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On X - Hadamard and $\mathcal{B}$- derivations[1]

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Abstract

Let F be an infinite dimensional complex Banach space endowed with a bounded shrinking basis X. We seek conditions to relate X- Hadamard derivations and $\mathcal{B}$-derivations supported on multiplier operators of F relative to X. It is seeing that in general the former class is larger than the first and some facts on basis problems are also considered.

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1 Introduction

Throughout this article by F we will denote a complex infinite dimensional Banach space endowed with a bounded shrinking basis $X = \{x_n\}_{n=1}^{\infty}$. Let $F\widehat{\otimes}F^*$ be the tensor product Banach space of F and F^*, i.e. the completion of the usual algebraic tensor product with respect to the following cross norm defined for $u \in F\widehat{\otimes}F^*$ as

$$||u||_n = \inf\left\{\sum_{j=1}^{n} ||x_j||||x_j^*|| : u = \sum_{j=1}^{n} x_j \otimes x_j^*\right\}.$$

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The space $F\widehat{\otimes}F^*$ is indeed a Banach algebra under the product so that

$$(x \otimes x^*)(y \otimes y^*) = \langle y, x^* \rangle (x \otimes x^*)$$

if $x, y \in F, x^*, y^* \in F^*$. Then $F\widehat{\otimes}F^*$ is isometric isomorphic to the Banach algebra $\mathcal{N}_{F^*}(F)$ of nuclear operators on F (cf. [5], Th. C.1.5, p. 256). This fact allows the transference of the investigation of properties and structure of bounded derivations to a more tractable frame which has essentially the same profile. For previous researches on this matter in a purely algebraic setting, in the frame of Hilbert spaces or on certain Banach algebras of operators the reader can see [1], [2], [3].

The class of bounded derivations on $F\widehat{\otimes}F^*$ denoted as $\mathcal{D}(F\widehat{\otimes}F^*)$ becomes a closed subspace of $\mathcal{B}(F\widehat{\otimes}F^*)$.

Example 1. *If $ad_v(u) = u \cdot v - v \cdot u$ for $u, v \in F\widehat{\otimes}F^*$ then $ad_v \in \mathcal{D}(F\widehat{\otimes}F^*)$. As usual, $\{ad_v\}_{v \in F\widehat{\otimes}F^*}$ is the set of inner derivations on $F\widehat{\otimes}F^*$.*

Example 2. *Let $\delta_F : \mathcal{B}(F) \to \mathcal{B}(F\widehat{\otimes}F^*), \delta_F(T) \triangleq \delta_T$ where δ_T is the unique linear bounded operator on $F\widehat{\otimes}F^*$ so that*

$$\delta_T(x \otimes x^*) = T(x) \otimes x^* - x \otimes T^*(x^*)$$

for all basic tensor $x \otimes x^* \in F\widehat{\otimes}F^*$. By the universal property on tensor products δ_F is well defined. Indeed, $\mathcal{R}(\delta_F) \subseteq \mathcal{D}(F\widehat{\otimes}F^*)$ and $\delta_F \in \mathcal{B}(\mathcal{B}(F), \mathcal{B}(F\widehat{\otimes}F^*))$.

Example 3. *$ad_{x \otimes x^*} = \delta_{x \odot x^*}$, where as usual $x \odot x^* \in \mathcal{B}(F)$ denotes the finite rank operator $(x \odot x^*)(y) = \langle y, x^* \rangle \cdot x$, with $x, y \in F$ and $x^* \in F^*$.*

Proposition 1. *(cf. [6], [7]) Let F be a Banach space, $\{x_n\}_{n=1}^\infty$ be a shrinking basis of F and let $\{x_n^*\}_{n=1}^\infty$ be its a.s.c.f.. The system of all basic tensor products $x_{\otimes x_m^*}$ is basis of $F \otimes F^*$, arranged into a single sequence as follows: If $m \in \mathbb{N}$ let $n \in \mathbb{N}$ so that $(n-1)^2 < m \leq n^2$ and then let's write $x_m = x_{\sigma_1(m)} \otimes x^*_{\sigma_2(m)}$, with*

$$\sigma(m) = \begin{cases} (m - (n-1)^2, n) & \text{if} \quad (n - 1^2) + 1 \leq m \leq (n-1)^2 + n, \\ (n, n^2 - m + 1) & \text{if} \quad (n-1)^2 + n \leq m \leq n^2. \end{cases}$$

Remark 1. *In particular, $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ becomes a bijective function. Since $F^*\widehat{\otimes}F \hookrightarrow (F\widehat{\otimes}F^*)^*$ we will also write $z_m^* = x_{\sigma_1(m)}^* \otimes x_{\sigma_2(m)}, \quad m \in \mathbb{N}$. Thus $\{z_m^*\}_{n=1}^\infty$ becomes the a.sc.f. of $\{z_m\}_{n=1}^\infty$.*

Theorem 1. *(cf. [4]) Let F be an infinite dimensional Banach space with a shrinking basis $\{x_n^*\}_{n=1}^\infty$. Given $\delta \in \mathcal{D}(F\widehat{\otimes}F^*)$ there are unique sequences $\{h_n\}_{n\in\mathbb{N}}$ and $\{g_u^v\}_{u,v\in\mathbb{N}}$ so that if $u, v \in \mathbb{N}$ then*

$$\delta(z_{\sigma^{-1}(u,v)}) = (h_u - h_v)z_{\sigma^{-1}(u,v)} + \sum_{n=1}^{\infty}(g_u^n \cdot z_{\sigma^{-1}(u,v)} - g_n^v \cdot z_{\sigma^{-1}(u,n)}).$$

Indeed, $h = h[\delta] = (\langle\delta(z_{n^2}), z_{n^2}^*\rangle)_{n\in\mathbb{N}}$ and $\eta = \eta[\delta] = (\eta_n^m)_{n,m=1}^\infty$, with

$$\eta_n^m = h_{n,1}^{\sigma^{-1}(m,1)} = h_{\sigma(n^2)}^{m^2} = \begin{cases} \langle\delta(z_{n^2}), z_{m^2}^*\rangle & \text{if } \ n \neq m \\ 0, & \text{if } \ n = m. \end{cases}$$

In the sequel we will say that they are the h and g sequences of δ.

Example 4. *Let $\{v_n\}_{n=1}^\infty \in \mathbb{C}^{\mathbb{N}}$ so that $v = \sum_{n=1}^{\infty} v_n \cdot z_n$ is a well defined element of $F\widehat{\otimes}F^*$. Then $h_{[ad_v]} = \{v_{n^2-n+1} - v_1\}_{n=1}^\infty, \eta_m[ad_v] = v_{m^2-n+1}$ if $1 \leq n < m$ and $\eta_n^m = v_{(n-1)^2+m}$ if $n > m$.*

Definition 1. *A derivation $\delta \in \mathcal{D}(F\widehat{\otimes}F^*)$ is said to be an X-Hadamard derivation if its g- sequence is null.*

We will denote the set of all those derivations as $\mathcal{D}_{\mathcal{X}}(F\widehat{\otimes}F^*)$. In [4] it is proved that the former is a complementary Banach subspace of $\mathcal{D}(F\widehat{\otimes}F^*)$.

Definition 2. *An operator $\delta \in \mathcal{D}(F\widehat{\otimes}F^*)$ will be called a $\mathcal{B}$-derivation if there exists $T \in \mathcal{B}(F)$ so that $\delta = \delta_T$ according to the notation of Example 2. We will denote the class of such derivations as $\mathcal{D}_{\mathcal{B}}(F\widehat{\otimes}F^*)$.*

Remark 2. *Any $\mathcal{B}$-derivations is infinitely supported because $\delta_T = \delta_{T+\lambda Id_F}$ if $T \in \mathcal{F}$ and $\lambda \in \mathbb{C}$. More precisely, $\ker(\delta_F = \mathbb{C} \cdot Id_F)$(see Lema 1 below).*

In Th. 2 will prove that any X-Hadamard derivation is a $\mathcal{B}$-derivation. In Proposition 2 and Proposition 3 we will analize necessary and sufficient conditions under which certain natural series of Hadamard derivations are realized as $\mathcal{B}$ derivations. It'll then be clear how h-sequences determine their structures since the corresponding supports become multiplier operators included by them.

2 X-Hadamard and $\mathcal{B}$-derivations

Lemma 1. $\ker(\delta_F) = \mathbb{C} \cdot Id_F$.

Proof. The inclusion $\supseteq$ is evident. Let $T \in \mathcal{B}(F)$ so that $\delta_T = 0$ and let $\lambda \in \sigma T$. If λ belongs to the compression spectrum of T let $x^* \in F^* - \{0\}$ so that $x^*\big|_{R(T-\lambda Id_F)} \equiv 0$. For all $x \in F$ we have

$$\langle x, T^*(x^*)\rangle = \langle T(x), x^*\rangle = \langle \lambda x, x^*\rangle = \langle x, \lambda x^*\rangle,$$

i.e. $(t^* - \lambda Id_{F^*})(x^*) = 0$. Moreover, since

$$(T(F) - \lambda x) \otimes x^* = x \otimes (T^*(x^*) - \lambda x^*) = 0,$$

the projective norm is a cross-norm and $x^* \neq 0$ then $T = \lambda Id_F$. If $\lambda \in \sigma_{ap}(T)$ we choose a sequence $\{y_n\}_{n=1}^{\infty}$ of unit vectors of F so that $T(y_n) - \lambda y_n \to 0$. If $y^* \in F^*$ then

$$\begin{aligned} 0 &= \lim_{n\to\infty} ||(T(y_n) - \lambda y_n) \otimes y^*||_\pi \\ &= \lim_{n\to\infty} ||y_n \otimes (T^*(y^*) - \lambda y^*)||_\pi \quad = \quad ||T^*(y^*) - \lambda y^*||. \end{aligned}$$

Reasoning as above we conclude that $T = \lambda Id_F$.

Lemma 2. *(i) If $r, s \in \mathbb{N}$ then*

$$h[\delta_{x_r \odot x_s^*}] = \begin{cases} \{0, -1, -1, \ldots\} & \text{if} \quad r = s = 1, \\ \{0, 0, ...\} & \text{if} \quad r \neq s, \\ e_r & \text{if} \quad r = s > 1. \end{cases} \tag{1}$$

(ii) If $r \neq s$ then $\eta[\delta_{x_r \odot x_s^*}] = e_s^r$ is the zero matrix elsewhere and has a one in the (s, r) entry. All derivations $\delta_{x_n \odot x_n^*}$ with $n \in \mathbb{N}$ are of Hadamard type.

Proof. (i) If $r, s, n \in \mathbb{N}$ we get

$$(2)\qquad \begin{aligned} \delta x_r \odot x_s^*(x_n \otimes x_m^*) = \; & (x_r \odot x_s^*)(x_n) \otimes x_m^* - x_n \otimes (x_r \odot x_s^*)^*(x_m^*) \\ & = [\langle x_n, x_s^* \rangle x_r] \otimes x_m^* - x_n \otimes [\langle x_r, x_m^* \rangle \cdot x_s^*] \\ & = \delta_s^n \cdot (x_r \otimes x_m^*) - \delta_r^m \cdot (x_n \otimes x_s^*). \end{aligned}$$

Letting $m = 1$ in (2) then

$$(3)\qquad \begin{aligned} \delta_{x_r \odot x_s^*}(x_n \otimes x_1^*) = \; & \sum_{p=1}^{\infty} h_{n,1}^p \cdot z_p \\ & = \delta_s^n \cdot (x_r \otimes x_1^*) - \delta_r^1 \cdot (x_n \otimes x_s^*) \\ & = \delta_s^n \cdot z_{r^2} - \delta_r^1 \cdot z_{\sigma^{-1}(n,s)}. \end{aligned}$$

If $r = s = 1$ by (3) is and the first assertion follows. If $r = s > 1$ by (3) is $\delta_{x_r \odot x_r^*}(x_n \otimes x_1^*) = \delta_r^n \cdot z_{r^2}$ and our thirth claim follows. Finally, if $r \neq s = n$ then (3) becomes

$$\delta_{x_r \odot x_s^*}(x_s \otimes x_1^*) = z_{r^2} - \delta_1^r \cdot z_{s^2 - s + 1}$$

and clearly $h_s[\delta_{x_r \odot x_s^*}] = 0$. If $s \notin \{r, n\}$ then

$$\delta_{x_r \odot x_s^*}(x_n \otimes x_1^*) = -\delta_1^r \cdot z_{\sigma^{-1}(n,s)}.$$

But $\sigma^{-1}(n, s) = n^2$ if and only if $s = 1$ and as $r \neq s$ then $h_n[\delta_{x_r \odot x_s^*}] = 0$.

(ii) If $r, s, n, m \in \mathbb{N}$ and $n \neq m$ then

$$(4)\qquad \begin{aligned} \eta_n^m[\delta_{x_r \odot x_s^*}] = \; & \langle \delta_X(x_r \odot x_s^*)(z_{n^2}), z_{m^2}^* \rangle \\ & = \langle \delta_X x_r \odot x_s^*)(x_n \otimes x_1^*), x_m^* \otimes x_1 \rangle \\ & = \langle \delta_s^n \cdot (x_r \otimes x_1^*) - (x_n) \otimes (x_r \otimes x_s^*)^*(x_1^*), x_m^* \otimes x_1 \rangle \\ & = \delta_s^n \cdot \delta_r{}^m. \end{aligned}$$

The conclusion is now clear.

Remark 3. *Given an elementary tensor $x \otimes x^* \in X \widehat{\otimes} X^*$ and $m \in \mathbb{N}$ we have*

$$\begin{aligned} \left(\sum_{n=1}^{m} \delta_{x_n \odot x_n^*} \right)(x \otimes x^*) = \; & \sum_{n=1}^{m} [\langle x, x_n^* \rangle (x_n \otimes x^*) - \langle x_n, x^* \rangle (x \otimes x_n^*)] \\ & = \left(\sum_{n=1}^{m} \langle x, x_{n^*} \rangle x_n \right) \otimes x^* - x \otimes \left(\sum_{n=1}^{m} \langle x_n, x^* \rangle x_n^* \right) \end{aligned}$$

and so $\lim_{m\to 0}\left(\sum_{n=1}^{m}\delta_{x_n\odot x_n^*}\right)(x\otimes x^*)\equiv 0$. *Since the basis* X *is assumed to be bounded then* $\rho=\inf_{n,p\in\mathbb{N}}||x_n||||x_p^*||$ *is positive (cf. [7], Corollary 3.1, p.20). Consequently, if* $n,m,p\in\mathbb{N}$ *and* $n\neq p$ *then*

$$(5)\qquad \begin{aligned}\left|\left|\delta_{x_n\odot x_m^*}\right|\right| \geq\ & \left|\left|\delta_{x_n\odot x_m^*}\left(\frac{x_m}{||x_m||}\otimes\frac{x_p^*}{||x_p^*||}\right)\right|\right|_\pi \\ =\ & \frac{||x_n||}{||x_m||}\geq \inf_{n\in\mathbb{N}}||x_n||/\sup_{m\in\mathbb{N}}||x_m||>0,\end{aligned}$$

i.e. the series $\sum_{n=1}^{\infty}\delta_{x_n\odot x_n^*}$ *is not convergent.*

Remark 4. *The set* $\{\delta_{x_n\odot x_n^*}\}_{n=1}^{\infty}$ *is linearly dependent. For, let* $\{c_n\}_{n=1}^{\infty}$ *be a sequence of scalars so that* $\sum_{n=1}^{\infty}c_n\cdot\delta_{x_n\odot x_n^*}\equiv 0$. *In particular, by (4) is* $\{c_n\}_{n=1}^{\infty}\in c_0$. *If* r,s *are two positive integers then*

$$\left[\sum_{n=1}^{\infty}c_n\cdot\delta_{x_n\odot x_n^*}\right](x_r\odot x_s^*)=(c_r-c_s)(x_r\odot x_s^*)=0,$$

i.e. $c_r=c_s$. *Hence* $\{c_n\}_{n=1}^{\infty}$ *becomes the constant zero sequence and the assertion follows.*

Theorem 2. *Every* X*-Hadamard derivation is a* $\mathcal{B}$*-derivation.*

Proof. If $\delta\in\mathcal{D}(F\widehat{\otimes}F^*)$ and $x\in F$ the series $\sum_{n=1}^{\infty}\langle x,x_n^*\rangle\cdot h_n[\delta]\cdot x_n$ converges. For, if $p,q\in\mathbb{N}$ then

$$\begin{aligned}\left|\left|\sum_{n=p}^{p+q}\langle x,x_n^*\rangle\cdot h_n[\delta]\cdot x_n\right|\right| =\ & \left|\left|\delta\left[\left(\sum_{n=p}^{p+q}\langle x,x_n^*\rangle\cdot x_n\right)\otimes\frac{x_1^*}{||x_1^*||}\right]\right|\right|_\pi \\ \leq\ & ||\delta||\left|\left|\sum_{n=p}^{p+q}\langle x,x_n^*\rangle\cdot x_n\right|\right|,\end{aligned}$$

i.e. the sequence of corresponding partial sums is a Cauchy sequence. So, it is defined a linear operator $M_{h[\delta]}:x\to\sum_{n=1}^{\infty}\langle x,x_n^*\rangle\cdot h_n[\delta]\cdot x_n$ that is bounded as a consequence of the Banach-Steinhauss theorem. Hence $h[\delta] \in M(f^*, X)$, i.e. $h[\delta]$ is a multiplier of F relative to the basis X. Analogously, if $x^* \in F^*$ the series $\sum_{n=1}^{\infty}\langle x_m, x^*\rangle \cdot h_m[\delta] \cdot x_m^*$ also converges because if $p, q \in \mathbb{N}$ we get

$$\begin{aligned}\Big\|\sum_{m=p}^{p+q}\langle x_m, x^*\rangle \cdot h_m[\delta] \cdot x_m^*\Big\| = & \Big\|\frac{x_1}{||x_1||} \otimes \sum_{m=p}^{p+q}\langle x_m, x^*\rangle \cdot h_m[\delta] \cdot x_m^*\Big\|_\pi \\ &= \Big\|\delta\left(\frac{x_1}{||x_1||} \otimes \sum_{m=p}^{p+q}\langle x_m, x^*\rangle \cdot x_m^*\right)\Big\|_\pi \\ &\leq ||\delta||\Big\|\sum_{n=p}^{p+q}\langle x_m, x^*\rangle \cdot x_m^*\Big\|.\end{aligned}$$

It is immediate that $M^*_{h[\delta]}(x^*) = \sum_{n=1}^{\infty}\langle x_m, x^*\rangle \cdot h_m[\delta] \cdot x_m^*$ for all $x^* \in F^*$ and $h[\delta]$ is also realizes as a multiplier of F^* relative to the basis X^*. Now, if $x \otimes x*$ is a fixed basic tensor in $F\widehat{\otimes}F^*$ we can write

$$\begin{aligned}\delta(x \otimes x^*) = & \textstyle\sum_{n=1}^{\infty}\langle x, x_n^*\rangle \sum_{n=1}^{\infty}\langle x_m, x^*\rangle(h_n[\delta] - h_m[\delta])(x_n \otimes x_{m^*}) \\ &= M_{h[\delta]}(x) \otimes x^* - x \otimes M^*_{h[\delta]}(x^*)\end{aligned}$$

and definitely $\delta = \delta_{M_{h[\delta]}}$.

Proposition 2. *Let* $\{\zeta\}_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}}$ *so that* $\delta = \sum_{n=1}^{\infty}\zeta_n \cdot \delta_{x_n \odot x_n^*}$ *is a well defined Hadamard derivation.*

(i) $h[\delta] \in c$, $\{\zeta\}_{n=1}^{\infty} \in c_0$ *and* $\zeta_m = h_m[\delta] - \lim_{n\to\infty} h_n[\delta]$ *if* $m \in \mathbb{N}$.

(ii) $\delta = \delta_S$ *where* $S \in \mathcal{B}(F)$ *is defined for* $x \in F$ *as*

$$S(x) = \sum_{n=1}^{\infty} h_n[\delta] \cdot \langle x, x_n^*\rangle \cdot x_n - x \cdot \lim_{n\to\infty} h_n[\delta].$$

Proof.

(i) If $m \in \mathbb{N}$ it is readily seeing that $\delta(x_m \otimes x_1^*) = (\zeta_m - \zeta_1) \cdot z_{m^2}$. Thus $h_1[\delta] = 0$ and $h_m[\delta] = \zeta_m - \zeta_1$ if $m > 1$. By (4) we have that $\{\zeta\}_{n=1}^{\infty} \in c_0$ and we get (ii).

(ii) Let $S_n = \sum_{k=1}^{n} (h_k[\delta] - \lim_{n\to\infty} x_k \odot x_k^*), n \in \mathbb{N}$. By the uniform boundedness principle and (ii) the sequence $\{S_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{B}(F)$. Whence, since $S_n(x) \to S(x)$ if $x \in F$ then $S \in \mathcal{B}(F)$. Indeed, if $n, m \in \mathbb{N}$ we get
$\delta_S(x_n \oplus x_m^*)=$
$= ((h_n[\delta] - \lim_{k\to\infty} h_k[\delta])x_n) \otimes x_m^* - x_n \otimes ((h_m[\delta] - \lim_{k\to\infty} h_k[\delta])x_m^*)$
$= (h_n[\delta] - h_m[\delta]) \cdot (x_n \otimes x_m^*)$
$= \delta(x_n \otimes x_m^*),$
i.e. $\delta = \delta_S$

Proposition 3. *Let $\{h_n\}_{n=1}^\infty \in M(F, \{x_n\}_{n=1}^\infty) \cap M(F^*, \{x_n^*\}_{n=1}^\infty) \cap c$ such that $h_1 = 0$. On writing $h_0 \triangleq \lim_{n\to\infty} h_n$ the series $\sum_{n=1}^\infty (h_n - h_0) \cdot \delta_{x_n \odot x_n^*}$ converges to a Hadamard derivation δ on $\hat{\otimes} F^*$ so that $h[\delta] = \{h_n\}_{n=1}^\infty$.*

Proof. If $S = \sum_{k=1}^{\infty} (h_k - h_0) \cdot x_k \odot x_k^*$ then $S \in \mathcal{B}(F)$ and

$$\|S\| \leq \|\{h_n\}_{n=1}^\infty\|_{M(F,\{x_n\}_{n=1}^\infty)} + |h_0|.$$

Let $S_n = \sum_{k=1}^{n} (h_k - h_0) \cdot x_k \oplus x_k 6*, n\mathbb{N}$. Given $x \in F$ the sequence $\{S_n(x)\}_{n=1}^\infty$ converges because $\{h_n\}_{n=1}^\infty$ is a multiplier of F and $(\{x_n\}_{n=1}^\infty)$ is an $F-$complete biorthogonal system. Therefore $\{\|S_n\|\}_{n=1}^\infty$ becomes bounded. Now if we fix an elementary tensor $y \otimes y^* \in F\hat{\otimes}F^*$ and $n \in \mathbb{N}$ then

$$\begin{aligned}
\|(\delta_{S_n} - \delta_S)(y \otimes y^*)\|_\pi &= \left\| \sum_{k>n} (h_k - h_0) \cdot \langle y, x_k^* \rangle \cdot x_k \otimes y^* \right.\\
&\quad \left. - y \otimes \sum_{k>n} (h_k - h_0) \cdot \langle x_k, y^* \rangle \cdot x_k^* \right\|_\pi \\
&\leq \left\| \sum_{k>n} (h_k - h_0) \cdot \langle y, x_k^* \rangle \cdot x_k \right\| \|y^*\| \\
&\quad + \|y\| \left\| \sum_{k>n} (h_k - h_0) \cdot \langle x_k, y^* \rangle \cdot x_k^* \right\|.
\end{aligned} \tag{6}$$

Since $\{h_n\}_{n=1}\infty \in M(F, \{x_n\}_{n=1}^{\infty}) \cap M(F^*, \{x_n^*\}_{n=1}^{\infty})$ and $\{x_n\}_{n=1}^{\infty}$ is a shrinking basis by (5) we see that $\lim_{n\to\infty}(\delta_{S_n} - \delta_S)(y \otimes y^*) = 0$. Indeed, as $F \otimes F^*$ is dense in $F \hat{\times} F^*$, $\{S_n\}_{n=1}^{\infty}$ is bounded and $\|\delta_T\| \leq 2\|T\|$ for all $T \in \mathcal{B}(F)$ then $\delta_S = \sum_{n=1}^{\infty}(h_n - h_0) \cdot \delta_{x_n \odot x_n^*}$.

Problem 1 *Giving $T \in \mathcal{B}(F)$ then $\eta[T] = \{\langle T(x_n), x_m^* \rangle\}_{n,m=1}^{\infty}$. So it is obvious that $\mathcal{D}_X(F\hat{\otimes}F^*)$. It would be desirable to decide if $\mathcal{D}_{\mathcal{B}}(F\hat{\otimes}F^*)$ is a Banach space.*

Remark 5. *Is $\{\delta_{x_n \otimes x_n^*}\}_{n=1}^{*}$ a basis of $\mathcal{D}_X(F\hat{\otimes}F^*)$?- In general this is not the case. For instance, let $F = l^p(\mathbb{N})$ with $1 < p < \infty$ and let $X = \{e_n\}_{n=1}^{\infty}$, where $e_n = \{\delta_{n,m}\}_{m=1}^{\infty}$ and $\delta_{n,m}$ the current Kronecker symbol if $n, m \in \mathbb{N}$. Then X is not only a shrinking basis, it is further an unconditional basis of F. Consequently, if $T(x) = \sum_{n=1}^{\infty}\langle x, e_{2n}^* \rangle$ for $x \in F$ then $T \in \mathcal{B}(F)$. It is readily seeing that δ_T is an X-Hadamard derivation. Since $h[\delta_T] = \{0, 1, 0, 1, ...\}$ by Prop. 2 $\{\delta_{e_n \odot e_n^*}\}_{n=1}^{\infty}$ can not be a basis of $\mathcal{D}_X(F\hat{\otimes}F^*)$.*

Problem 2 *Is $\{\delta_{x_n \odot x_n^*}\}_{n=1}^{\infty}$ a sequence basis?- Can be be constructed a basis of $\mathcal{D}_X((F\hat{\otimes}F^*)$?-*

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