

Common fixed points of Ciric quasi-contractive operators through Ishikawa iteration process with errors ¹

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Abstract

We establish a general theorem to approximate common fixed points of Ciric quasi-contractive operators on a normed space through the modified Ishikawa iteration process with errors in the sense of Xu [21]. Our result generalizes and improves upon, among others, the corresponding results of [2, 15].

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1 Introduction and Preliminaries

Let C be a nonempty convex subset of a normed space E , $T : C \rightarrow C$ be a mapping and $F(T)$ be the set of fixed points.

Let $\{b_n\}$ and $\{b'_n\}$ be two sequences in $[0, 1]$.

The Mann iteration process is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ (see [12]):

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$$(1) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)x_n + b_nTx_n, \quad n \geq 0. \end{cases}$$

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$(2) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)x_n + b_nTy_n, \\ y_n = (1 - b'_n)x_n + b'_nTx_n, \quad n \geq 0 \end{cases}$$

is known as the Ishikawa iteration process [7].

Liu [11] introduced the concept of Ishikawa iteration process with errors by the sequence $\{x_n\}_{n=0}^{\infty}$ defined as follows:

$$(3) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)y_n + b_nTy_n + u_n, \\ y_n = (1 - b'_n)x_n + b'_nTx_n + v_n, \quad n \geq 0 \end{cases}$$

where $\{b_n\}$ and $\{b'_n\}$ are sequences in $[0, 1]$ and $\{u_n\}$ and $\{v_n\}$ satisfy

$\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$. This surely contains both (1) and (2). Also

this contains the Mann process with error terms

$$(4) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)x_n + b_nTx_n + u_n, \quad n \geq 0. \end{cases}$$

In 1998, Xu [21] introduced more satisfactory error terms in the sequence defined by:

$$(5) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = a_nx_n + b_nTy_n + c_nu_n, \\ y_n = a'_nx_n + b'_nTx_n + c'_nv_n, \quad n \geq 0 \end{cases}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C . Clearly, this iteration process contains the processes (1), (2) as its special cases. Also it contains the Mann process with error terms:

$$(6) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 0. \end{cases}$$

For two self mappings S and T of C , the Ishikawa iteration processes have been generalized by Das and Debata [6] as follows

$$(7) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n) x_n + b_n S y_n, \\ y_n = (1 - b'_n) x_n + b'_n T x_n, \quad n \geq 0. \end{cases}$$

They used this iteration process to find the common fixed points of quasi-nonexpansive mappings in a uniformly convex Banach space. Takahashi and Tamura [19] studied it for the case of two nonexpansive mappings under different conditions in a strictly convex Banach space.

Recently, Agarwal et al [1] studied the iteration process for two nonexpansive mappings using errors in the sense of Xu [21]:

$$(8) \quad \begin{cases} x_0 \in C, \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \quad n \geq 0, \end{cases}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C .

Clearly, this iteration process contains all the processes (1-7) as its special cases.

We recall the following definitions in a metric space (X, d) . A mapping $T : X \rightarrow X$ is called an a -contraction if

$$(9) \quad d(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X,$$

where $a \in (0, 1)$.

The map T is called Kannan mapping [8] if there exists $b \in (0, \frac{1}{2})$ such that

$$(10) \quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.$$

A similar definition is due to Chatterjea [3]: there exists a $c \in (0, \frac{1}{2})$ such that

$$(11) \quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X.$$

Combining these three definitions, Zamfirescu [22] proved the following important result.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ a mapping for which there exists the real numbers a, b and c satisfying $a \in (0, 1)$, $b, c \in (0, \frac{1}{2})$ such that for any pair $x, y \in X$, at least one of the following conditions holds:*

$$(z_1) \quad d(Tx, Ty) \leq ad(x, y),$$

$$(z_2) \quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)],$$

$$(z_3) \quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)].$$

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 2, \dots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

Remark 1.2. *The conditions $(z_1) - (z_3)$ can be written in the following equivalent form*

$$d(Tx, Ty) \leq h \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\},$$

$\forall x, y \in X; 0 < h < 1$. Thus, a class of mappings satisfying the contractive conditions $(z_1) - (z_3)$ is a subclass of mappings satisfying the following condition

$$(C) \quad d(Tx, Ty) \leq h \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\},$$

$$0 < h < 1.$$

The class of mappings satisfying (C) is introduced and investigated by Ciric [4] in 1971.

Remark 1.3. A mapping satisfying (C) is commonly called Ciric generalized contraction.

In 2004, Berinde [2] introduced a new class of operators on an arbitrary Banach space E satisfying

$$(12) \quad \|Tx - Ty\| \leq h \|x - y\| + 2h \|Tx - x\|$$

for any $x, y \in E, 0 \leq h < 1$.

It may be noted that (12) is equivalent to

$$(13) \quad \|Tx - Ty\| \leq h \|x - y\| + L \min\{\|Tx - x\|, \|Ty - y\|\},$$

for any $x, y \in E, 0 \leq h < 1$ and $L \geq 0$.

He proved that this class is wider than the class of Zamfiresu operators and used the Ishikawa iteration process (2) to approximate fixed points of this class of operators in an arbitrary Banach space given in the form of following theorem:

Theorem 1.4. Let C be a nonempty closed convex subset of an arbitrary Banach space E and $T : C \rightarrow C$ be an operator satisfying (12). Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process (2) and $x_0 \in C$, where $\{b_n\}$ and $\{b'_n\}$ are sequences of positive numbers in $[0, 1]$ with $\{b_n\}$ satisfying $\sum_{n=0}^{\infty} b_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T .

In this paper, a convergence theorem of Rhoades [16] regarding the approximation of fixed points of some quasi contractive operators in uniformly convex Banach spaces using the Mann iteration process, is extended to the approximation of common fixed points of some Ciric quasi-contractive operators in normed spaces using the iteration process (8).

2 Main Results

In this paper we shall consider a class of mappings satisfying the following condition

$$d(Tx, Ty) \leq h \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, d(x, Ty), d(y, Tx) \right\}, 0 < h < 1.$$

This class of mappings is a subclass of mappings satisfying the following condition

$$(QC) \quad d(Tx, Ty) \leq h \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

$0 < h < 1$. The class of mappings satisfying (QC) is introduced and investigated by Ciric [5] in 1974 and a mapping satisfying is commonly called Ciric quasi contraction.

The following lemma is proved in [20].

Lemma 2.1. *If there exists a positive integer N such that for all $n \geq N$, $n \in \mathbb{N}$,*

$$\rho_{n+1} \leq (1 - \alpha_n)\rho_n + b_n,$$

then

$$\lim_{n \rightarrow \infty} \rho_n = 0,$$

where $\alpha_n \in [0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $b_n = o(\alpha_n)$.

Following [2, 15], we obtain such a result without employing any fixed point theorem.

Theorem 2.2. Let C be a nonempty closed convex subset of a normed space E . Let $S, T : C \rightarrow C$ be two operators satisfying the condition

$$\|Sx - Ty\| \leq h \max\{\|x - y\|, \frac{\|x - Sx\| + \|y - Ty\|}{2}, \|x - Ty\|, \|y - Sx\|\}, \quad (CR)$$

$\forall x, y \in C; 0 < h < 1$. Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process (8). If $F = F(S) \cap F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} b_n = \infty$, $c_n = o(b_n)$ and $\lim_{n \rightarrow \infty} c'_n = 0$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of S and T .

Proof. Since S and T are Ciric operators, so by [4], we know that S and T have a unique fixed point in C , say w . Consider $x, y \in C$. If from (CR),

$$\begin{aligned} \|Sx - Ty\| &\leq \frac{h}{2} [\|x - Sx\| + \|y - Ty\|] \\ &\leq \frac{h}{2} [\|x - Sx\| + \|y - x\| + \|x - Sx\| + \|Sx - Ty\|], \end{aligned}$$

then

$$\left(1 - \frac{h}{2}\right) \|Sx - Ty\| \leq \frac{h}{2} \|x - y\| + h \|x - Sx\|,$$

which yields (using the fact that $0 < h < 1$)

$$(14) \quad \|Sx - Ty\| \leq \frac{\frac{h}{2}}{1 - \frac{h}{2}} \|x - y\| + \frac{h}{1 - \frac{h}{2}} \|x - Sx\|.$$

If from (CR),

$$\|Sx - Ty\| \leq h \|x - Ty\|,$$

then

$$(15) \quad \|Sx - Ty\| \leq \frac{h}{1 - h} \|x - Sx\|.$$

Also, if from (CR),

$$\|Sx - Ty\| \leq h \|y - Sx\|,$$

then we have

$$(16) \quad \begin{aligned} \|Sx - Ty\| &\leq h \|y - Sx\| \\ &\leq h \|x - y\| + h \|x - Sx\|. \end{aligned}$$

Denote

$$\begin{aligned} h &= \max \left\{ h, \frac{\frac{h}{2}}{1 - \frac{h}{2}} \right\} = h, \\ L &= \max \left\{ h, \frac{h}{1 - \frac{h}{2}}, \frac{h}{1 - h} \right\} = \frac{h}{1 - h}. \end{aligned}$$

Thus, in all cases,

$$(17) \quad \begin{aligned} \|Sx - Ty\| &\leq h \|x - y\| + L \|x - Sx\| \\ &= h \|x - y\| + \frac{h}{1 - h} \|x - Sx\|, \end{aligned}$$

holds for all $x, y \in C$.

Also from (CR) with $y = w = Tw$, we have

$$\begin{aligned} \|Tx - w\| &\leq h \max \left\{ \|x - w\|, \frac{\|x - Sx\|}{2}, \|x - w\|, \|w - Sx\| \right\} \\ &= h \max \left\{ \|x - w\|, \frac{\|x - Sx\|}{2}, \|w - Sx\| \right\} \\ &\leq h \max \left\{ \|x - w\|, \frac{\|x - w\| + \|w - Sx\|}{2}, \|w - Sx\| \right\} \\ &\leq h \max \{ \|x - w\|, \|w - Sx\| \}. \end{aligned}$$

If we suppose that $\max\{\|x - w\|, \|w - Sx\|\} = \|Sx - w\|$, then we have $\|Sx - w\| \leq h \|w - Sx\|$, which is impossible for $Sx \neq w$. Thus

$$(18) \quad \|Sx - w\| \leq h \|x - w\|.$$

Assume that $F \neq \phi$ and $w \in F$, then

$$M = \max \left\{ \sup_{n \geq 0} \{ \|u_n - w\| \}, \sup_{n \geq 0} \{ \|v_n - w\| \} \right\}.$$

Using (8), we have

$$\begin{aligned}
 \|x_{n+1} - w\| &= \|a_n x_n + b_n S y_n + c_n u_n - (a_n + b_n + c_n)w\| \\
 &= \|a_n(x_n - w) + b_n(S y_n - w) + c_n(u_n - w)\| \\
 &\leq a_n \|x_n - w\| + b_n \|S y_n - w\| + c_n \|u_n - w\| \\
 (19) \qquad &\leq (1 - b_n) \|x_n - w\| + b_n \|S y_n - w\| + M c_n.
 \end{aligned}$$

Now for $x = y_n$, (18) gives

$$(20) \qquad \|S y_n - w\| \leq h \|y_n - w\|.$$

In a similar fashion, we can get

$$\begin{aligned}
 \|y_n - w\| &= \|a'_n x_n + b'_n T x_n + c'_n v_n - (a'_n + b'_n + c'_n)w\| \\
 &= \|a'_n(x_n - w) + b'_n(T x_n - w) + c'_n(v_n - w)\| \\
 &\leq a'_n \|x_n - w\| + b'_n \|T x_n - w\| + c'_n \|v_n - w\| \\
 (21) \qquad &\leq (1 - b'_n) \|x_n - w\| + b'_n \|T x_n - w\| + M c'_n.
 \end{aligned}$$

Again by (16), if $x = w$ and $y = x_n$, we get

$$(22) \qquad \|T x_n - w\| \leq h \|x_n - w\|.$$

From (19-22), we obtain

$$\begin{aligned}
 \|x_{n+1} - w\| &\leq (1 - b_n) \|x_n - w\| + b_n h [(1 - b'_n) \|x_n - w\| \\
 &\quad + b'_n h \|x_n - w\| + M c'_n] + M c_n \\
 &= [1 - (1 - h)(1 + h b'_n) b_n] \|x_n - w\| + h M b_n c'_n + M c_n
 \end{aligned}$$

which, by the inequality

$$1 - (1 - h^2) b_n \leq 1 - (1 - h)(1 + h b'_n) b_n \leq 1 - (1 - h) b_n,$$

yields

$$\|x_{n+1} - w\| \leq [1 - (1 - h) b_n] \|x_n - w\| + h M b_n c'_n + M c_n.$$

By Lemma 2.1, with $\sum_{n=1}^{\infty} b_n = \infty$, $c_n = o(b_n)$ and $\lim_{n \rightarrow \infty} c'_n = 0$, we get that $\lim_{n \rightarrow \infty} \|x_n - w\| = 0$. Consequently $x_n \rightarrow w \in F$ and this completes the proof.

Remark 2.3. For $S = T$, (17) reduces to

$$(23) \quad \|Tx - Ty\| \leq h \|x - y\| + \frac{h}{1-h} \|x - Tx\|,$$

holds for all $x, y \in C$.

Corollary 2.4. Let C be a nonempty closed convex subset of a normed space E . Let $S, T : C \rightarrow C$ be two operators satisfying (CR). Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process (7). If $F = F(S) \cap F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a common fixed point of S and T .

Corollary 2.5. Let C be a nonempty closed convex subset of a normed space E . Let $T : C \rightarrow C$ be an operator satisfying (23). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process (5). If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$, $c_n = o(b_n)$ and $\lim_{n \rightarrow \infty} c'_n = 0$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .

Corollary 2.6. Let C be a nonempty closed convex subset of a normed space E . Let $T : C \rightarrow C$ be an operator satisfying (23). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process (6). If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$ and $c_n = o(b_n)$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .

Corollary 2.7. Let C be a nonempty closed convex subset of a normed space E . Let $T : C \rightarrow C$ be an operator satisfying (23). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process (2). If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T .

Corollary 2.8. *Let C be a nonempty closed convex subset of a normed space E . Let $T : C \rightarrow C$ be an operator satisfying (23). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process (1). If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .*

Remark 2.9. *1. The Chatterjea's and the Kannan's contractive conditions (11) and (10) are both included in the class of Zamfirescu operators and so their convergence theorems for the Ishikawa iteration process are obtained in Corollary 2.4.*

2. Theorem 4 of Rhoades [16] in the context of Mann iteration on a uniformly convex Banach space has been extended in Corollary 2.5.

3. In Corollary 2.5, Theorem 8 of Rhoades [17] is generalized to the setting of normed spaces.

4. Our result also generalizes Theorem 5 of Osilike [13] and Theorem 2 of Osilike [14].

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