2}(1+\rho^2),\sigma+\mu\le 0$, for $z\in U, n\ge 1$.

If $p\in H[1,n]$ and

$$\operatorname{Re}\psi(p(z),zp'(z),z^2p''(z);z)>0,\quad z\in U$$

then

$$\operatorname{Re}p(z)>0,\quad z\in U.$$

Definition 1.3 (1)**.** *Let α, $\beta\in\mathbb{R}$, $n\in\mathbb{N}^*$, $f\in A_n$ with*

$$\frac{f(z)f'(z)}{z}\ne 0,\, 1-\alpha+\alpha\frac{zf'(z)}{f(z)}\ne 0,\, z\in U.$$

We say that the function f is in the class $M^n_{\alpha,\beta}$ if the function $F:U\to\mathbb{C}$, defined as

$$F(z)=f(z)\left[\frac{zf'(z)}{f(z)}\right]^{\alpha(1-\beta)}\cdot\left[1-\alpha+\alpha\frac{zf'(z)}{f(z)}\right]^{\beta}$$

is a starlike function on the unit disc U.

Remark 1.1. ([1])

1. *If $\beta=0$ then $F(z)=f(z)\left[\frac{zf'(z)}{f(z)}\right]^{\alpha}$, $z\in\mathcal{U}$ and $M^1_{\alpha,0}=M_\alpha$ (the class of α-convex functions).*
2. *If $\beta=1$ then $F(z)=(1-\alpha)f(z)+\alpha zf'(z)$, $z\in\mathcal{U}$ and $M^1_{\alpha,1}=P_\alpha$ (the class of α-starlike functions defined by N.N. Pascu).*

3. *If* $\alpha = 0$ *then* $F(z) = f(z)$, $z \in \mathcal{U}$ *and* $M^1_{0,\beta} = S^*$ *(the class of starlike functions).*

4. *If* $\alpha = 1$ *then* $F(z) = zf'(z)$, $z \in \mathcal{U}$ *and* $M^1_{1,\beta} = K$ *(the class of convex functions).*

Remark 1.2. ([1]) *For all real numbers* α, β *satisfying the condition* $\alpha\beta(1-\alpha) \geq 0$ *we have*

$$M^n_{\alpha,\beta} \subset S^*.$$

2 A subclass of starlike functions

Definition 2.1. *Let* $\alpha \geq 0$ *and* $f \in A$ *such that*

$$\frac{f(z)f'(z)}{z} \neq 0,\ \alpha + \frac{zf'(z)}{f(z)} \neq 0, z \in U.$$

We say that the function f *is in the class* N_α *if the function* $F : U \to \mathbb{C}$ *given by*

$$F(z) = zf'(z)\left(\alpha + \frac{zf'(z)}{f(z)}\right)$$

is starlike in U*.*

Theorem 2.1. *For each real number* $\alpha \geq 0$ *we have*

$$N_\alpha \subset S^*.$$

Proof. Let $f \in N_\alpha$, $f \in A$ with $\dfrac{f(z)f'(z)}{z} \neq 0$ and $\alpha + \dfrac{zf'(z)}{f(z)} \neq 0, z \in U$. We denote $\dfrac{zf'(z)}{f(z)} = p(z), z \in U$. We have $p \in H[1,1]$ and $F(z) = zf'(z) \cdot (\alpha + p(z))$. (We make the remark that $F(0) = 0$ and $F'(0) = \alpha + 1 \neq 0$).

For $z \in U \backslash \{0\}$ we apply the logarithm to the equality $F(z) = zf'(z)(\alpha + p(z))$ and we obtain:

$$\log F(z) = \log z + \log f'(z) + \log(\alpha + p(z)).$$

If we derive the above equality(with respect to the independent variable z) and, afterwards, we multiply the result with z, we will obtain:

$$(1) \qquad \frac{zF'(z)}{F(z)} = 1 + \frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{\alpha + p(z)}.$$

But $\dfrac{zf'(z)}{f(z)} = p(z)$ implies that $zf'(z) = p(z)f(z)$ and deriving this equality we obtain

$$f'(z) + zf''(z) = p'(z)f(z) + p(z)f'(z) \mid : f'(z) \neq 0,$$

so

$$1 + \frac{zf''(z)}{f'(z)} = p'(z) \cdot z \cdot \frac{1}{p(z)} + p(z).$$

We will replace the last equality in (1) and we will have:

$$\frac{zF'(z)}{F(z)} = \frac{zp'(z)}{p(z)} + p(z) + \frac{zp'(z)}{\alpha + p(z)},\ z \in U \setminus \{0\}.$$

We make the remark that the above equality is also verified for $z = 0$.

We denote

$$(2) \qquad \psi(p(z), zp'(z); z) = p(z) + zp'(z)\left(\frac{1}{p(z)} + \frac{1}{\alpha + p(z)}\right)$$

From Definition 2.1 we know that the function F is starlike, so

$$(3) \qquad \operatorname{Re} \frac{zF'(z)}{F(z)} > 0, z \in U.$$

Using the notation (2) the condition (3) is equivalent with

$$\operatorname{Re} \psi(p(z), zp'(z); z) > 0, \quad z \in U.$$

Making the calculus we have:

$$\operatorname{Re} \psi(is, t) = \operatorname{Re} \left[is + t\left(\frac{1}{is} + \frac{1}{\alpha + is}\right)\right] =$$

$$= \operatorname{Re}\left[is + t\Big(\frac{-is}{s^2} + \frac{\alpha - is}{\alpha^2 + s^2}\Big)\right] = \frac{t\alpha}{\alpha^2 + s^2} \le \frac{-\alpha(1+s^2)}{2(\alpha^2+s^2)} \le 0,$$

for all $t \le -\frac{1}{2}(1+s^2)$ and $s \in \mathbb{R}$.

Consequently, we have obtained $\operatorname{Re}\psi(is,t) \le 0$ for all $s \in \mathbb{R}$ and $t \le -\frac{1+s^2}{2}$ and

$$\operatorname{Re}\psi(p(z), zp'(z); z) > 0,\ z \in U,\ p \in H[1,1],$$

from where it results that

$$\operatorname{Re} p(z) > 0,\ z \in U.$$

So, returning to the notation $\frac{zf'(z)}{f(z)} = p(z)$ we obtain

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0, z \in U,$$

and that means that $f \in S^*$. So, $N_\alpha \subset S^*$.

3 A subclass of convex functions of order α

Definition 3.1. *Let $\alpha \in [0,1)$ and $f \in A$ with*

$$\frac{f(z)f'(z)}{z} \ne 0, \quad 1 + \frac{zf''(z)}{f'(z)} \ne 0,\ z \in U.$$

We say that the function f is in the class $N(\alpha)$ if the function $F : U \to \mathbb{C}$ given by

$$F(z) = zf'(z)\Big(1 + \frac{zf''(z)}{f'(z)}\Big),$$

is starlike of order α.

Theorem 3.1. *For $\alpha \in [0,1)$ we have*

$$N(\alpha) \subset K(\alpha).$$

Proof. Let $f \in N(\alpha)$. We denote $1+\dfrac{zf''(z)}{f'(z)}=(1-\alpha)p(z)+\alpha p(z)$. We have $p \in H[1,1]$ and $F(z)=zf'(z)[(1-\alpha)p(z)+\alpha]$. Using the logarithmic derivation and the multiplying with z we obtain:

$$\frac{zF'(z)}{F(z)}=1+\frac{zf''(z)}{f'(z)}+\frac{(1-\alpha)p'(z)\cdot z}{(1-\alpha)p(z)+\alpha}=$$

$$=(1-\alpha)p(z)+\alpha+\frac{zp'(z)(1-\alpha)}{(1-\alpha)p(z)+\alpha}$$

which is equivalent with

$$\frac{zF'(z)}{F(z)}-\alpha=(1-\alpha)p(z)+\frac{(1-\alpha)zp'(z)}{(1-\alpha)p(z)+\alpha}. \tag{4}$$

We denote

$$\psi(p(z),zp'(z);z)=(1-\alpha)p(z)+\frac{zp'(z)(1-\alpha)}{(1-\alpha)p(z)+\alpha},\ z\in U. \tag{5}$$

We know that $f \in N(\alpha)$, so F is starlike of order α, and hence

$$\operatorname{Re}\frac{zF'(z)}{F(z)}>\alpha,\ z\in U. \tag{6}$$

Using (4) and the notation (5), the condition (6) is equivalent with

$$\operatorname{Re}\psi(p(z),zp'(z);z)>0,\ z\in U.$$

Making the calculus we have

$$\operatorname{Re}\psi(is,t)=\operatorname{Re}\left[(1-\alpha)is+\frac{t(1-\alpha)}{(1-\alpha)is+\alpha}\right]=$$

$$=\frac{\alpha(1-\alpha)t}{(1-\alpha)^2s^2+\alpha^2}\le-\frac{\alpha(1-\alpha)(1+s^2)}{2[(1-\alpha)^2s^2+\alpha^2]}\le 0$$

for $\alpha\in[0,1)$, $s\in\mathbb{R}$ and $t\le-\frac{1}{2}(1+s^2)$.

Consequently, we have obtained $\operatorname{Re}\psi(is,t)\le 0$ for all $s\in\mathbb{R}$ and $t\le -\dfrac{1+s^2}{2}$ and

$$\operatorname{Re}\psi(p(z),zp'(z);z)>0,\ z\in U,\ p\in H[1,1],$$

from where it results that

$$\operatorname{Re}p(z)>0, z\in U.$$

Returning to the notation $1+\dfrac{zf''(z)}{f'(z)}=(1-\alpha)p(z)+\alpha$ and using the inequality $\operatorname{Re}p(z)>0, z\in U$ we obtain $\operatorname{Re}\left(1+\dfrac{zf''(z)}{f'(z)}\right)=(1-\alpha)\operatorname{Re}p(z)+\alpha>\alpha$ for $\alpha\in[0,1)$, so $f\in K(\alpha)$.
Finally we have $N(\alpha)\subset K(\alpha)$.

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