General Mathematics Vol. 12, No. 4 (2004), 23–28

Smooth dependence on parameters for some functional-differential equations

Olaru Ion Marian

Abstract

I this paper we study the smooth dependence on parameters for the following equation:

$$x'(t,\lambda) = f(t, x(t,\lambda), x(g(t),\lambda),\lambda),\ t \in [a,b]$$

by using theorem of fibre contraction.

2000 Mathematical Subject Classification: 34K10, 47H10

Keywords: Picard operators, weakly Picard operators, fixed points, data dependence

1 Introduction

The purpose of this paper is to study the following problem:

$$x'(t,\lambda) = f(t, x(t,\lambda), x(g(t),\lambda),\lambda),\ t \in [a,b] \tag{1}$$

$$x(t,\lambda) = \varphi(t,\lambda),\ t \in [a_1,a], \lambda \in J \tag{2}$$

where

(H_0) $a_1 \leq a < b$, $J \subset \mathbb{R}$, a compact interval.

(H_1) $g \in (C[a,b],[a_1,b]), \varphi \in C^1([a_1,a] \times J), f \in C^1([a,b] \times \mathbb{R}^{2n} \times J)$.

(H_2) Exists $L_f > 0$ such that $\left|\left|\dfrac{\partial f}{\partial u_i}(t,u_1,u_2,\lambda)\right|\right|_{\mathbb{R}^n} \leq L_f$, for all $t \in [a,b]$, $u_i \in \mathbb{R}^n$, $i = \overline{1,2}$, $\lambda \in J$.

$(H_3) 2L_f(b-a) < 1$

We note $X = C([a_1,b] \times J)$.

The problem (1) + (2) is equivalent with

$$(3) \qquad x(t,\lambda) = \begin{cases} \varphi(a,\lambda) + \int\limits_a^t f(s,x(s,\lambda),x(g(s),\lambda))ds, & t \in [a,b] \\ \varphi(t,\lambda), & t \in [a_1,a] \end{cases}$$

We consider the following operator: $B : X \longrightarrow X$,

$$B(x)(t,\lambda) = \begin{cases} \varphi(a,\lambda) + \int\limits_a^t f(s,x(s,\lambda),x(g(s),\lambda))ds, & t \in [a,b] \\ \varphi(t,\lambda) & ,t \in [a_1,a] \end{cases}$$

We need the theorem of fiber contractions(see [3]).

Theorem 1. *Let (X,d) and (Y,ρ) be two metric space and*

$$A : X \times Y \to X \times Y, A(x,y) = (B(x), C(x,y)).$$

We suppose that

(i)(Y,ρ) is a complete metric space .

(ii) The operator $B : X \to X$ is weakly Picard operator.

(iii)There exists $a \in [0,1)$ such that $C(x,\cdot)$ is an a-contraction,for all $x \in X$.

(iv)If $(x^*,y^*) \in F_A$ then $C(\cdot,y^*)$ is continuous in x^*.

Then A is weakly Picard operator.If B is Picard operator then A is a Picard operator.

2 Main results

Differentiability with respect the parameters was studied in [2],[4].

For $x \in C[a_1, b] \times J$ we have:

$$|x|_C = \max_{(t,\lambda)\in[a_1,b]\times J} ||f(t,\lambda)||_{\mathbb{R}^n}.$$

$(X, |\cdot|_C)$ has a structure of Banach space.

Proposition 1. *We suppose that the conditions* $(H_0), (H_1), (H_2), (H_3),$ *are satisfied. Then:*

i) the problem (1) + (2) has an unique solutions $x^* \in X$

ii) $x^*(t, \cdot) \in C^1(J),$ *for all* $t \in [a_1, b_1]$.

Proof. From

$||B(x)(t,\lambda) - B(y)(t,\lambda)||_{\mathbb{R}^n} \leq$

$$\leq \int_a^t ||f(s, x(s,\lambda), x(g(s),\lambda), \lambda) - f(s, y(s,\lambda), y(g(s),\lambda), \lambda)||_{\mathbb{R}^n} ds \leq$$

$$\leq L_f \int_a^t ||x(s,\lambda) - y(s,\lambda)||_{\mathbb{R}^n} + ||x(g(s),\lambda) - y(g(s),\lambda)||_{\mathbb{R}^n} ds \leq$$

$$\leq 2L_f(b-a)||x-y||_C$$

we have that the B is Picard operator.It follow that there exists a unique solution $x^*(t,\lambda) \in X$.

We have

$$(4) \quad x^*(t,\lambda) = \begin{cases} \varphi(a,\lambda) + \int_a^t f(s, x^*(s,\lambda), x^*(g(s),\lambda), \lambda) ds, & t \in [a,b] \\ \varphi(t,\lambda) & , t \in [a_1, a] \end{cases}$$

We consider the operator $C : X \times X \to X$.

$$C(x,y)(t,\lambda) = \begin{cases} \dfrac{\partial \varphi}{\partial \lambda}(a,\lambda) + \int\limits_a^t \dfrac{\partial f}{\partial u_1}((s,x(s),\lambda),x(g(s),\lambda),\lambda)\cdot y(s,\lambda)ds + \\ + \int\limits_a^t \dfrac{\partial f}{\partial u_2}(s,x(s,\lambda),x(g(s),\lambda),\lambda)y(g(s),\lambda)ds + \\ + \int\limits_a^t \dfrac{\partial f}{\partial \lambda}(s,x(s,\lambda),x(g(s),\lambda),\lambda)ds,\ t\in[a,b] \\ \dfrac{\partial \varphi}{\partial \lambda}(t,\lambda),\ t\in[a_1,a] \end{cases}$$

We show that $C(x,\cdot) : X \to X$ is a contraction.

$$||C(x,y)(t,\lambda) - C(x,z)(t,\lambda)||_{\mathbb{R}^n} \le$$

$$\le L_f \int_a^t \left(||y(s,\lambda) - z(s,\lambda)||_{\mathbb{R}^n} + ||y(g(s),\lambda) - z(g(s),\lambda)||_{\mathbb{R}^n}\right) ds \le$$

$$\le 2L_f(b-a)||y-z||_C$$

From fiber contractions theorem we have that the operator $A : X\times X \to X \times X$,

$$A(x,y) = (B(x), C(x,y))$$

is a Picard operator.

So, the sequences

$$x_{n+1} = B(x_n),\ n\in\mathbb{N}$$

$$y_{n+1} = C(x_n,y_n),\ n\in\mathbb{N}$$

converges uniformly (with respect to $t\in[a_1,b], \lambda\in J$) to $(x^*,y^*)\in F_A$, for all $x_0, y_0 \in X$.

If we take $x_0 = 0, y_0 = \dfrac{\partial x_0}{\partial \lambda} = 0$, then

$$x_1(t,\lambda) = B(x_0)(t,\lambda) = \begin{cases} \varphi(a,\lambda) + \int\limits_a^t f(s,0,0,\lambda)ds,\ t\in[a,b] \\ \varphi(t,\lambda) \qquad\qquad\qquad\qquad ,\ t\in[a_1,a] \end{cases}$$

$$y_1(t,\lambda) = C(x_0, y_0)(t,\lambda) =$$

$$= \begin{cases} \dfrac{\partial \varphi}{\partial \lambda}(a,\lambda) + \int\limits_a^t \dfrac{\partial f}{\partial \lambda}(s,0,0,\lambda)ds, \ t \in [a,b] \\ \dfrac{\partial \varphi}{\partial \lambda}(t,\lambda), \ t \in [a_1, a] \end{cases} = \dfrac{\partial x_1}{\partial \lambda}(t,\lambda).$$

We suppose that $y_n(t,\lambda) = \dfrac{\partial x_n}{\partial \lambda}(t,\lambda)$.

We show that $y_{n+1}(t,\lambda) = \dfrac{\partial x_{n+1}}{\partial \lambda}(t,\lambda)$.

$$x_{n+1}(t,\lambda) = B(x_n)(t,\lambda) =$$

$$= \begin{cases} \varphi(a,\lambda) + \int\limits_a^t f(s, x_n(s,\lambda), x_n(g(s),\lambda))ds, \ t \in [a,b] \\ \varphi(t,\lambda) \qquad , \ t \in [a_1, a] \end{cases}$$

$$y_{n+1}(t,\lambda) = C(x_n, y_n)(t,\lambda) =$$

$$= \begin{cases} \dfrac{\partial \varphi}{\partial \lambda}(a,\lambda) + \int\limits_a^t \dfrac{\partial f}{\partial u_1}(s, x_n(s,\lambda), x_n(g(s),\lambda),\lambda) y_n(s,\lambda)ds + \\ + \int\limits_a^t \dfrac{\partial f}{\partial u_2}(s, x_n(s,\lambda), x_n(g(s),\lambda),\lambda) y_n(s,\lambda)ds + \\ + \int\limits_a^t \dfrac{\partial f}{\partial \lambda}(s, x_n(s,\lambda), x_n(g(s),\lambda),\lambda)ds, \ t \in [a,b] \\ \dfrac{\partial \varphi}{\partial \lambda}(t,\lambda) \qquad , t \in [a_1, a] \end{cases} =$$

$$= \begin{cases} \dfrac{\partial \varphi}{\partial \lambda}(a,\lambda) + \int\limits_a^t \dfrac{\partial f}{\partial u_1}(s, x_n(s,\lambda), x_n(g(s),\lambda),\lambda) \dfrac{\partial x_n}{\partial \lambda}(s,\lambda)ds + \\ + \int\limits_a^t \dfrac{\partial f}{\partial u_2}(s, x_n(s,\lambda), x_n(g(s),\lambda),\lambda) \dfrac{\partial x_n}{\partial \lambda}(s,\lambda)ds + \\ + \int\limits_a^t \dfrac{\partial f}{\partial \lambda}(s, x_n(s,\lambda), x_n(g(s),\lambda),\lambda)ds, \ t \in [a,b] \\ \dfrac{\partial \varphi}{\partial \lambda}(t,\lambda), \ t \in [a, a_1] \end{cases} =$$

$$= \frac{\partial x_{n+1}}{\partial \lambda}(t,\lambda).$$

Thus,

$$x_n \longrightarrow x^* \text{ as } n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \longrightarrow y^* \text{ as } n \to \infty.$$

These imply that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} = y^*$.

References

[1] L. Ladeira,J. K. Hale, *Differentiability with respect to delays*, J. Differential equations, 92, 1991,14-26.

[2] I. A. Rus, *Functional-Differential equation of mixed point*, via weakly Picard operator Seminar on mixed point theory Cluj-Napoca, vol. 3, 2002,335-345.

[3] I. A. Rus, *Weakly Picard operators and applications*, Seminar on fixed point theory Cluj-Napoca, vol. 2, 2001,41-57.

[4] J.Sotomayor,*Smooth dependence of solutions of differential equations on initial data:a simple proof*, Bul. Soc. Bras. Mat., 1973,55-59

"Lucian Blaga" University of Sibiu
Faculty of Sciences
Department of Mathematics
Str. Dr. I. Raţiu, no. 5-7
550012 - Sibiu, Romania
E-mail: *olaruim@yahoo.com*