General Mathematics Vol. 13, No. 3 (2005), 37–46

Differential Subordination Defined by Sălăgean Operator

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

By using the Sălăgean operator $D^n f(z)$, $z \in U$, we introduce a class of holomorphic functions, denoted by $S^n_\alpha(\beta)$, and we obtain some inclusion relations and differential subordination.

2000 Mathematics Subject Classification: Primary 30C80, Secondary 30C45, 30A20, 34A40.

Keywords and phrases: Holomorphic function, convex function, α-convex function (Mocanu function), dominant.

1 Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. We let $\mathcal{H}[a,n]$ denote the class of analytic functions in U of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \quad z \in U.$$

We let

$$A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots,\ z \in U\}$$

with $A_1 = A$.

For α real, let

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{1}$$

The class of α-convex functions (Mocanu functions) in the unit disc, are defined by

$$M_\alpha = \{f \in A;\ \mathrm{Re}\ J(\alpha, f; z) > 0,\ z \in U\}.$$

If f and g are analytic in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $t \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We use the following subordination results.

Lemma A. (Hallenbeck and Ruscheweyh [2, p.71]) *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex with* $\mathrm{Re}\ \gamma \geq 0$. *If $p \in \mathcal{H}(U)$, with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and is the best dominant.

(The definition of best dominant is given in [1]).

Lemma B. (Miller and Mocanu [1]) *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z),$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = g(0) + p_n z^n + \dots$$

is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec g(z), \quad z \in U,$$

and this results is sharp.

Definition 1. (see [3]) For $f \in A$ and $n \in \mathbb{N}^* \cup \{0\}$ the operator $D^n f$ is defined by

$$D^0 f(z) = f(z)$$

$$D^{n+1} f(z) = z[D^n f(z)]', \quad z \in U.$$

Remark 1. We have

$$D^n f(z) = (K * K * K * ... * K * f)(z)$$

where $*$ stands for convolution, $K(z) = \dfrac{z}{(1-z)^2}$ is the Koebe function and

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, \quad z \in U.$$

2 Main results

Definition 2. We denote

$$J_1(\alpha, f; z) = zJ(\alpha, f; z), \quad z \in U,$$

where $J(\alpha, f; z)$ is given by (1).

Definition 3. If $0 \leq \beta < 1$, $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}^* \cup \{0\}$, let $S_\alpha^n(\beta)$ denote the class of functions $f \in A$ which satisfy the inequality

$$\mathrm{Re}\,[D^n J_1(\alpha, f; z)]' > \beta, \quad z \in U.$$

Theorem 1. *If* $0 \leq \beta < 1$, $\alpha \in \mathbb{R}$ *and* $n \in \mathbb{N}^* \cup \{0\}$, *then*

$$S_\alpha^{n+1}(\beta) \subset S_\alpha^n(\delta),$$

where

$$\delta = \delta(\beta) = 2\beta - 1 + 2(1-\beta)\ln 2.$$

Proof. Let $f \in S_\alpha^{n+1}(\beta)$. By using the properties of the operator $D^n f$ we have

$$D^{n+1} J_1(\alpha, f; z) = z[D^n J_1(\alpha, f; z)]', \quad z \in U. \tag{2}$$

Differentiating (2), we obtain

$$[D^{n+1} J_1(\alpha, f; z)]' = [D^n J_1(\alpha, f; z)]' + z[D^n J_1(\alpha, f; z)]'', \quad z \in U. \tag{3}$$

If we let $p(z) = [D^n J_1(\alpha, f; z)]'$, then (3) becomes

$$[D^{n+1} J_1(\alpha, f; z)]' = p(z) + zp'(z), \quad z \in U. \tag{4}$$

Since $f \in S_{\alpha}^{n+1}(\beta)$, by using Definition 3 we have

$$\text{Re}\,[p(z) + zp'(z)] > \beta, \quad z \in U, \tag{5}$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1 + (2\beta - 1)z}{1 + z} \equiv h(z), \quad z \in U.$$

By using Lemma A, we have

$$p(z) \prec q(z) \prec h(z)$$

i.e.

$$[D^n J_1(\alpha, f; z)]' \prec q(z),$$

where

$$q(z) = \frac{1}{z}\int_0^z \frac{1 + (2\beta - 1)t}{1 + t}dt = 2\beta - 1 + 2(1 - \beta)\frac{\ln(1 + z)}{z}, \quad z \in U.$$

The function q is convex and is the best dominant.

Hence $p(z) \prec q(z)$, $z \in U$, it results that

$$\text{Re}\, p(z) > \text{Re}\, g(1) = 2\beta - 1 + 2(1 - \beta)\ln 2$$

from which we deduce that $S_{\alpha}^{n+1}(\alpha) \subset S_{\alpha}^{n}(\delta)$.

Theorem 2. *Let g be a convex function, $g(0) = 1$, and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in U.$$

If $f \in S_{\alpha}^{n}(\beta)$ and verifies the differential subordination

$$[D^{n+1} J_1(\alpha, f; z)]' \prec h(z), \quad z \in U, \tag{6}$$

then

$$[D^n J_1(\alpha, f; z)]' \prec g(z), \quad z \in U,$$

and this result is sharp.

Proof. By using the properties of the operator $D^n f$ we have

$$D^{n+1} J_1(\alpha, f; z) = z[D^n J_1(\alpha, f; z)]', \quad z \in U,$$

we obtain

$$[D^{n+1} J_1(\alpha, f; z)]' = [D^n J_1(\alpha, f; z)]' + z[D^n J_1(\alpha, f; z)]''.$$

If we let

$$p(z) = [D^n J_1(\alpha, f; z)]',$$

then we obtain

$$[D^{n+1} J_1(\alpha, f; z)]' = p(z) + zp'(z), \quad z \in U,$$

and (6), becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z), \quad z \in U.$$

By using Lemma B, we have

$$p(z) \prec g(z), \quad z \in U$$

i.e.

$$[D^n J_1(\alpha, f; z)]' \prec g(z)$$

and this result is sharp.

Theorem 3. *Let g be a convex function, $g(0) = 1$ and*

$$h(z) = g(z) + zp'(z), \quad z \in U.$$

If $f \in S_\alpha^n(\beta)$ *and verifies the differential subordination*

(7) $$[D^n J_1(\alpha, f; z)]' \prec h(z), \quad z \in U,$$

then

$$\frac{D^n J_1(\alpha, f; z)}{z} \prec g(z), \quad z \in U,$$

and this result is sharp.

Proof. We let

(8) $$p(z) = \frac{D^n J_1(\alpha, f; z)}{z}, \quad z \in U,$$

and we obtain

(9) $$D^n J_1(\alpha, f; z) = zp(z), \quad z \in U.$$

By differentiating (9), we obtain

$$[D^n J_1(\alpha, f; z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (7), becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z) \equiv h(z), \quad z \in U.$$

By using Lemma B, we have

$$p(z) \prec g(z), \quad z \in U,$$

i.e.

$$\frac{D^n J_1(\alpha, f; z)}{z} \prec g(z), \quad z \in U,$$

and this result is sharp.

Theorem 4. *Let* $h \in \mathcal{H}(U)$*, with* $h(0) = 1$*,* $h'(0) \neq 0$*, which verifies the inequality*

$$\text{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $f \in S_\alpha^n(\beta)$ and verifies the differential subordination

$$[D^{n+1}J_1(\alpha, f; z)]' \prec h(z), \quad z \in U, \tag{10}$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [2, Corollary 2.6.g.2, p.66] shows that the function g is convex. From

$$D^{n+1}J_1(\alpha, f; z) = z[D^n J_1(\alpha, f; z)]', \quad z \in U,$$

we obtain

$$[D^{n+1}J_1(\alpha, f; z)]' = [D^n J_1(\alpha, f; z)]' + z[D^n J_1(\alpha, f; z)]'', \quad z \in U.$$

If we let

$$p(z) =]D^n J_1(\alpha, f; z)]',$$

then we obtain

$$[D^{n+1}J_1(\alpha, f; z)]' = p(z) + zp'(z), \quad z \in U.$$

By using Lemma A, we have

$$p(z) \prec g(z) = \frac{1}{z}\int_0^z h(t)dt,$$

i.e.

$$[D^n J_1(\alpha, f; z)]' \prec \frac{1}{z}\int_0^z h(t)dt, \quad z \in U.$$

Theorem 5. *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\text{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $f \in S^n_\alpha(\beta)$ and verifies the differential subordination

$$[D^n J_1(\alpha, f; z)]' \prec h(z), \quad z \in U, \tag{11}$$

then

$$\frac{D^n J_1(\alpha, f; z)}{z} \prec g(z), \quad z \in U,$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [2, Corollary 2.6.g.2, p.66] shows that the function g is convex.

We let

$$p(z) = \frac{D^n J_1(\alpha, f; z)}{z}, \quad z \in U,$$

and we obtain

$$D^n J_1(\alpha, f; z) = zp(z), \quad z \in U.$$

By differentiating, we obtain

$$[D^n J_1(\alpha, f; z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (11) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

By using Lemma A, we have

$$p(z) \prec g(z), \quad z \in U,$$

i.e.

$$\frac{D^n J_1(\alpha, f; z)}{z} \prec g(z), \quad z \in U,$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U,$$

is convex and is the best dominant.

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