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On the diophantine equations of type $a^x + b^y = c^z$

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Dedicated to Professor Emil C. Popa on his 60th birthday

Abstract

In this paper we study some diophantine equations of type $a^x + b^y = c^z$.

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The diophantine equations of type $a^x + b^y = c^z$ have been extensively studied in certian particular cases (see [1] - [6]). For example, for $b > a$ and $\max(a, b, c) > 13$, Z. Cao in [2] and [3] proved that this equation can have at most one solution with $z > 1$.

Another result (see [6]) says that if a, b, c are not powers of two, then the diophantine equations $a^x + b^y = c^z$ can have at most a finite number of solutions.

The aim of this paper is to find elementary solutions for some diophantine equations of this type.

1 The equation of type $p^x + p^y = p^z$, where p is prime number.

If $p = 2$ and $x = y < z$, then the diophantine equation becomes $2^{x+1} = 2^z$, where we get $z = x + 1$, $x \in \mathbb{N}$. Therefore, in this case we have the solutions $(k, k, k + 1)$, k natural number.

For $x < y < z$ we have $2^x(1 + 2^{y-x}) = 2^z$, that is $1 + 2^{y-x} = 2^{z-x}$, contradiction, since the left side is $\equiv 1 \pmod 2$ and the right side is $\equiv 0 \pmod 2$.

If $p \geq 3$, then $p^x + p^y$ is even number and p^z is odd number, hence the diophantine equation has no solutions.

In conclusion, we have:

Theorem 1. *If $p = 2$ the diophantine equation $2^x + 2^y = 2^z$ has the solutions $(x, y, z) = (k, k, k + 1)$, $k \in \mathbb{N}$.*

If $p \geq 3$ the diophantine equation

$$p^x + p^y = p^z \tag{1}$$

has no solutions.

2 The equation of type (1) $p^x + p^y = (2p)^z$, where p is prime number

We consider nine cases

2.1 $x = y$. The diophantine equation becomes

$$2p^x = 2^z \cdot p^z. \tag{2}$$

If $p \neq 2$, then we obtain $z = 1$ and $x = z$, that is we have the solution $(x, y, z) = (1, 1, 1)$.

If $p = 2$, then the equation (2) takes the form $2^x = 2^{2z-1}$, where $x = 2z-1$. Then, we find the solutions $(x, y, z,) = (2k-1, 2k-1, k)$, $k \in \mathbb{N}/\{0\}$.

2.2 $x = z$**.** The diophantine equation (1) takes the form

$$p^y = p^x(2^x - 1). \tag{3}$$

If $p = 2$ and $x > 1$, then the equation (3) is impossible because p^y is even number and $2^x - 1$ is an odd number. For $p = 2$ and $x = 1$ we obtain the solution $(x, y, z) = (1, 1, 1)$. If $p \geq 3$, then from (3) it results $2^x - 1 = p^t$, $t \in \mathbb{N} - \{0\}$. This equation has the solution only for $t = 1$ ([6]) and p is prime Mersenne number. For $p = 2^a - 1 = M_a$, $a \in \mathbb{N} - \{0\}$, where M_a is prime Mersenne number, the diophantine equation (3) has the solution $(x, y, z) = (a, a + 1, a)$.

Examples 2.2.1. For $p = 3$ we have $p = 2^2 - 1 = M_2$, hence the diophantine equation $3^x + 3^y = 6^z$ has the solution $(x, y, z) = (2, 3, 2)$ ([4]).

2.2.2 $p = 7$**.** For $p = 7$ we have $p = 2^3 - 1 = M_3$, hence the diophantine equation $7^x + 7^y = 14^z$ has the solution $(x, y, z) = (3, 4, 3)$.

2.2.3. $p = 31 = 2^5 - 1 = M_5$**.** For $p = 31 = 2^5 - 1 = M_5$, we find the solution $(x, y, z) = (5, 6, 5)$ for the diophantine equation $31^x + 31^y = 62^z$.

2.3. $y = z$**.** Using the symmetry of the equation in x and y, it follows that this case is similar to the case 2.2.

2.4. $x < y < z$**.** The diophantine equation (1) is equivalent to

$$p^x(1 + p^{y-x}) = 2^z \cdot p^z$$

or

$$1 + p^{y-x} = 2^z \cdot p^{z-x}$$

Hence $1 + p^{y-x} \equiv 1 \pmod{\text{p}}$ and $2^z \cdot p^{z-x} \equiv 0 \pmod{\text{p}}$, it results the equation has no solutions in this case.

2.5. $y < x < z$**.** This case is analogous with 2.4.

2.6. $y < z < x$**.** The equation (1) is equivalent to

$$p^y(p^{x-y}+1) = 2^z \cdot p^z$$

or

$$p^{x-y}+1 = 2^z \cdot p^{z-y}$$

which is impossible.

2.7. $x < z < y$**.** This case is similar to 2.6.

2.8. $z < x < y$**.** The equation (1) is equivalent to $p^{x-z}+p^{y-z} = 2^z$ or

$$p^{x-z}(1+p^{y-z}) = 2^z. \tag{4}$$

For $p \geq 3$ we have $p^{x-z}(1+p^{y-x}) \equiv 0$ (mod p) and $2^z \not\equiv 0$ (mod p), thus the equation (4) is impossible.

For $p = 2$ we have $2^{x-z}(1+2^{y-x}) = 2^z$ which is impossible hence $1+2^{y-x}$ is on odd number and 2^z is an even number.

2.9. $z < y < x$**.** This is analogous with 2.8.

In fine we proved:

Theorem 2. *i) For every p prime, the diophantine equation (1) has the solution* $(x, y, z) = (1, 1, 1)$

ii) For $p = 2$ *the diophantine equation (1) has the solutions* $(x, y, z) = (2k-1, 2k-1, k)$*,* $k \in \mathbb{N} - \{0\}$*.*

iii) For $p = 2^a - 1 = M_a$*, a integer positive,* $a \geq 2$*, and* M_a *prime Mersenne's number, the equation has the solutions* $(x, y, z) = (a, a+1, a)$ *and* $(x, y, z) = (a+1, a, a)$*.*

3 The diophantine equation (5) $p^x+q^y = (pq)^z$, with p and q two given primes.

We distinguish five cases.

3.1. $x = 0$**.** The given equation becames

(6) $$1 + q^y = (pq)^z.$$

If $y \geq 1$ and $z \geq 1$, then (6) is impossible because $1 + q^y \equiv 1$ (mod q) and $(pq)^z \equiv 0$ (mod q).

If $y = 0$, then (6) is equivalent to $2 = (pq)^z$, which is impossible.

If $z = 0$, then from (6) we obtain $q^y = 0$, which is impossible.

3.2. $y = 0$**.** This case is similar with the case 3.1.

3.3. $z = 0$**.** The diophantine equation $p^x + q^y = 1$ has no solutions in natural numbers.

Now, we consider $x \geq 1, y \geq 1, z \geq 1$.

3.4. $1 \leq x \leq z$**.** The equation (5) is equivalent to

(7) $$q^y = p^x(p^{z-x} \cdot q^z - 1).$$

If $p \neq q$, then (7) is impossible.

If $p = q$ and $x \neq y$, then (7) is also impossible.

If $p = q$ and $x = y$, the equation (7) takes the form $p^{2z-x} = 2$, which it is possible only if $p = 2$ and $2z - x = 1$.

It follows that for $p = q = 2$ the equation (5) has the solutions $(x, y, z) = (2k - 1, 2k - 1, k)$, k arbitrary positive integer.

3.5. $x > z \geq 1$**.** We write the equation (5) under the form

(8) $$q^y = p^z(q^z - p^{x-z}).$$

For $p \neq q$ the equation (8) is impossible.

If $p = q$, then from (8) it results

(9) $$p^y = p^z(p^z - p^{x-z}).$$

The diophantine equation (9) is possible only if $z > x - z$, that is $x < 2z$. Then the equation (9) is equivalent to

$$p^y = p^{2z}(1 - p^{x-2z}),$$

which is impossible.

As a conclusion we obtained:

Theorem 3. *The diophantine equation (5) has solutions only if $p = q = 2$. These are $(x, y, z) = (2k - 1, 2k - 1, k)$, k arbitrary positive integer.*

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