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On some constants in approximation by Bernstein operators

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Abstract

We estimate the constants $\sup\limits_{x\in(0,1)} \sup\limits_{f\in C[0,1]\setminus\Pi_1} \frac{|B_n(f,x)-f(x)|}{\omega_2\left(f,\sqrt{\frac{x(1-x)}{n}}\right)}$ and $\inf\limits_{x\in(0,1)} \sup\limits_{f\in C[0,1]\setminus\Pi_1} \frac{|B_n(f,x)-f(x)|}{\omega_2\left(f,\sqrt{\frac{x(1-x)}{n}}\right)}$, where B_n is the Bernstein operator of degree n and ω_2 is the second order modulus of continuity.

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1 Introduction

Denote by $B[0,1]$, the space of bounded real functions on the interval $[0,1]$, with the sup-norm: $\|\cdot\|$ and by $C[0,1]$, the subspace of continuous functions.

The Bernstein operators $B_n : B[0,1] \to \mathbf{R}^{[0,1]}$, $n \in \mathbf{N}$ are given by:

$$B_n(f,x) = \sum_{j=0}^{n} p_{n,j}(x) \cdot f\left(\frac{j}{n}\right), \quad f \in B[0,1],\ x \in [0,1], \tag{1}$$

where

$$p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}. \tag{2}$$

Consider the monomial functions $e_j(t) = t^j,\ t \in [0,1],\ j = 0,1,2\ldots$. The set of linear functions is denoted by Π_1.

In this paper we are interested in estimating the degree of approximation by Bernstein operators in terms of the second order modulus and the argument $\sqrt{\frac{x(1-x)}{n}}$. The quantity $\sqrt{\frac{x(1-x)}{n}}$, $n \in \mathbf{N}$, $x \in [0,1]$ plays an important role in such estimates, since $B_n((e_1 - xe_0)^2, x) = \frac{x(1-x)}{n}$. Recall that the second order modulus of a function $f \in B[0,1]$ is defined for $h > 0$ by:

$$\omega_2(f,h) = \sup\{|f(x+\rho) - 2f(x) + f(x-\rho)|,\ x \pm \rho \in [0,1],\ 0 < \rho \leq h\}. \tag{3}$$

More precisely we are concerning with the evaluation of the constants:

$$C_n^{\sup} = \sup_{x\in(0,1)} \sup_{f\in C[0,1]\setminus\Pi_1} \frac{|B_n(f,x) - f(x)|}{\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right)}; \tag{4}$$

$$C_n^{\inf} = \inf_{x\in(0,1)} \sup_{f\in C[0,1]\setminus\Pi_1} \frac{|B_n(f,x) - f(x)|}{\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right)}. \tag{5}$$

In the definitions of these constants we can replace the space $C[0,1]$, by the space $B[0,1]$, since $\sup_{f\in C[0,1]\setminus\Pi_1} \frac{|B_n(f,x)-f(x)|}{\omega_2\left(f,\sqrt{\frac{x(1-x)}{n}}\right)} = \sup_{f\in B[0,1]\setminus\Pi_1} \frac{|B_n(f,x)-f(x)|}{\omega_2\left(f,\sqrt{\frac{x(1-x)}{n}}\right)}$.

In connection with these constants, mention the constant

$$\sup_{f\in C[0,1]\setminus\Pi_1} \frac{\|B_n(f) - f\|}{\omega_2\left(f, \frac{1}{\sqrt{n}}\right)} = 1, \tag{6}$$

proved in [5] and also the constant studied in [1].

2 The estimate of $C_n^{\sup}$

In order to derive an upper inequality for $C_n^{\sup}$ we use a general result for estimating the positive linear operators, [2], [6]. Here we give it only in a particular form as follows:

Theorem A *If $L : C[0,1] \to \mathbf{R}^{[0,1]}$ is a linear positive operator, satisfying the properties: $L(e_j) = e_j$, $j = 0, 1$, then for any $f \in C[0,1]$, $x \in [0,1]$ and $0 < h \leq \frac{1}{2}$, we have:*

$$|L(f,x) - f(x)| \leq \left(1 + \frac{1}{2h^2} \cdot L((e_1 - xe_0)^2, x)\right) \omega_2(f,h). \tag{7}$$

Lemma 1 *For any $n \in \mathbf{N}$ we have*

$$\sup_{x\in(0,1)} \sup_{f\in C[0,1]\setminus\Pi_1} \frac{|B_n(f,x) - f(x)|}{\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right)} \leq \frac{3}{2}. \tag{8}$$

Proof. We apply Theorem A to the operator $L = B_n$ and the argument $h = \sqrt{\frac{x(1-x)}{n}}$.

Remark 1 *In [3], see also [6], it is given, in the same conditions like in Theorem A, the following estimate:*

$$|L(f,x) - f(x)| \leq \left[1 + \frac{1}{2(1-b)^2} L\left(\left(\left|\frac{e_1 - xe_0}{h}\right|^p - b\right)^2, x\right)\right] \cdot \omega_2(f,h),$$

for $f \in B[0,1]$, $x \in [0,1]$, $0 < h \leq \frac{1}{2}$, $p \geq 1$, $b \in [0,1)$ and it was shown that in certain cases it leads to better estimates then applying (7). However it is not possible to derive from it a better estimate for Bernstein operators, using $\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right)$. From this estimate, for $p = 2$ and $b = 0$ and from the relation $B_n((e_1 - xe_0)^4, x) = \left(\frac{3}{n^2} - \frac{6}{n^3}\right)(x(1-x))^2 + \frac{1}{n^3} \cdot x(1-x)$ we can obtain, *immediately, only the inequality:* $|B_n(f,x) - f(x)| \le \frac{11}{8} \cdot \omega_2\left(f, \frac{\sqrt[4]{x(1-x)}}{\sqrt{n}}\right)$, $n \ge 2$. *This is the correct form of the misprinted formula* $|B_n(f,x) - f(x)| \le \frac{11}{8} \cdot \omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right)$, *appearing in* [6].

In order to obtain an inverse inequality we fix $n \in \mathbf{N}$ and take a variable number $p \in \mathbf{N}$, $p \ge 2$. Denote $m = np$. There is an unique number $0 < x_p < \frac{1}{2}$, such that $\sqrt{\frac{x_p(1-x_p)}{n}} = \frac{1}{m}$. We have $x_p < \frac{1}{m}$.

Consider the linear piecewise function $f_p \in C[0,1]$ with the knots: $0 < x_p < \frac{1}{m} < \frac{2}{m} < \ldots < 1$, which take in the knots the following values: $f_p\left(\frac{k}{m}\right) = \frac{k^2-2k}{2}$, $0 \le k \le m$, $f_p(x_p) = \frac{m}{2} \cdot x_p - 1$ and is linear on the intervals $[0, x_p]$, $\left[x_p, \frac{1}{m}\right]$, $\left[\frac{1}{m}, \frac{2}{m}\right], \ldots, \left[\frac{m-1}{m}, 1\right]$. Note that f_p is linear on the whole interval $\left[x_p, \frac{2}{m}\right]$. More explicitly we have the representation:

$$f_p(t) = \begin{cases} \left(\frac{m}{2} - \frac{1}{x_p}\right) t, & t \in [0, x_p], \\ \frac{m}{2} \cdot t - 1, & t \in \left[x_p, \frac{2}{m}\right], \\ \frac{2k-1}{2} \cdot mt - \frac{k^2+k}{2}, & t \in \left[\frac{k}{m}, \frac{k+1}{m}\right],\ 2 \le k \le m-1. \end{cases} \tag{9}$$

Lemma 2 *For all* $n, p \in \mathbf{N}$, $p \ge 2$ *we have*

$$\omega_2\left(f_p, \sqrt{\frac{x_p(1-x_p)}{n}}\right) = 1. \tag{10}$$

Proof. The relation is equivalent to $\omega_2\left(f_p, \frac{1}{m}\right) = 1$. Consider a number $0 < \rho \le \frac{1}{m}$ and consider three points $0 \le u < v < w \le 1$, such that $u = v - \rho$, $w = v + \rho$. Denote $\Delta_\rho^2 f_p(u) = f_p(w) - 2f_p(v) + f_p(u)$. We ignore the case when the three points u, v, w belong to a same interval ended by the knots $0 < x_p < \frac{2}{m} < \frac{3}{m} < \ldots < 1$, because, then $\Delta_\rho^2 f_p(u) = 0$. It remains the following seven cases:

Case 1: $u, v \in [0, x_p]$, $w \in \left[x_p, \frac{2}{m}\right]$. We have:

$$\Delta_\rho^2 f_p(u) = \left(\frac{m}{2} - \frac{1}{x_p}\right)(v-\rho) - 2\left(\frac{m}{2} - \frac{1}{x_p}\right)v + \frac{m}{2}\cdot(v+\rho) - 1 = \frac{w}{x_p} - 1.$$

Hence $\Delta_\rho^2 f_p(u) = \frac{2v-u}{x_p} - 1 \leq \frac{2v}{x_p} - 1 \leq 1$ and $\Delta_\rho^2 f_p(u) \geq 0$.

Case 2: $u \in [0, x_p]$, $v, w \in \left[x_p, \frac{2}{m}\right]$. We have:

$$\Delta_\rho^2 f_p(u) = \left(\frac{m}{2} - \frac{1}{x_p}\right)(v-\rho) - 2\left(\frac{m}{2}\cdot v - 1\right) + \frac{m}{2}\cdot(v+\rho) - 1 = -\frac{u}{x_p} + 1.$$

Hence $\Delta_\rho^2 f_p(u) \leq 1$ and $\Delta_\rho^2 f_p(u) \geq 0$.

Case 3: $u \in [0, x_p]$, $v \in \left[x_p, \frac{2}{m}\right]$, $w \in \left[\frac{2}{m}, \frac{3}{m}\right]$. We have:

$$\Delta_\rho^2 f_p(u) = \left(\frac{m}{2} - \frac{1}{x_p}\right)(v-\rho) - 2\left(\frac{m}{2}\cdot v - 1\right) + \frac{3m}{2}\cdot(v+\rho) - 3 = mw - \frac{u}{x_p} - 1.$$

Hence $\Delta_\rho^2 f_p(u) \leq m\left(u + \frac{2}{m}\right) - \frac{u}{x_p} - 1 = \left(m - \frac{1}{x_p}\right)u + 1 \leq 1$ and $\Delta_\rho^2 f_p(u) \geq m \cdot \frac{2}{m} - 1 - 1 = 0$.

Case 4: $u, v \in \left[x_p, \frac{2}{m}\right]$, $w \in \left[\frac{2}{m}, \frac{3}{m}\right]$. We have:

$$\Delta_\rho^2 f_p(u) = \frac{m}{2}(v-\rho) - 1 - 2\left(\frac{m}{2}\cdot v - 1\right) + \frac{3}{2}\cdot m(v+\rho) - 3 = mw - 2.$$

Hence $\Delta_\rho^2 f_p(u) \leq 1$ and $\Delta_\rho^2 f_p(u) \geq 0$.

Case 5: There is an integer $1 \leq k \leq n-2$, such that $u, v \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$, $w \in \left[\frac{k+1}{m}, \frac{k+2}{m}\right]$. We have:

$$\begin{aligned}\Delta_\rho^2 f_p(u) &= \frac{2k-1}{2}\cdot m(v-\rho) - \frac{k^2+k}{2} - 2\left(\frac{2k-1}{2}\cdot mv - \frac{k^2+k}{2}\right)\\ &\quad + \frac{2k+1}{2}\cdot m(v+\rho) - \frac{k^2+3k+2}{2}\\ &= mw - k - 1.\end{aligned}$$

Hence $\Delta_\rho^2 f_p(u) \leq 1$ and $\Delta_\rho^2 f_p(u) \geq 0$.

Case 6: There is an integer $1 \le k \le n-2$, such that $u \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$, $v, w \in \left[\frac{k+1}{m}, \frac{k+2}{m}\right]$. We have:

$$\begin{aligned}
\Delta_\rho^2 f_p(u) &= \frac{2k-1}{2} \cdot m(v-\rho) - \frac{k^2+k}{2} - 2\left(\frac{2k+1}{2} \cdot mv - \frac{k^2+3k+2}{2}\right) \\
&\quad + \frac{2k+1}{2} \cdot m(v+\rho) - \frac{k^2+3k+2}{2} \\
&= -mu + k + 1.
\end{aligned}$$

Hence $\Delta_\rho^2 f_p(u) \le 1$ and $\Delta_\rho^2 f_p(u) \ge 0$.

Case 7: There is an integer $1 \le k \le n-3$, such that $u \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$, $v \in \left[\frac{k+1}{m}, \frac{k+2}{m}\right]$, $w \in \left[\frac{k+2}{m}, \frac{k+3}{m}\right]$. We have:

$$\begin{aligned}
\Delta_\rho^2 f_p(u) &= \frac{2k-1}{2} \cdot m(v-\rho) - \frac{k^2+k}{2} - 2\left(\frac{2k+1}{2} \cdot mv - \frac{k^2+3k+2}{2}\right) \\
&\quad + \frac{2k+3}{2} \cdot m(v+\rho) - \frac{k^2+5k+6}{2} \\
&= 2m\rho - 1.
\end{aligned}$$

Hence $\Delta_\rho^2 f_p(u) \le 1$. Also, since in this case $\rho \ge \frac{1}{2m}$, it follows $\Delta_\rho^2 f_p(u) \ge 0$.

Since in all the cases we obtain $0 \le \Delta_\rho^2 f_p(u) \le 1$, relation (10) is proved.

Lemma 3 *For all $n, p \in \mathbf{N}$, $p \ge 2$ we have:*

$$B_n(f_p, x_p) - f_p(x_p) = \frac{3}{2} - \frac{3}{2} \cdot npx_p + \frac{1}{2}(npx_p)^2. \tag{11}$$

Proof. Consider the function $g_p(t) = \frac{1}{2}(mt)^2 - mt$, $t \in [0,1]$. Since f_p coincides with g_p on the knots $\frac{k}{n} = \frac{kp}{m}$, $0 \le k \le n$, we have $B_n(f_p) = B_n(g_p)$. We obtain

$$\begin{aligned}
B_n(f_p, x_p) - f_p(x_p) &= \frac{m^2}{2}\left(x_p^2 + \frac{x_p(1-x_p)}{n}\right) - mx_p - \frac{m}{2} \cdot x_p + 1 \\
&= \frac{3}{2} - \frac{3}{2} \cdot mx_p + \frac{1}{2}(mx_p)^2.
\end{aligned}$$

The main result is the following:

Theorem 1 *For any $n \in \mathbf{N}$ we have*

$$C_n^{\sup} = \frac{3}{2}. \tag{12}$$

Proof. Fix $n \in \mathbf{N}$. From the definition of x_p and from $m = np$ we obtain $npx_p = \frac{1}{p(1-x_p)}$. Since $x_p < \frac{1}{m} \leq \frac{1}{2}$, it follows $\lim\limits_{p\to\infty} npx_p = 0$. Then, from Lemma 2 and Lemma 3 we obtain

$$\lim_{p\to\infty} \frac{|B_n(f_p, x_p) - f_p(x_p)|}{\omega_2\left(f_p, \sqrt{\frac{x_p(1-x_p)}{n}}\right)} = \frac{3}{2}.$$

Since $f_p \in C[0,1]$ it follows

$$\sup_{x\in(0,1)} \sup_{f\in C[0,1]\setminus\Pi_1} \frac{|B_n(f,x) - f(x)|}{\omega_2\left(f, \sqrt{\frac{x(1-x)}{n}}\right)} \geq \frac{3}{2}.$$

By taking into account Lemma 1 the theorem is proved.

3 The estimate of $C_n^{\inf}$

First we mention two auxiliary results:

Theorem B([3]) *Let $F : B[0,1] \to \mathbf{R}$ be o functional with equidistant knots of the form $F(f) := \sum\limits_{k=0}^{n} f\left(\frac{k}{n}\right) \nu_k$, $f \in B[0,1]$, where $\nu_k \in \mathbf{R}$, $0 \leq k \leq n$. For any irrational number $x \in (0,1)$ and any $h > 0$ we have*

$$\sup_{f\in C[0,1]\setminus\Pi_1} \frac{|F(f) - f(x)|}{\omega_2(f,h)} \geq 1. \tag{13}$$

For any function $f : [0,1] \to \mathbf{R}$ and any points $a < b < c$ from $[0,1]$, denote:

$$\Delta(f; a, b, c) = \frac{b-a}{c-a} \cdot f(c) + \frac{c-b}{c-a} \cdot f(a) - f(b). \tag{14}$$

Theorem C ([2]) *For any $f \in B[0,1]$ and any points $a < b < c$ from the interval $[0,1]$, if we denote $h = \frac{c-a}{2}$ we have:*

$$|\Delta(f; a, b, c)| \le \omega_2(f, h). \tag{15}$$

The main result of this section is the following

Theorem 2 *For any $n \in \mathbf{N}$, we have*

$$C_n^{\inf} \ge 1. \tag{16}$$

and

$$\limsup_{n\to\infty} C_n^{\inf} \le \frac{3}{2} - \frac{1}{e} = 1,13\ldots. \tag{17}$$

Proof. Relation (16) follows from Theorem B. For proving relation (17) we consider $n \in \mathbf{N}$, $n \ge 4$ and define y_n to be the unique point $y_n \in \left(0, \frac{1}{2}\right)$, such that $\sqrt{\frac{y_n(1-y_n)}{n}} = \frac{1}{n}$. We obtain $y_n = \frac{1-\sqrt{1-\frac{4}{n}}}{2} = \frac{2}{n\left(1+\sqrt{1-\frac{4}{n}}\right)}$. Hence $\frac{1}{n} < y_n < \frac{2}{n}$ and $\lim\limits_{n\to\infty} n y_n = 1$.

Let an arbitrary function $f \in C[0,1]$. In order to estimate the fraction $\frac{|B_n(f,y_n)-f(y_n)|}{\omega_2\left(f,\frac{1}{n}\right)}$ it is sufficient to consider that $f(0) = 0 = f\left(\frac{1}{n}\right)$. Indeed, otherwise we can replace the function f by the function $g(t) = f(t) + n\left(f(0) - f\left(\frac{1}{n}\right)\right)t - f(0)$, $t \in [0,1]$, since $B_n(f) - f = B_n(g) - g$ and $\omega_2(f,h) = \omega_2(g,h)$, for any $0 \le h \le \frac{1}{2}$. Moreover we have $g(0) = 0 = g\left(\frac{1}{n}\right)$.

Also we can suppose that $B_n(f,y_n) - f(y_n) \geq 0$, since otherwise we can replace f by the function $g = -f$.

Let $a \in \mathbf{R}$ be such that $f\left(\frac{2}{n}\right) = a\omega_2\left(f,\frac{1}{n}\right)$.

The following relation can be proved easily by induction.

$$f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \leq (k-1+a)\omega_2\left(f,\frac{1}{n}\right),\ 1 \leq k \leq n-1. \tag{18}$$

Indeed, for $k=1$ we take into account that $f\left(\frac{1}{n}\right)=0$ and the definition of a. Then, if we suppose (18) true for $1 \leq k \leq n-2$, we have

$$\begin{aligned} f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right) &= f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \\ &\quad + \left(f\left(\frac{k+2}{n}\right) - 2f\left(\frac{k+1}{n}\right) + f\left(\frac{k}{n}\right)\right) \\ &\leq (k+a)\omega_2\left(f,\frac{1}{n}\right). \end{aligned}$$

Then, for $2 \leq k \leq n$ we obtain

$$\begin{aligned} f\left(\frac{k}{n}\right) &= f\left(\frac{1}{n}\right) + \sum_{j=1}^{k-1}\left(f\left(\frac{j+1}{n}\right) - f\left(\frac{j}{n}\right)\right) \\ &\leq \sum_{j=1}^{k-1}(j-1+a)\omega_2\left(f,\frac{1}{n}\right) \\ &= \left(\frac{k^2-k}{2} + (k-1)(a-1)\right)\omega_2\left(f,\frac{1}{n}\right). \end{aligned}$$

It follows

$$\begin{aligned} \frac{B_n(f,y_n)}{\omega_2\left(f,\frac{1}{n}\right)} &\leq \sum_{k=2}^{n}\left(\frac{k^2-k}{2} + (k-1)(a-1)\right)p_{n,k}(y_n) \\ &= B_n\left(\frac{n^2}{2}\cdot e_2 - \frac{n}{2}\cdot e_1, y_n\right) + (a-1)[B_n(ne_1 - e_0, y_n) + p_{n,0}(y_n)] \\ &= \frac{(ny_n)^2}{2} + \frac{1}{2} - \frac{ny_n}{2} + (a-1)(ny_n - 1 + p_{n,0}(y_n)). \end{aligned}$$

We consider now two cases.

Case 1: $a \geq 0$. From the relation

$$\Delta\left(f; 0, y_n, \frac{2}{n}\right) = \frac{ny_n}{2} \cdot f\left(\frac{2}{n}\right) + \left(1 - \frac{ny_n}{2}\right) f(0) - f(y_n)$$

and from Theorem C we obtain $f(y_n) \geq \left(\frac{ny_n}{2} \cdot a - 1\right) \omega_2\left(f, \frac{1}{n}\right)$. Consequently we obtain

$$\frac{B_n(f, y_n) - f(y_n)}{\omega_2\left(f, \frac{1}{n}\right)} \leq \frac{(ny_n)^2}{2} + \frac{3}{2} - ny_n + (a-1)\left(\frac{ny_n}{2} - 1 + p_{n,0}(y_n)\right).$$

Since $\lim\limits_{n\to\infty} ny_n = 1$ it follows $\lim\limits_{n\to\infty}(1-y_n)^n = \frac{1}{e}$. Hence $\lim\limits_{n\to\infty} \frac{ny_n}{2} - 1 + p_{n,0}(y_n) = -\frac{1}{2} + \frac{1}{e} < 0$. Then there is $n_0 \in \mathbf{N}$, sufficiently greater such that $\frac{ny_n}{2} - 1 + p_{n,0}(y_n) < 0$, for all $n \geq n_0$. Since $a \geq 0$ and $B_n(f, y_n) - f(y_n) \geq 0$, we obtain, for $n \geq n_0$:

$$\begin{aligned}\frac{|B_n(f, y_n) - f(y_n)|}{\omega_2\left(f, \frac{1}{n}\right)} &\leq \frac{(ny_n)^2}{2} + \frac{3}{2} - ny_n - \left(\frac{ny_n}{2} - 1 + p_{n,0}(y_n)\right) \\ &= \frac{(ny_n)^2}{2} + \frac{5}{2} - \frac{3}{2} \cdot ny_n - p_{n,0}(y_n).\end{aligned}$$

Case 2: $a \leq 0$. From the relation

$$\Delta\left(f; \frac{1}{n}, y_n, \frac{2}{n}\right) = (2 - ny_n) f\left(\frac{1}{n}\right) + (ny_n - 1) f\left(\frac{2}{n}\right) - f(y_n)$$

and from Theorem C we obtain: $f(y_n) \geq ((ny_n - 1)a - 1)\omega_2\left(f, \frac{1}{n}\right)$. Consequently we arrive to

$$\frac{B_n(f, y_n) - f(y_n)}{\omega_2\left(f, \frac{1}{n}\right)} \leq \frac{(ny_n)^2}{2} + \frac{5}{2} - \frac{3}{2} \cdot ny_n + (a-1) p_{n,0}(y_n).$$

Since $a \leq 0$ and $B_n(f, y_n) - f(y_n) \geq 0$ we obtain the same upper bound as in Case 1:

$$\frac{|B_n(f, y_n) - f(y_n)|}{\omega_2\left(f, \frac{1}{n}\right)} \leq \frac{(ny_n)^2}{2} + \frac{5}{2} - \frac{3}{2} \cdot ny_n - p_{n,0}(y_n).$$

Finally, since

$$\lim_{n\to\infty}\left(\frac{(ny_n)^2}{2}+\frac{5}{2}-\frac{3}{2}\cdot ny_n-p_{n,0}(y_n)\right)=\frac{3}{2}-\frac{1}{e},$$

we obtain relation (17).

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