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Multivalued Sakaguchi functions

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Abstract

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form $f(z) = z + a_2z^2 + \cdots$ which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. In 1959 [5], K. Sakaguchi has considered the subclass of $\mathcal{A}$ consisting of those $f(z)$ which satisfy $\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0$, where $z \in \mathbb{U}$. We call such a functions "Sakaguchi Functions". Various authors have investigated this class ([4], [5], [6]). Now we consider the class of functions of the form $f(z) = z^\alpha(z + a_2z^2 + \cdots + a_nz^n + \cdots)$ $(0 < \alpha < 1)$, that are analytic and multivalued in $\mathbb{U}$, we denote the class of these functions by $\mathcal{A}_\alpha$, and we consider the subclass of $\mathcal{A}_\alpha$ consisting of those $f(z)$ which satisfy $\mathrm{Re}\left(\frac{z\mathrm{D}_z^\alpha f(z)}{\mathrm{D}_z^\alpha f(z)-\mathrm{D}_z^\alpha f(-z)}\right) > 0$ $(z \in \mathbb{U})$, where $\mathrm{D}_z^\alpha f(z)$ is the fractional derivative of order α of $f(z)$. We call such a functions "Multivalued Sakaguchi Functions" and denote the class of those functions by $\mathcal{S}_s^\alpha$.

The aim of this paper is to investigate some properties of the class $\mathcal{S}_s^\alpha$.

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1 Introduction

Let $\mathcal{A}_\alpha$ denote the class of functions $f(z)$ of the form

$$f(z) = z^\alpha \left(z + \sum_{n=2}^{\infty} a_n z^n \right) \quad (0 < \alpha < 1),$$

that are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} | |z| < 1\}$. Let Ω be the class of analytic functions $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$. Also, denote by $\mathcal{P}$ the class of functions $p(z)$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in $\mathbb{U}$ and satisfy $\mathrm{Re} p(z) > 0$ for every $z \in \mathbb{U}$.

For analytic functions $g(z)$ in $\mathbb{U}$, we recall here the fractional calculus (fractional integrals and fractional derivatives) given by Owa [3], also by Srivastava and Owa [7].

Definition 1. *The fractional integral of order λ for an analytic function $g(z)$ in $\mathbb{U}$ is defined by*

$$\mathrm{D}_z^{-\lambda} g(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{g(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0),$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2. *The fractional derivative of order λ for an analytic function $g(z)$ in $\mathbb{U}$ is defined by*

$$\mathrm{D}_z^{\lambda} g(z) = \frac{d}{dz}(\mathrm{D}_z^{\lambda-1} g(z)) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{g(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1),$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 3. *Under the hypotheses of Definition 2, the fractional derivative of order* $(n+\lambda)$ *for an analytic function* $g(z)$ *in* $\mathbb{U}$ *is defined by*

$$\mathrm{D}_z^{\lambda+n} g(z) = \frac{d^n}{dz^n}(\mathrm{D}_z^{\lambda} g(z)) \quad (0 \leq \lambda < 1, n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}).$$

Remark 1. *From the definitions of the fractional calculus, we see that*

$$\mathrm{D}_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda} \quad (\lambda > 0, k > 0),$$

$$\mathrm{D}_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \quad (0 \leq \lambda < 1, k > 0),$$

$$\mathrm{D}_z^{n+\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)} z^{k-n-\lambda} \quad (0 \leq \lambda < 1, k > 0, n \in \mathbb{N}_0, k-n \neq -1, -2, \cdots).$$

Therefore we say that for any real λ

$$\mathrm{D}_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \quad (k > 0, k-\lambda \neq -1, -2, \cdots).$$

Applying the fractional calculus, we introduce the subclass of $\mathcal{A}_\alpha$.

Definition 4. *A function* $f \in \mathcal{A}_\alpha$ *is said to be Sakaguchi function if* $f(z)$ *satisfies*

$$\mathrm{Re}\left(\frac{z\mathrm{D}_z^{\alpha} f(z)}{\mathrm{D}_z^{\alpha} f(z) - \mathrm{D}_z^{\alpha} f(-z)}\right) = p(z) \quad (z \in \mathbb{U})$$

for some $p(z) \in \mathcal{P}$. *The subclass of* $\mathcal{A}_\alpha$ *consisting of such functions is denoted by* $\mathcal{S}_s^{\alpha}$.

Further, for analytic functions $h(z)$ and $s(z)$ in $\mathbb{U}$, $h(z)$ is said to be subordinate to $s(z)$ if there exists $w(z) \in \Omega$ such that $h(z) = s(w(z))$ $(z \in \mathbb{U})$. We denote this subordination by $h(z) \prec s(z)$. In particular, if $s(z)$ is univalent in $\mathbb{U}$, then the subordination $h(z) \prec s(z)$ is equivalent to $h(0) = s(0)$ and $h(\mathbb{U}) \subset s(\mathbb{U})$ (see [1]).

2 Main Results

To consider some properties for the class $\mathcal{S}_s^\alpha$, we need the following lemma by Jack [2].

Lemma 1. *Let $w(z)$ be a non-constant and analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_1 \in \mathbb{U}$, then we have*

$$z_1 w'(z_1) = k w(z_1),$$

where k is real and $k \geq 1$.

Definition 5. *Let us call any transformation which reduces a multivalued function to a single valued a filter for this function.*

Lemma 2. *Let α be a real number such that $0 < \alpha < 1$, and let*

$$f(z) = z^\alpha + \left(z + \sum_{n=2}^{\infty} a_n z^n \right)$$

be an analytic and multivalued function in the open unit disc $\mathbb{U}$. Then the $\alpha-$fractional derivative D_z^α is a filter f. Moreover, this filter regularizes f.

Propertie 1. *Using the rule for the fractional calculus of the power function z^α and the linear property of the fractional derivatives, we get after simple calculations*

$$\begin{aligned} D_z^\alpha f(z) &= D_z^\alpha (z^{\alpha+1} + a_2 z^{\alpha+2} + \cdots + a_n z^{\alpha+n} + \cdots) \\ &= \frac{\Gamma(\alpha+2)}{\Gamma(2)} z + a_2 \frac{\Gamma(\alpha+3)}{\Gamma(3)} z^2 + \cdots + a_n \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} z^n + \cdots \\ &= b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots . \end{aligned} \tag{4}$$

The inequality (4) *shows that* $D_z^\alpha f(z)$ *is regular and analytic in* $\mathbb{U}$.

Conversely, consider the fractional differential equaliton

$$D_z^\alpha f(z) = s(z) \quad (0 < \alpha < 1). \tag{5}$$

Let us first take the initial condition $f(0) = 0$. *Assume that the function* $s(z)$ *can be expanded in a Taylor series converging for* $|z| < 1$, *i.e.,*

$$s(z) = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} z^n \quad (z \in \mathbb{U}). \tag{6}$$

Using the rule for the fractional calculus of the power function z^α *we write*

$$D_z^\lambda z^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\lambda)} z^{\alpha-\lambda} \quad (0 < \alpha < 1). \tag{7}$$

Taking into account the formula (7) *we can look for a solution of the equation* (5) *in the form of the following power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} z^{\alpha+n} \quad (0 < \alpha < 1). \tag{8}$$

Substituting (8) *and* (6) *into the equation* (5) *and using* (7) *we get*

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} z^n = s(z) = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} z^n. \tag{9}$$

Comparing the coefficients of the both series in (9), *we get*

$$a_n = \frac{s^{(n)}(0)}{n!} \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} = \frac{s^{(n)}(0)}{\Gamma(\alpha+n+1)}. \tag{10}$$

Therefore under the above assumption, the solution of the equation (5) *is*

$$f(z) = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{\Gamma(\alpha+n+1)} z^{\alpha+n}.$$

On the other hand, since the solution $f(z)$ satisfies the assumed initial condition, we can directly apply $\alpha-th$ order fractional integration to both sides of the equation $D_z^\alpha f(z) = s(z)$, and an application of the composition law the fractional derivative gives

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{\Gamma(\alpha+n+1)} z^{\alpha+n} = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} \frac{n!}{\Gamma(\alpha+n+1)} z^{\alpha+n} \\
&= \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} z^{\alpha+n} = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} D_z^{-\alpha} z^n \\
&= D_z^{-\alpha} \left(\sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} z^n \right) = D_z^{-\alpha} s(z).
\end{aligned} \tag{12}
$$

Therefore we have

$$D_z^\alpha f(z) = s(z) \Leftrightarrow f(z) = D_z^{-\alpha} s(z).$$

Theorem 1. *If $f \in \mathcal{S}_s^\alpha$ then the odd starlike function*

$$F(z) = \mathrm{D}_z^\alpha f(z) - \mathrm{D}_z^\alpha f(-z) = 2 \left(\frac{\Gamma(\alpha+2)}{\Gamma(2)} z + \sum_{k=2}^{\infty} a_{2k-1} \frac{\Gamma(\alpha+2k)}{\Gamma(2k)} z^{2k-1} \right)$$

satisfies

$$\left(\frac{z\mathrm{D}_z^{\alpha+1} f(z)}{\mathrm{D}_z^\alpha f(z) - \mathrm{D}_z^\alpha f(-z)} + \frac{z\mathrm{D}_z^{\alpha+1} f(-z)}{\mathrm{D}_z^\alpha f(z) - \mathrm{D}_z^\alpha f(-z)} - 1 \right) \prec \frac{2z^2}{1-z^2} = F_1(z) \tag{13}$$

and this result is sharp because the extremal function is the solution of the fractional differential equation

$$\mathrm{D}_z^\alpha f(z) - \mathrm{D}_z^\alpha f(-z) = \frac{2z}{1-z^2}. \tag{14}$$

Propertie 2. *We define the function*

$$\frac{D_z^\alpha f(z) - D_z^\alpha f(-z)}{2\Gamma(\alpha+2)z} = (1-w(z))^{-2} \quad (z \in \mathbb{U}, w(z) \neq 1),$$

then $w(z)$ *is analytic in* $\mathbb{U}$, $w(0)=0$ *and*

$$\frac{zF'(z)}{F(z)}=\frac{zD_z^{\alpha+1}f(z)}{D_z^{\alpha}f(z)-D_z^{\alpha}f(-z)}+\frac{zD_z^{\alpha+1}f(-z)}{D_z^{\alpha}f(z)-D_z^{\alpha}f(-z)}-1=\frac{2zw'(z)}{1-w(z)} \tag{15}$$

Now, it is easy to realize that the subordination (13) *is equivalent to* $|w(z)|<1$ *for all* $z\in\mathbb{U}$. *Indeed, assume the contrary: then, there exists a* $z_1\in\mathbb{U}$, *such that* $|w(z_1)|=1$. *Then, by Lemma 1,* $z_1w'(z_1)=kw(z_1)$ *for some real* $k\geq 1$. *For such* z_1 *we have (form* (14)*)*

$$\begin{aligned}\frac{z_1F'(z_1)}{F(z_1)}&=\frac{z_1D_z^{\alpha+1}f(z_1)}{D_z^{\alpha}f(z_1)-D_z^{\alpha}f(-z_1)}+\frac{z_1D_z^{\alpha+1}f(-z_1)}{D_z^{\alpha}f(z_1)-D_z^{\alpha}f(-z_1)}-1\\&=\frac{2kw(z_1)}{1-w(z_1)}=F_1(w(z_1))\notin F_1(\mathbb{U}),\end{aligned} \tag{16}$$

because $|w(z_1)|=1$ *and* $k\geq 1$. *But this contradicts* (13), *so the assumption is wrong, i.e,* $|w(z)|<1$ *for every* $z\in\mathbb{U}$.

The sharpness of this result follows from the fact that

$$F(z)=D_z^{\alpha}f(z)-D_z^{\alpha}f(-z)=\frac{2z}{1-z^2}\Rightarrow$$

$$\frac{zF'(z)}{F(z)}=\frac{zD_z^{\alpha+1}f(z)}{D_z^{\alpha}f(z)-D_z^{\alpha}f(-z)}+\frac{zD_z^{\alpha+1}f(-z)}{D_z^{\alpha}f(z)-D_z^{\alpha}f(-z)}-1=\frac{2z^2}{1-z^2}$$

Corollary 1. *If* $f(z)\in\mathcal{S}_s^{\alpha}$, *then*

$$\left|\left(\frac{2\Gamma(\alpha+2)z}{\mathrm{D}_z^{\alpha}f(z)-\mathrm{D}_z^{\alpha}f(-z)}\right)^{\frac{1}{2}}-1\right|<1.$$

This inequality is the Marx-Strohhacker inequality for the class $\mathcal{S}_s^{\alpha}$.

Propertie 3. *This corollary is a simple consequence of Theorem 1.*

Corollary 2. *If* $f(z)\in\mathcal{S}_s^{\alpha}$, *then*

$$\frac{\Gamma(\alpha+2)r}{2(1+r^2)}\leq|\mathrm{D}_z^{\alpha}f(z)-\mathrm{D}_z^{\alpha}f(-z)|\leq\frac{\Gamma(\alpha+2)r}{2(1-r^2)}. \tag{18}$$

Propertie 4. *If $F(z)$ is an odd starlike function, then [1]*

$$\frac{r}{1+r^3} \leq |F(z)| \leq \frac{r}{1-r^3},$$

for $|z| = r$, so by Theorem 1 we obtain (18). *This result is sharp because the extremal function is the solution of the fractional differential equation is given* (14).

Corollary 3. *If $f(z) \in \mathcal{S}_s^\alpha$, then*

$$\frac{\Gamma(\alpha+2)(1-r)}{(1+r^2)(1+r)} \leq |\mathrm{D}_z^\alpha f(z)| \leq \frac{\Gamma(\alpha+2)(1+r)}{(1-r^2)(1-r)}. \tag{19}$$

for $|z| = r$.

Propertie 5. *By the definition of the class $\mathcal{S}_s^\alpha$ and Caratheodory functions we have*

$$\frac{zD_z^\alpha f(z)}{D_z^\alpha f(z) - D_z^\alpha f(-z)} = p(z) \Leftrightarrow zD_z^\alpha f(z) = D_z^\alpha f(z) - D_z^\alpha f(-z) \tag{20}$$

for some $p(z) \in \mathcal{P}$. On the other hand, the well known Cratheodory's inequality [1]

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r}, \tag{21}$$

together with (18), (20) *and* (21) *yields* (19) *after simple calculations.*

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