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On the means of sequences [1]

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Abstract

In this paper we investigate the invariancy of a class of real sequences with respect to the transformation $A : a \to A(a)$.

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1

We consider the set of real sequences K, the set K_m of all sequences which are convex of order m ($m \in \mathbb{N}$) and the operator $\Delta^r : K \to \mathbb{R}$, $r \in \mathbb{R}$, defined by

$$(1) \qquad \Delta^r a_n = (-1)^{[r]} \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} a_{n+k},$$

with the convention $\Delta^0 a_n = a_n$ for every $n \in \mathbb{N}$, where:

$$(x)_l = x(x+1)...(x+l-1), l \in \mathbb{N}, (x)_0 = 1$$

$[r]$-represent integer part of the real number r.

Definition 1.1. *We say that a real sequence $(a_n)_{n=1}^{\infty}$ is of M_r class if and only if*

$$(2) \qquad \Delta^r a_n \geq 0 \textit{ for every } n \in \mathbb{N} \text{ (see [5]).}$$

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If $r \in \mathbb{N}$ then $M_r = K_r\left(\Delta^r a_n = \sum\limits_{k=0}^{r}(-1)^{r-k}\binom{r}{k}a_{n+k}\right)$

Property. For real numbers r, r_1, r_2 we have:

i) $\Delta^{r+1}a_n = \Delta^r a_{n+1} - \Delta^r a_n$, for every $n \in \mathbb{N}$

ii) $\Delta^{r_1+r_2}a_n = \Delta^{r_1}(\Delta^{r_2}a_n) = \Delta^{r_2}(\Delta^{r_1}a_n)$ $(see[5])$

Let $A(a) = (A_n(a))_{n=1}^{\infty}$, be the sequence of the means, that is $A_n(a) = \frac{1}{n+1}\sum\limits_{k=0}^{n}a_k,\ n = 0, 1, 2, ...$

If S denotes a certain class of real sequences, then a_n interesting problem is to investigate if this class is invariant with respect to the transformation $A : a \to A(a)$; in other words if $A(S) \subseteq S$. For instance, it is well-known that $A(S_0) \subseteq S_0$, S_0 being the class of all real sequences which are convergent.

In [1]-[4] it is shown that from the n-th order convexity of $a = (a_n)$ follows the convexity, of the same order, of the sequence $A(a) = (A_n(a))$, i.e. $A(K_m) \subseteq K_m$.

We shall find a representation of $\Delta^r A_n(a)$ as a positive linear combinations of $\Delta^r a_0, \Delta^r a_1, ..., \Delta^r a_n$.

Theorem 1.1. *For $r \geq 0$ and $n = 0, 1, 2, ...$ the equality*

$$\Delta^r A_n(a) = \sum_{k=0}^{n} c_k(n,r), \Delta^r a_k \text{ with} \tag{3}$$

$$c_k(n,r) = \begin{cases} \dfrac{n!}{(r+1)_{n+1}}, & k = 0 \\ \dfrac{n!}{(r+2)_n}\cdot\dfrac{(r+2)_{k-1}}{k!}, & k = 1, 2, ...n. \end{cases} \tag{4}$$

is verified.

Proof. For $k = 0, 1, 2, ...$ we have:

$$a_k = (k+1)A_k(a) - kA_{k-1}(a)$$

$$\Delta^r a_k = (-1)^{[r]}\sum_{i=0}^{\infty}\frac{(-r)_i}{i!}a_{k+i} =$$

$$= (-1)^{[r]}\sum_{i=0}^{\infty}\frac{(-r)_i}{i!}\left[(k+i+1)A_{k+i}(a) - (k+i)A_{k+i-1}(a)\right] =$$

$$= (-1)^{[r]}\left[\sum_{i=0}^{\infty}\frac{(-r)_i}{i!}(k+i+1)A_{k+1}(a) - \sum_{i=0}^{\infty}\frac{(-r)_i}{i!}(k+1)A_{k+1-1}(a)\right] =$$

$$= (-1)^{[r]} \left[\sum_{i=0}^{\infty} \frac{(-r)_i}{i!} (k+i+1) A_{k+i}(a) - \right.$$

$$\left. - \sum_{i=0}^{\infty} \frac{(-r)_{i+1}}{(i+1)!} (k+i+1) A_{k+i}(a) - k A_{k-1}(a) \right] =$$

$$= (-1)^{[r]} \left\{ \sum_{i=0}^{\infty} (k+i+1) A_{k+1}(a) \left[\frac{(-r)_i}{i!} - \frac{(-r)_{i+1}}{(i+1)!} \right] - k A_{k-1}(a) \right\} =$$

$$= (-1)^{[r]} \left[\sum_{i=0}^{\infty} \frac{(-r)_i}{i!} (k+i+1) A_{k+i}(a) \frac{1+r}{i+1} - k A_{k-1}(a) \right] =$$

$$= (-1)^{[r]} \left[(1+r) \sum_{i=0}^{\infty} \frac{(-r)_i}{i!} A_{k+1}(a) + k \sum_{i=0}^{\infty} \frac{(-r)_i}{i!} \cdot \frac{1+r}{i+1} A_{k+i}(a) - k A_{k-1}(a) \right] =$$

$$= (1+r) \Delta^r A_k(a) - k(-1)^{[r]} \left[\sum_{i=0}^{\infty} \frac{(-r-1)_{i+1}}{(i+1)!} A_{k+i}(a) - A_{k-1}(a) \right] =$$

$$= (1+r) \Delta^r A_k(a) - k(-1)^{[r]} \left[\sum_{i=0}^{\infty} \frac{(-r-1)_i}{i!} A_{k+i-1}(a) - A_{k-1}(a) \right] =$$

$$= (1+r) \Delta^r A_k(a) - k(-1)^{[r]} \sum_{i=0}^{\infty} \frac{(-r-1)_i}{i!} A_{k-1+i} =$$

$$= (1+r) \Delta^r A_k(a) + k \Delta^{r+1} A_{k-1}(a).$$

From property ii), $\Delta^{r+1} A_{k-1}(a) = \Delta^r A_k(a) - \Delta^r A_{k-1}(a)$.
We obtain

$$\Delta^r a_k = (1+r+k) \Delta^r A_k(a) - k \Delta^r A_{k-1}(a), \tag{5}$$

$$\frac{(r+2)_{k-1}}{k!} \Delta^r a_k = \frac{(r+2)_k}{k!} \Delta^r A_k(a) - \frac{(r+2)_{k-1}}{(k-1)!} \Delta^r A_{k-1}(a). \tag{6}$$

By summing these equalities we obtain

$$\sum_{k=1}^{n} \frac{(r+2)_{k-1}}{k!} \Delta^r a_k = \frac{(r+2)_n}{n!} \Delta^r A_n(a) - \Delta^r A_0(a).$$

In virtue of (5), for $k = 0$, $\Delta^r A_0(a) = \dfrac{1}{r+1} \Delta^r a_0$.

We obtain

$$\Delta^r A_n(a) = \frac{1}{r+1} \cdot \frac{n!}{(r+2)_n} \Delta^r a_0 + \frac{n!}{(r+2)_n} \sum_{k=1}^{n} \frac{(r+2)_{k-1}}{k!} \Delta^r a_k,$$

$$\Delta^r A_n(a) = \frac{n!}{(r+1)_{n+1}} \Delta^r a_0 + \frac{n!}{(r+2)_n} \sum_{k=1}^{n} \frac{(r+2)_{k-1}}{k!} \Delta^r a_k.$$

Theorem 1.2. *Let* $a = (a_n), A(a) = (A(a_n))$*; then:*
i) $A_n(M_r) \subseteq M_r$
ii) if there exists $C \in R$*, such that*

$$|\Delta^r(a_n)| < C, \; n = 0, 1, 2, ...$$

then for $n = 1, 2, ...$

$$|\Delta^r A_n(a)| < \frac{C}{r+1}$$

Proof. The assertions i), ii) are consequences of the equalities (3) and (4).

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