General Mathematics Vol. 14, No. 4 (2006), 113–126

Meromorphic Starlike Functions with Alternating and Missing Coefficients [1]

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In memoriam of Associate Professor Ph. D. Luciana Lupaş

Abstract

In this paper we introduced a new class $\Omega_p(n, \alpha)$ of meromorphic starlike univalent functions with alternating coefficients in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$. We obtain among other results are the coefficient inequalities, distortion theorem and class preserving integral operator.

2000 Mathematics Subject Classification: 30C45

Keywords: meromorphic function, Starlike functions, Distortion theorem

[1] *Received 24 September, 2006*

Accepted for publication (in revised form) 9 November, 2006

1 Introduction

Let A_p denote the class of functions of the form:

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k}, \quad (a_{-1} \neq 0, \quad p \in \mathbf{N} = \{1, 2, 3, \ldots\}) \tag{1}$$

which are regular in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$.

Define

$$D^0 f(z) = f(z) \tag{2}$$

$$\begin{aligned} Df(z) = D^1 f(z) &= \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} (p+k+2) a_{p+k} z^{p+k} \\ &= \frac{(z^2 f(z))'}{z} \end{aligned} \tag{3}$$

$$D^2 f(z) = D(D^1 f(z)), \tag{4}$$

and for $n = 1, 2, 3, \ldots$

$$\begin{aligned} D^n f(z) &= D(D^{n-1} f(z)) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} (p+k+2) a_{p+k} z^{p+k}, \\ &= \frac{(z^2 D^{n-1} f(z))'}{z} \end{aligned} \tag{5}$$

Let $B_n(\alpha)$, denote the class consisting functions in A_p satisfying

$$\operatorname{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - 2 \right\} < -\alpha, \qquad (z \in U^*, 0 \leq \alpha < 1, n \in \mathbf{N}_o = \mathbf{N} \cup \{0\}). \tag{6}$$

Let Λ_p be the subclass of A_p which consisting of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} (-1)^{p+k-1} a_{p+k} z^{p+k}, \quad (a_{-1} > 0; a_{p+k} \geq 0, \quad p \in \mathbf{N}). \tag{7}$$

Further let

$$\Omega_p(n,\alpha) = B_n(\alpha) \cap \Lambda_p. \tag{8}$$

We note that, in [1] Uralegaddi and Somanatha define a class $B_n(\alpha)$, which consists functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-1} \neq 0)$$

which are analytic in U^* and obtained inclusion relation, the class preserving integral. In the same year, Uralegaddi and Ganigi [2] considered meromorphic starlike functions with alternating coefficients. Further, recently in [4], Aouf and Darwish also considered meromorphic starlike univalent functions with alternating coefficients and obtained coefficient inequalities, distortion theorem and integral operators. The class of meromorphic functions have been studied by various authors and among are Darwish [3], Aouf and Hossen [5], and Mogra *et.al* [6], few to mention.

In the present paper, we consider functions of the forms (7) and obtain basic properties, which include for example the coefficient inequalities, distortion theorem, closure theorem and integral operators. Finally, the class preserving integral operator of the form

$$F_{c+1}(z) = (c+1)\int_0^1 u^{c+1} f(uz)du \qquad (0 \leq u < 1, 0 < c < \infty) \tag{9}$$

is considered. Techniques used are similar to those of Aouf and Hossen [5].

2 Coefficient Inequality

Theorem 2.1. *Let the function $f(z)$ be defined by (1). If*

$$\sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)|a_{p+k}| \le (1-\alpha)|a_{-1}|, \tag{10}$$

then $f(z) \in B_n(\alpha)$.

Proof. It suffices to show that

$$\left|\frac{\frac{D^{n+1}f(z)}{D^nf(z)}-1}{\frac{D^{n+1}f(z)}{D^nf(z)}-(3-2\alpha)}\right| < 1, \qquad |z|<1, \tag{11}$$

we have

$$\left|\frac{\frac{D^{n+1}f(z)}{D^nf(z)}-1}{\frac{D^{n+1}f(z)}{D^nf(z)}-(3-2\alpha)}\right| =$$

$$= \left|\frac{\sum\limits_{k=0}^{\infty}(p+k+2)^n(p+k+1)a_{p+k}z^{p+k}}{(2-2\alpha)a_{-1}-\sum\limits_{k=0}^{\infty}(p+k+2)^n(p+k+2\alpha-1)a_{p+k}z^{p+k}}\right| \le$$

$$\le \frac{\sum\limits_{k=0}^{\infty}(p+k+2)^n(p+k+1)|a_{p+k}|}{(2-2\alpha)a_{-1}-\sum\limits_{k=0}^{\infty}(p+k+2)^n(p+k+2\alpha-1)|a_{p+k}|}.$$

The last expression is bounded by 1 if

$$\sum_{k=0}^{\infty}(p+k+2)^n(p+k+1)|a_{p+k}| \le (2-2\alpha)a_{-1}-\sum_{k=0}^{\infty}(p+k+2)^n(p+k+2\alpha-1)|a_{p+k}|,$$

which reduces to

$$\sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)|a_{p+k}| \le (1-\alpha)|a_{-1}|. \tag{12}$$

But (12) is true by hypothesis. Hence the result follows.

Theorem 2.2. *Let the function $f(z)$ be defined by (7). Then $f(z) \in \Omega_p(n,\alpha)$ if and only if*

$$\sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)|a_{p+k}| \leq (1-\alpha)a_{-1}, \tag{13}$$

Proof. In view of Theorem 2.1, it suffices to prove the "only if" part. Let us assume that $f(z)$ defined by (7) is in $\Omega_p(n,\alpha)$. Then

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}-2\right\} = \mathrm{Re}\left\{\frac{D^{n+1}f(z)-2D^nf(z)}{D^nf(z)}\right\},$$

(14)

$$= \mathrm{Re}\left\{\frac{a_{-1}-\sum\limits_{k=0}^{\infty}(p+k+2)^n(p+k)a_{p+k}z^{p+k}}{-a_{-1}-\sum\limits_{k=0}^{\infty}(-1)^{p+k-1}(p+k+2)^na_{p+k}z^{p+k}}\right\} < -\alpha, \qquad |z|<1.$$

Choose value of z on the real axis so that $\frac{D^{n+1}f(z)}{D^nf(z)}-2$ is real. Upon clearing the denominator in (14) and letting $z \to 1$ through real values, we obtain

$$a_{-1}-\sum_{k=0}^{\infty}(p+k+2)^n(p+k)a_{p+k} \geq \left(a_{-1}+\sum_{k=0}^{\infty}(p+k+2)^na_{p+k}\right)\alpha.$$

Thus

$$\sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)a_{p+k} \leq (1-\alpha)a_{-1}.$$

Hence the result follows.

Corollary 1 *Let the function $f(z)$ be defined by (7) be in $\Omega_p(n,\alpha)$. Then*

$$a_{p+k} \leq \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}, \tag{15}$$

The result is sharp for the function

$$f(z) = \frac{a_{-1}}{z}+(-1)^{p+k-1}\frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}z^{p+k}, \quad (k \geq 1). \tag{16}$$

3 Distortion Theorem

Theorem 3.1. *Let the function* $f(z)$ *be defined by (7) be in* $\Omega_p(n,\alpha)$. *Then for* $0 < |z| < 1$, *we have*

$$\frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}r \leq |f(z)| \leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}r, \tag{17}$$

where equality holds for the function

$$f(z) = \frac{a_{-1}}{z} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}z^p, \qquad (z = ir, r), \tag{18}$$

and

$$\frac{a_{-1}}{r^2} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} \leq |f'(z)| \leq \frac{a_{-1}}{r^2} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}, \tag{19}$$

where equality holds for the function $f(z)$ *given by (18) at* $z = \mp ir, \mp r$.

Proof. In view of Theorem 2.2, we have

$$\sum_{k=0}^{\infty} a_{p+k} \leq \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}. \tag{20}$$

Thus, for $0 < |z| < 1$,

$$\begin{aligned} |f(z)| &\leq \frac{a_{-1}}{r} + r\sum_{k=0}^{\infty} a_{p+k} \\ &\leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}r \end{aligned} \tag{21}$$

and

$$\begin{aligned} |f(z)| &\geq \frac{a_{-1}}{r} - r\sum_{k=0}^{\infty} a_{p+k} \\ &\geq \frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}r. \end{aligned} \tag{22}$$

Thus (17) follows.

Since

$$(p+3)^n(p+\alpha+1)\sum_{k=0}^{\infty}(p+k)|a_{p+k}| \leq \sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)|a_{p+k}| \leq (1-\alpha)|a_{-1}|$$

where $\frac{(p+k+2)^n(p+k+\alpha)}{p+k}$ is an increasing function of k, from Theorem 2.2, it follows that

$$\sum_{k=0}^{\infty}(p+k) \leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}. \tag{23}$$

Hence

$$\begin{aligned} |f'(z)| &\leq \frac{a_{-1}}{r^2} + \sum_{k=0}^{\infty}(p+k)a_{p+k}r^{p+k-1} \\ &\leq \frac{a_{-1}}{r} + \sum_{k=0}^{\infty}(p+k)a_{p+k} \\ &\leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} \qquad (24) \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq \frac{a_{-1}}{r^2} - \sum_{k=0}^{\infty}(p+k)a_{p+k}r^{p+k-1} \\ &\geq \frac{a_{-1}}{r} - \sum_{k=0}^{\infty}(p+k)a_{p+k} \\ &\geq \frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} \qquad (25) \end{aligned}$$

Thus (19) follows. It can be easily seen that the function $f(z)$ defined by (18) is extremal for the theorem.

4 Closure Theorem

Let the function $f_j(z)$ be defined for $j \in \{1, 2, 3, \ldots, m\}$, by

$$f_j(z) = \frac{a_{-1,j}}{z} + \sum_{k=0}^{\infty} (-1)^{p+k-1} a_{p+k,j} z^{p+k}, \quad (a_{-1,j} > 0; a_{p+k,j} \geq 0) \tag{26}$$

for $z \in U^*$.

Now, we shall prove the following result for the closure of function in the class $\Omega_p(n, \alpha)$.

Theorem 4.1. *Let the functions $f_j(z)$ be defined by (26) be in the class $\Omega_p(n, \alpha)$ for every $j \in \{1, 2, 3, \ldots, m\}$. Then the function $F(z)$ defined by*

$$F(z) = \frac{b_{-1}}{z} + \sum_{k=0}^{\infty} (-1)^{p+k-1} b_{p+k} z^{p+k}, \quad (b_{-1} > 0; b_{p+k} \geq 0, \quad p \in \mathbf{N}) \tag{27}$$

is a member of the class $\Omega_p(n, \alpha)$, where

$$b_{-1} = \frac{1}{m} \sum_{j=1}^{\infty} a_{-1,j} \quad \text{and} \quad b_{p+k} = \frac{1}{m} \sum_{j=1}^{\infty} a_{p+k,j} \qquad (k = 1, 2, \ldots). \tag{28}$$

Proof. Since $f_j(z) \in \Omega_p(n, \alpha)$, it follows from Theorem 2.2, that

$$\sum_{k=0}^{\infty} (p + k + 2)^n (p + k + \alpha) |a_{p+k,j}| \leq (1 - \alpha) |a_{-1,j}|, \tag{29}$$

for every $j \in \{1, 2, 3, \ldots, m\}$. Hence,

$$\sum_{k=0}^{\infty} (p + k + 2)^n (p + k + \alpha) b_{p+k} =$$

$$= \sum_{k=0}^{\infty} (p + k + 2)^n (p + k + \alpha) \left(\frac{1}{m} \sum_{j=1}^{\infty} a_{p+k,j} \right) =$$

$$= \frac{1}{m}\sum_{j=1}^{\infty}\left(\sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)a_{p+k,j}\right) \leq$$

$$\leq (1-\alpha)\left(\frac{1}{m}\sum_{j=1}^{\infty}a_{-1,j}\right) = (1-\alpha)b_{-1},$$

which (in view of Theorem 2.2) implies that $F(z) \in \Omega_p(n,\alpha)$.

Theorem 4.2. *The class $\Omega_p(n,\alpha)$ is closed under convex linear combination.*

Proof. Let the $f_j(z)(j=1,2)$ defined by (26) be in the class $\Omega_p(n,\alpha)$, it is sufficient to prove that the function

$$H(z) = \lambda f_1(z) + (1-\lambda)f_2(z) \qquad (0 \leq \lambda \leq 1) \tag{30}$$

is also in the class $\Omega_p(n,\alpha)$. Since, for $(0 \leq \lambda \leq 1)$,

$$H(z) = \frac{\lambda a_{-1,1} + (1-\lambda)a_{-1,2}}{z} - \sum_{k=0}^{\infty}\left\{\lambda a_{p+k,1} + (1-\lambda)a_{p+k,2}\right\} z^{p+k}, \tag{31}$$

we observe that

$$\sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)\left\{\lambda a_{p+k,1} + (1-\lambda)a_{p+k,2}\right\} =$$

$$= \lambda\sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)\lambda a_{p+k,1} + (1-\lambda)\sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)\lambda a_{p+k,2} \leq$$

$$\leq (1-\alpha)\left\{\lambda a_{-1,1} + (1-\lambda)a_{-1,2}\right\}$$

with the aid of Theorem 2.2. Hence $H(z) \in \Omega_p(n,\alpha)$.
This completes the proof of Theorem 4.2.

Theorem 4.3. *Let*

$$(32)\quad f_o(z) = \frac{a_{-1}}{z}$$

$$(33)\quad f_{p+k}(z) = \frac{a_{-1}}{z} + (-1)^{p+k-1}\frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}z^{p+k}, \quad (k \geq 1).$$

Then $f(z) \in \Omega_p(n,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z), \tag{34}$$

where

$$\lambda_{p+k}(k \geq 0) \quad and \quad \sum_{k=0}^{\infty} \lambda_k = 1.$$

Proof. Let

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z), \quad where \quad \lambda_{p+k}(k \geq 0) \quad and \quad \sum_{k=0}^{\infty} \lambda_k = 1.$$

Then

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z) = \lambda_o f_o(z) + \sum_{k=1}^{\infty} \lambda_{p+k} f_{p+k}(z) = \left(1 - \sum_{k=0}^{\infty} \lambda_{p+k}\right) +$$

$$+ \sum_{k=0}^{\infty} \lambda_{p+k} \left\{ \frac{a_{-1}}{z} + (-1)^{p+k-1} \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)} z^{p+k} \right\} =$$

$$= \frac{a_{-1}}{z} + (-1)^{p+k-1} \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)} z^{p+k}$$

Since

$$\sum_{k=0}^{\infty} (p+k+2)^n(p+k+\alpha) \cdot \frac{(1-\alpha)a_{-1}\lambda_{p+k}}{(p+k+2)^n(p+k+\alpha)}$$

$$= (1-\alpha)a_{-1} + \sum_{k=0}^{\infty} \lambda_{p+k} = (1-\alpha)a_{-1}(1-\lambda_o)$$

$$\leq (1-\alpha)a_{-1}, \tag{35}$$

by Theorem 2.2, $f(z) \in \Omega_p(n, \alpha)$. Conversely, we suppose that $f(z)$ defined by (7) is in the class $\Omega_p(n, \alpha)$. Then by using (15), we get

$$a_{p+k} \leq \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}, \quad (k \geq 1). \tag{36}$$

Setting

$$\lambda_{p+k} = \frac{(p+k+2)^n(p+k+\alpha)}{(1-\alpha)a_{-1}} a_{p+k}, \quad (k \geq 1). \tag{37}$$

and

$$\lambda_o = 1 - \sum_{k=0}^{\infty} \lambda_{p+k}, \tag{38}$$

we have (34). This completes the proof of the Theorem 4.3.

5 Integral Operator

In his section we consider integral transforms of functions in the class $\Omega_p(n, \alpha)$.

Theorem 5.1. *Let the function $f(z)$ be defined by (7) be in the class $\Omega_p(n, \alpha)$, then the integral transforms*

$$F_{c+1}(z) = (c+1)\int_0^1 u^{c+1} f(uz)du \qquad (0 \leq u < 1, 0 < c < \infty), \tag{39}$$

are in $\Omega_p(n, \alpha)$, where

$$\delta(\alpha, c, p) = \frac{(p+k+3)(p+\alpha+1) - (p+1)(1-\alpha)(c+1)}{(p+k+3)(p+\alpha+1) + (c+1)(1-\alpha)}. \tag{40}$$

The result is sharp for the function

$$f(z) = \frac{a_{-1}}{z} + (-1)^{p+k-1} \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} z^p. \tag{41}$$

Proof. Let

$$F_{c+1}(z) = (c+1)\int_0^1 u^{c+1} f(uz)du = \frac{a_{-1}}{z} - \sum_{k=0}^{\infty} \frac{(c+1)}{p+k+c+2} a_{p+k} z^{p+k}. \tag{42}$$

In view of Theorem 2.2, it is sufficient to show that

$$\sum_{k=0}^{\infty} \frac{(p+k+2)^n (p+k+\delta)}{(1-\delta)a_{-1}} \left(\frac{c+1}{p+k+c+2} \right) a_{p+k} \leq 1. \tag{43}$$

Since $f(z) \in \Omega_p(n, \alpha)$, we have

$$\sum_{k=0}^{\infty} \frac{(p+k+2)^n (p+k+\alpha)}{(1-\alpha)a_{-1}} a_{p+k} \leq 1. \tag{44}$$

Thus (42) will be satisfy if

$$\frac{(p+k+\delta)(c+1)}{(1-\delta)(p+k+c+2)} \leq \frac{p+k+\alpha}{1-\alpha} \quad \text{for each} \quad k,$$

or

$$\delta \leq \frac{(p+k+c+2)(p+k+\alpha) - (p+k)(1-\alpha)(c+1)}{(p+k+c+2)(p+k+\alpha) + (c+1)(1-\alpha)}. \tag{45}$$

For each α, p, and c fixed, let

$$F(k) = \frac{(p+k+c+2)(p+k+\alpha) - (p+k)(1-\alpha)(c+1)}{(p+k+c+2)(p+k+\alpha) + (c+1)(1-\alpha)}.$$

Then

$$F(k+1) - F(k) =$$
$$= \frac{(c+1)(1-\alpha)(p+k+1)(p+k+2)}{[(p+k+c+2)(p+k+\alpha) + (c+1)(1-\alpha)][(p+k+c+3)(p+k+\alpha+1) + (c+1)(1-\alpha)]}$$

for each k. Hence, $F(k)$ is an increasing function of k.

Since

$$F(1) = \frac{(p+c+3)(p+\alpha+1) - (p+1)(1-\alpha)(c+1)}{(p+c+3)(p+\alpha+1) + (c+1)(1-\alpha)},$$

the result follows.

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