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Farthest Points in Normed Linear Spaces [1]

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Abstract

In this paper we established a characterization of farthest points in a normed linear spaces. We also provide some application of farthest points in the space $C(Q)$ and $C_R(Q)$.

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1 Introduction

The concept of farthest points in normed linear spaces has been investigated by Franchetti and Singer [4]. They obtained some results on characterization and existence of farthest points in normed linear spaces in terms of bounded linear functionals. Section 2 gives some fundamental concepts of farthest points. A characterization of farthest points in normed linear spaces are provided in Section 3. Section 4 delineates some applications of farthest points. Some basic properties of farthest point operator are established in Section 5.

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2 Preliminaries

Definition 2.1 [4]. Let G be a bounded non-empty subset of a real normed linear space E and $x \in E$. An element $g_0 \in G$ is called a farthest point to x in G if

$$||g_0 - x|| = \sup_{g \in G} ||g - x||, \text{ for all } g \in G.$$

The set of all farthest points to x from G is denoted by $F_G(x)$, for all $g \in G$.
Example 2.2. Let $E = R^2$, the set $G = \{(x_1, x_2) : 0 \leq x_1 \leq 2, -1 \leq x_2 \leq 1\}$ and $x = (2, 2)$. Then $(0, -1) \in F_G(x)$. If $x = (1, 2)$, then $(0, -1)$ and $(2, -1)$ belong to $F_G(x)$.
Definition 2.3. Let A be a closed convex set of a topological linear space L. A non-empty subset $M \subsetneqq A$ is said to be an extremal subset of A, if a proper convex combination $\lambda x + (1 - \lambda)y, 0 < \lambda < 1$, of two points x and y of A lies in M only if both x and y are in M.

An extremal subset of A consisting of just one point is called an extremal point of A. The set of all extremal points of A is denoted by $\sigma(A)$.
Definition 2.4. Let E^* denote the conjugate space of E, that is, the space of all continuous linear functionals on E, endowed with the usual vector operations and with the norm

$$||f|| = \sup_{\substack{x \in E \\ ||x|| \leq 1}} |f(x)|.$$

Let S_{E^*} represent an unit cell defined by

$$S_{E^*} = \{f \in E^* : ||f|| \leq 1\}.$$

Definition 2.5. Let Q be a compact space and $C(Q)$ be the space of all numerical continuous functions on Q, endowed with the usual vector operations and with the norm

$$||x|| = \max_{q \in Q} |x(q)|.$$

We shall denote by $C_R(Q)$ the space of all continuous real-valued functions on Q, endowed with the usual vector operations and with the norm

$$||x|| = \max_{q \in Q} |x(q)|.$$

The following results are required to prove the main result of this paper.

Lemma 2.6 [7]. *Let M be an extremal subset of a closed convex set A in a topological linear space L. Then*

$$\sigma(M) = \sigma(A) \cap M$$

Lemma 2.7 [7]. *Let E be a normed linear space and let F be a non-empty convex subset of the set $\{x \in E : ||x|| = 1\}$. Then the set*

$$M_F = \bigcap_{x \in F} \{f \in E^* : ||f|| = 1, f(x) = 1\}$$

is a non-empty extremal subset of the cell S_{E^*} endowed with weak topology $\sigma(E^*, E)$.

Corollary 2.8 [7]. *Let E be a normed linear space and $x \in E, x \neq 0$. Then the set*

$$M_x = \{f \in E^* : ||f|| = 1, f(x) = ||x||\}$$

is a non-empty extremal subset of the cell S_{E^*} endowed with $\sigma(E^*, E)$.

A characterization of farthest points is presented in the next section.

3 Characterizations of Farthest Points

Theorem 3.1. *Let G be a bounded subset of a normed linear space $E, x \in E$ and $g_0 \in G$. Then $g_0 \in F_G(x)$ if and only if there exists an $f_0 \in E^*$ such that*

$$f_0 \in \sigma(S_{E^*}) \tag{3.1}$$

$$f_0(g_0 - x) = \sup_{g \in G} ||g - x|| \tag{3.2}$$

Proof. Let $g_0 \in F_G(x)$. Then by the theorem 3.1[4], there exists $f \in E^*$ such that $||f|| = 1$,
$f(g_0 - x) = \sup_{g \in G} ||g - x||$. By Corollary 2.8, the set

$$M = \{f \in E^* : ||f|| = 1, f(g_0 - x) = \sup_{g \in G} ||g - x||\}$$

is a non-empty extremal subset of the cell S_{E^*} endowed with $\sigma(E^*, E)$. So, by the Krein - Milman theorem, the set $\sigma(M)$ is non-empty and hence, by Lemma 2.6, there exists an $f_0 \in E^*$ such that (3.1) and (3.2) hold.
Conversely, assume that (3.1) and (3.2) hold. Then, by (3.1) and (3.2)

$$||g_0 - x|| \geq ||g - x||, \text{ for all } g \in G.$$

Whence it follows that $g_0 \in F_G(x)$.

Corollary 3.2. *Let G be a bounded subset of a normed linear space E, $x \in E$ and $g_0 \in G$. Then the following statements are equivalent:*

(a) $g_0 \in F_G(x)$.

(b) *There exists an $f \in E^*$ such that f satisfies*

$$|f(g_0 - x)| = \sup_{g \in G} ||g - x|| \tag{3.3}$$
$$|f(g_0 - x)| \geq |f(g - x)|, \text{ for all } g \in G. \tag{3.4}$$

(c) *There exists an $f \in E^*$ such that f satisfies* (3.3) and

$$\mathrm{Re}[f(g_0 - g)\overline{f(g_0 - x)}] \geq 0. \tag{3.5}$$

Proof. Let $g_0 \in F_G(x)$. Then by Theorem 3.1, we have (3.3) and

$$|f(g_0 - x)| \geq ||g - x|| \geq |f(g - x)|, \text{ for all } g \in G.$$

Which proves (3.4). Thus, (a) $\Rightarrow$ (b).

To prove (b) $\Rightarrow$ (c) assume that (b) holds.
Then, by (3.4),

$$|f(g_0 - x)|^2 \geq |f(g - x)|^2 = |f(g - g_0)|^2 + |f(g_0 - x)|^2 +$$

$$+2 \ \mathrm{Re} f(g - g_0)\overline{f(g_0 - x)} \geq$$

$$\geq |f(g_0 - x)|^2 + 2\mathrm{Re} f(g - g_0)\overline{f(g_0 - x)}$$

whence it follows that $\mathrm{Re}[f(g_0 - g)\overline{f(g_0 - x)}] \geq 0$.

(c) $\Rightarrow$ (a) is trivial.

Let G be a non-empty bounded subset of a normed linear space E and for each $b > 0$, the b extension of G denoted by G_b and defined by

$$G_b = \{x \in E : d(x, G) = \sup \parallel x - g \parallel \leq b\}, \ b \neq 0.$$

Proposition 3.3. Let $G \subset E$, $x_0 \in E$ and $b \leq \sup\limits_{g \in G} \parallel x_0 - g \parallel$. Then $\mathcal{F}(x_0, G) = \mathcal{F}(x_0, G_b)$.

Proof. For each $z \in E$ such that $\sup\limits_{g \in G} \parallel z - g \parallel \geq b$,

$$\sup_{g \in G_b} \parallel z - g \parallel = \sup_{g \in G} \parallel z - g \parallel + b. \tag{3.6}$$

Let $z \in \mathcal{F}(G, x_0)$. Then

$$\sup_{g \in G} \parallel z - g \parallel = \sup_{g \in G} \parallel x_0 - g \parallel + \parallel z - x_0 \parallel \geq b$$

By (3.6),

$$\sup_{g \in G_b} \parallel z - g \parallel = \sup_{g \in G} \parallel z - g \parallel + b$$

and

$$\sup_{g \in G_b} \parallel x_0 - g \parallel = \sup_{g \in G} \parallel x_0 - g \parallel + b$$

Hence

$$\begin{aligned}\sup_{g\in G_b} \| z-g \| &= \sup_{g\in G} \| z-g \| +b \\ &= \| z-x_0 \| + \sup_{g\in G} \| x_0-g \| +b \\ &= \| z-x_0 \| + \sup_{g\in G_b} \| x_0-g \|\end{aligned}$$

which implies $\mathcal{F}(G, x_0) \subseteq \mathcal{F}(G_b, x_0)$.

Let $z \in \mathcal{F}(G_b, x_0),\ z \neq x_0$. Then

$$\sup_{g\in G_b} \| z-g \| = \| z-x_0 \| + \sup_{g\in G_b} \| x_0-g \| \geq b$$

Therefore,

$$\begin{aligned}\sup_{g\in G} \| z-g \| &= \sup_{g\in G_b} \| z-g \| -b \\ &= \sup_{g\in G_b} \| x_0-g \| + \| z-x_0 \| -b \\ &= \| z-x_0 \| + \sup_{g\in G} \| x_0-g \|\end{aligned}$$

which implies $\mathcal{F}(G_b, x_0) \subseteq \mathcal{F}(G, x_0)$.

Hence the result follows.

Some applications of farthest points are explored in the next section.

4 Applications of Farthest Points in the Spaces $C(Q)$ and $C_R(Q)$.

Theorem 4.1. *Let $E = C(Q)$ (Q compact), G be a bounded subset of $E, x \in E$ and $g_0 \in G$. Then $g_0 \in F_G(x)$ if and only if there exists a Radon measure μ (real or complex) on Q such that*

$$|\mu|(Q) = 1, \tag{4.1}$$

$$\frac{d\mu}{d|\mu|} \in C(Q), \tag{4.2}$$

$$g_0(q) - x(q) = \left[\operatorname{sign}\frac{d\mu}{d|\mu|}(q)\right] \sup_{g\in G}\max_{q\in Q}|g(q)-x(q)| \; q \in S(\mu), \tag{4.3}$$

where (4.2) *is meant in the sense that* $\frac{d\mu}{d|\mu|}$ *can be made continuous on* Q *by changing its values on a set of* $|\mu|$ *- measure zero, in* (4.3) *is taken this continuous function* $\frac{d\mu}{d|\mu|}$ *and* $S(\mu)$ *is the carrier of the measure* μ.

Proof. By theorem 3.1 [4], we have $g_0 \in F_G(x)$ if and only if there exists a Radon measure μ on Q such that we have (4.1) and

$$\int_Q [g_0(q) - x(q)]d\mu(q) = \sup_{g\in G}\max_{q\in Q}|g(q)-x(q)| \tag{4.4}$$

We shall now show that these conditions are equivalent to (4.1)-(4.3). Assume first that we have (4.1) and (4.4). Then from (4.4), (4.1) and $x - g_0 \neq 0$ it follows that

$$\frac{d\mu}{d|\mu|}(q) = \frac{\overline{g_0(q)-x(q)}}{\sup_{g\in G}\max_{q\in Q}|g(q)-x(q)|} \qquad |\mu| - \text{ a.e. on } Q \tag{4.5}$$

Indeed, assume the contrary, that is that there exists a set $A \subset Q$ with $|\mu|(A) > 0$, such that

$$\frac{d\mu}{d|\mu|}(q) \neq \frac{\overline{g_0(q)-x(q)}}{\sup_{g\in G} \| g - x \|} \qquad |\mu| - \text{ a.e. on } A. \tag{4.6}$$

Then

$$\operatorname{Re}\left(\frac{d\mu}{d|\mu|}(q)[g_0(q)-x(q)]\right) < \sup_{g\in G}||g-x|| \qquad |\mu| - \text{ a.e. on } A,$$

since otherwise, by taking into account that we have

$$\left|\frac{d\mu}{d|\mu|}(q)\right| = 1 \qquad |\mu| - \text{ a.e on } Q, \tag{4.7}$$

there would exist a set $A_1 \subset A$ with $|\mu|(A_1) > 0$ such that

$$\begin{aligned} \sup_{g\in G} ||g-x|| &\leq \operatorname{Re}\left(\frac{d\mu}{d|\mu|}(q)[g_0(q)-x(q)]\right) \\ &\leq \left|\frac{d\mu}{d|\mu|}(q)[g_0(q)-x(q)]\right| \\ &\leq ||g_0-x|| \\ &\leq \sup_{g\in G} ||g-x|| \qquad |\mu| - \text{a.e on } A_1, \end{aligned}$$

which implies that $\frac{d\mu}{d|\mu|}(q)[g_0(q)-x(q)]$ is real and positive $|\mu|$ − a.e on A_1, hence equal to $||g_0 - x||$ $|\mu|$ a.e. on A_1, and thus

$$\frac{d\mu}{d|\mu|}(q)[g_0(q)-x(q)] = \sup_{g\in G} ||g-x|| \qquad |\mu| - \text{a.e. on } A_1$$

So

$$\frac{d\mu}{d|\mu|}(q) = \frac{\sup_{g\in G} ||g-x||}{g_0(q)-x(q)} = \frac{\overline{g_0(q)-x(q)}}{\sup_{g\in G} ||g-x||} \qquad |\mu| - \text{a.e on } A_1 \ ,$$

which contradicts the hypothesis. Consequently, we obtain

$$\begin{aligned} \operatorname{Re}\int_Q [g_0(q)-x(q)]d\mu(q) &= \operatorname{Re}\int_Q [g_0(q)-x(q)]\frac{d\mu}{d|\mu|}(q)d|\mu|(q) \\ &= \int_Q \operatorname{Re}\left([g_0(q)-x(q)]\frac{d\mu}{d|\mu|}(q)\right) d|\mu|(q) \\ &< \int_Q \sup_{g\in G} ||g-x|| d|\mu|(q) \\ &= \sup_{g\in G} ||g-x||, \end{aligned}$$

which contradicts (4.4). Hence we obtain (4.5).

By changing the values of $\frac{d\mu}{d|\mu|}$ so as to have (4.5) every where on Q, we will have (4.2), whence, taking into account (4.7), there follow the relations

$$\left|\frac{d\mu}{d|\mu|}(q)\right| = 1 \qquad (q \in S(\mu)), \tag{4.8}$$

and thus, by (4.5) (every where on Q), we obtain (4.3).

Conversely, assume that we have (4.1) - (4.3). Then by (4.3), (4.2), (4.7) and (4.1), it follows that

$$\begin{aligned}
\int_Q [g_0(q) - x(q)] d\mu(q) &= \sup_{g\in G} ||g-x|| \int_Q \left[\operatorname{sign}\frac{d\mu}{d|\mu|}(q)\right] d\mu(q) \\
&= \sup_{g\in G} ||g-x|| \int_Q \left[\operatorname{sign}\frac{d\mu}{d|\mu|}(q)\right] \frac{d\mu}{d|\mu|}(q) d|\mu|(q) \\
&= \sup_{g\in G} ||g-x|| \int_Q \left|\frac{d\mu}{d|\mu|}(q)\right| d|\mu|(q) \\
&= \sup_{g\in G} ||g-x|| \; |\mu|(Q) \\
&= \sup_{g\in G} ||g-x||,
\end{aligned}$$

that is (4.4), which completes the proof.

Theorem 4.2. *Let $E = C_R(Q)$ (Qcompact), G be a bounded subset of $E, x \in E$ and $g_0 \in G$. Then $g_0 \in F_G(x)$ if and only if there exist two disjoint sets $Y^+_{g_0}$ and $Y^-_{g_0}$ closed in Q, and a Radon measure μ on Q, with the following properties:*

$$|\mu|(Q) = 1, \tag{4.9}$$

μ is non-decreasing on $Y^+_{g_0}$, non-increasing on $Y^-_{g_0}$ and

$$S(\mu) \subset Y^+_{g_0} \cup Y^-_{g_0} \qquad (S(\mu) - \text{ the carrier of } \mu), \tag{4.10}$$

$$g_0(q) - x(q) = \begin{cases} \sup_{g\in G} \max_{q\in Q} |g(q) - x(q)| & \text{for } \; q \in Y^+_{g_0} \\ -\sup_{g\in G} \max_{q\in Q} |g(q) - x(q)| & \text{for } \; q \in Y^-_{g_0} \end{cases}. \tag{4.11}$$

Proof. By Theorem 4.1, we have $g_0 \in F_G(x)$ if and only if there exists a real Radon measure μ on Q such that we have (4.1) and (4.4). We shall show that these conditions are equivalent to (4.9) - (4.11).

Assume that we have (4.1) and (4.4). Now we define

$$Y_{g_0}^{+} = \left\{ q \in Q : g_0(q) - x(q) = \sup_{g \in G} \max_{q \in Q} |g(q) - x(q)| \right\} , \tag{4.12}$$

$$Y_{g_0}^{-} = \left\{ q \in Q : g_0(q) - x(q) = -\sup_{g \in G} \max_{q \in Q} |g(q) - x(q)| \right\} . \tag{4.13}$$

Then, $Y_{g_0}^{+}$ and $Y_{g_0}^{-}$ are disjoint and closed in Q and we have (4.11).

To prove (4.10), let μ be decreasing on $Y_{g_0}^{+}$. Then there would exist a set $A \subset Y_{g_0}^{+}$ with $|\mu|(A) > 0$, such that $\mu(A) < |\mu|(A)$. So

$$\begin{aligned} \int_A [g_0(q) - x(q)] d\mu(q) &= \sup_{g \in G} ||g - x|| \mu(A) \\ &< |\mu|(A) \sup_{g \in G} ||g - x|| \\ &= \int_A \sup_{g \in G} ||g - x|| d|\mu|(q), \end{aligned}$$

Whence, taking into account (4.1),

$$\int_Q [g_0(q) - x(q)] d\mu(q) < \int_Q \sup_{g \in G} ||g - x|| d|\mu|(q) = \sup_{g \in G} ||g - x||,$$

which contradicts (4.4). Hence μ is non-decreasing on $Y_{g_0}^{+}$.

Similarly it can be shown that μ is non-increasing on $Y_{g_0}^{-}$.

If there exists a $q_0 \in S(\mu)$ such that $q_0 \notin Y_{g_0}^{+} \cup Y_{g_0}^{-}$, then

$$|g_0(q_0) - x(q_0)| < \sup_{g \in G} ||g - x||,$$

then, taking an open neighbourhood U of q_0 such that

$$|g_0(q_0) - x(q)| < \sup_{g \in G} ||g - x|| \quad (q \in U),$$

We would have $|\mu|(U) > 0$ (since $q_0 \in S(\mu)$) and

$$\begin{aligned} \int_U [g_0(q) - x(q)] d\mu(q) &\leq \int_U |g_0(q) - x(q)| d|\mu|(q) \\ &< \int_U \sup_{g \in G} ||g - x|| d|\mu|(q) \end{aligned}$$

Whence, by (4.1),

$$\begin{aligned}\int_Q [g_0(q)-x(q)]d\mu(q) &< \int_Q \sup_{g\in G}||g-x||d|\mu|(q)\\ &= \sup_{g\in G}||g-x||,\end{aligned}$$

which contradicts (4.4). Thus (4.1) and (4.4) imply (4.9) - (4.11).

Conversely, assume that there exist two disjoint closed sets $Y^+_{g_0}$ and $Y^-_{g_0}$ in Q and a real Radon measure μ on Q such that we have (4.9) - (4.11). Then, by (4.10), (4.11) and (4.9), we have

$$\begin{aligned}\int_Q [g_0(q)-x(q)]d\mu(q) &= \int_{S(\mu)\cap Y^+_{g_0}} \sup_{g\in G}\max_{q\in Q}|g(q)-x(q)|d\mu(q)\\ &\quad + \int_{S(\mu)\cap Y^-_{g_0}} \left(-\sup_{g\in G}\max_{q\in Q}|g(q)-x(q)|\right)d\mu(q)\\ &= \int_Q \sup_{g\in G}\max_{q\in Q}|g(q)-x(q)|d|\mu|(q)\\ &= \sup_{g\in G}\ \max_{q\in Q}|g(q)-x(q)|,\end{aligned}$$

which gives (4.4). Thus (4.9)-(4.11) imply (4.1) and (4.4).

5 Operator F_G and Farthest Approximations

Let E be a normed linear space, G be a nonempty bounded subset of E and $D(F_G)$ denote domain of F_G. Then define a mapping $F_G : D(F_G) \to G$ by $F_G(x) \in \mathcal{F}_G(x)$ ($x \in D(F_G)$). In general $D(F_G) \neq E$.

Theorem 5.1. *Let E be a normed linear space and G be a nonempty bounded subset of E. Then*

(1) $\Big|||x-F_G(x)||-||y-F_G(y)||\Big| \le ||x-y||$

(2) $||x-y|| \le ||x-F_G(x)||+||y-F_G(y)||$

(3) $||x - F_G(x)|| \geq ||x||$

(4) If G_1 *is a nonempty subset of* G, *then* $||x - F_G(x)|| \geq ||x - F_{G_1}(x)||$, *for all* $x \in (D(F_G) \cap D(F_{G_1}))$

Proof. (1) By definition of farthest points, we have

$$\begin{aligned} ||x - F_G(x)|| \geq ||x - F_G(y)|| &= ||x - y + y - F_G(y)|| \\ &\geq ||y - F_G(y)|| - ||x - y|| \end{aligned}$$

$$\Rightarrow ||x - y|| \geq ||y - F_G(y)|| - ||x - F_G(x)||$$

interchanging x and y, we have

$$||x - y|| \geq ||x - F_G(x)|| - ||y - F_G(y)||$$

hence

$$\Big| ||x - F_G(x)|| - ||y - F_G(y)|| \Big| \leq ||x - y||$$

also

(2)
$$\begin{aligned} ||x - F_G(x)|| \geq ||x - F_G(y)|| &= ||x - y + y - F_G(y)|| \\ &\geq ||x - y|| - ||y - F_G(y)|| \end{aligned}$$

$$\Rightarrow ||x - y|| \leq ||x - F_G(x)|| + ||y - F_G(y)||$$

(3) By definition of farthest points, we have

$$||x - F_G(x)|| \geq ||x - g||, \text{ for all } g \in G.$$

In particular

$$||x - F_G(x)|| \geq ||x||, \quad \text{if} \quad 0 \in G$$

(4)
$$\begin{aligned} ||x - F_G(x)|| &= \sup_{g \in G} ||x - g|| \\ &\geq \sup_{g \in G_1} ||x - g||, \text{ since } G_1 \subset G \\ &= ||x - F_{G_1}(x)||, \text{ for all } x \in (D(F_G) \cap D(F_{G_1})) \end{aligned}$$

Theorem 5.2. *Let E be a normed linear space, G be a nonempty bounded subset of E and $g_0 \in G$. Then the set $F_G^{-1}(g_0)$ is closed and $x \in F_G^{-1}(g_0) \Rightarrow \alpha x + (1-\alpha)g_0 \in F_G^{-1}(g_0)$ (α-scalar).*

Proof. Let $x_n \in F_G^{-1}(g_0)$ and $x \in E$ such that

$$\lim_{n\to\infty} x_n = x.$$

Then, since norm is a continuous function and $g_0 \in F_G(x_n)$, we have

$$\begin{array}{llll} & \lim_{n\to\infty} ||x_n - g_0|| & \geq \lim_{n\to\infty} ||x_n - g||, & \text{for all } g \in G \\ \Rightarrow & ||\lim_{n\to\infty}(x_n - g_0)|| & \geq ||\lim_{n\to\infty}(x_n - g)||, & \text{for all } g \in G \\ \Rightarrow & ||x - g_0|| & \geq ||x - g||, & \text{for all } g \in G \\ \Rightarrow & g_0 \in F_G(x) & \Rightarrow x \in F_G^{-1}(g_0) & \end{array}$$

which proves that $F_G^{-1}(g_0)$ is closed.

Now let $x \in F_G^{-1}(g_0)$ be arbitrary and α be an arbitrary scalar and let $y = \alpha x + (1-\alpha)g_0$.

If $\alpha = 0$, then $y = g_0 \in F_G^{-1}(g_0)$. If $\alpha \neq 0$ then for every $g \in G$, by taking into account that $x \in F_G^{-1}(g_0)$,

$$\begin{aligned} ||y - g|| &= ||\alpha x + (1-\alpha)g_0 - g|| \\ &= \alpha||x - (1 - \frac{1}{\alpha})g_0 - g|| \\ &\leq \alpha||x - g_0|| \\ &= ||\alpha x + (1-\alpha)g_0 - g_0|| \\ &= ||y - g_0|| \end{aligned}$$

whence it follows that $g_0 \in \mathcal{F}_G(y)$ if $y \in \mathcal{F}_G^{-1}(g_0)$.

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