General Mathematics Vol. 12, No. 4 (2004), 39–48

Certain preserving properties of the generalized Alexander operator

Mugur Acu

Abstract

In this paper we give certain preserving properties of the generalized Alexander operator on some subclasses of n-uniformly functions.

2000 Mathematical Subject Classification: 30C45

1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in A : f \text{ is univalent in } U\}$.

In [8] the subfamily T of S consisting of functions f of the form

$$(1) \qquad f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0, j = 2, 3, ...,\ z \in U$$

was introduced.

Let define the Alexander operator $I^p : A \to \mathcal{H}(U)$

$$I^0 f(z) = f(z)$$
$$I^1 f(z) = If(z) = \int_0^z \frac{f(t)}{t} dt$$
$$I^p f(z) = I(I^{p-1} f(z)), \quad p = 1, 2, 3, ...$$

We have for $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$,

$$I^p f(z) = z + \sum_{j=2}^{\infty} \frac{1}{j^p} a_j z^j, p = 1, 2, 3, ...$$

Now we can define the generalized Alexander operator

$$I^\lambda : A \to \mathcal{H}(U), \quad I^\lambda f(z) = z + \sum_{j=2}^{\infty} \frac{1}{j^\lambda} a_j z^j, \tag{2}$$

with $\lambda \in \mathbb{R}, \ \lambda \geq 0$, where $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$.

The purpose of this paper is to show that the class of n-uniform starlike functions of type α and order γ with negative coefficients, the class of n-uniform close to convex functions of type α and order γ with negative coefficients and some subclasses of functions with negative coefficients, which derive from the above mentioned classes, are preserved by the generalized Alexander operator.

2 Preliminary results

Let D^n be the Sălăgean differential operator (see [6]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

Remark 2.1 *If* $f \in T, f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j = 2, 3, ...,\ z \in U$ *then* $D^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j$.

Theorem 2.1 *[6] If* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0,\ j = 2, 3, ...,\ z \in U$ *then the next assertions are equivalent:*

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, *where* $T^* = T \bigcap S^*$ *and* S^* *is the well-known class of starlike functions.*

Definition 2.1 *[6] Let* $\gamma \in [0, 1)$ *and* $n \in \mathbb{N}$, *then*

$$S_n(\gamma) = \left\{ f \in A : Re \frac{D^{n+1} f(z)}{D^n f(z)} > \gamma\, ,\ z \in U \right\}$$

is the set of n-starlike functions of order γ.

We denote by $T_n(\gamma)$ the subclass $T \bigcap S_n(\gamma)$.

Definition 2.2 *[3] Let* $f \in A$. *We say that* f *is n-close to convex of order* γ *with respect to a half-plane, and denote by* $CC_n(\gamma)$ *the set of these functions, if there exists* $g \in S_n(0)$ *so that*

$$Re \frac{D^{n+1} f(z)}{D^n g(z)} > \gamma,\ \ z \in U,$$

where $n \in \mathbb{N}, \gamma \in [0, 1)$.

Definition 2.3 *[1] Let* $f \in T$*,* $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$*,* $a_j \geq 0$*,* $j = 2, 3, ...,$ $z \in U$*. We say that* f *is in the class* $CCT_n(\gamma), \gamma \in [0, 1)$*,* $n \in \mathbb{N}$*, with respect to the function* $g \in T_n(0)$*, if:*

$$Re \frac{D^{n+1} f}{D^n g} > \gamma\,, \;\; z \in U.$$

Theorem 2.2 *[1] Let* $\gamma \in [0, 1)$ *and* $n \in \mathbb{N}$*. The function* $f \in T$ *of the form (1) is in* $CCT_n(\gamma)$*, with respect to the function* $g \in T_n(0)$*,* $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$*,* $b_j \geq 0$*,* $j = 2, 3, ...,$ *if and only if*

(3)
$$\sum_{j=2}^{\infty} j^n [j a_j + (2 - \alpha) b_j] < 1 - \gamma\,.$$

Definition 2.4 *[4] Let* $f \in A$*, we say that* f *is n-uniform starlike function of type* α *if*

$$Re \left(\frac{D^{n+1} f(z)}{D^n f(z)} \right) \geq \alpha \cdot \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right|, z \in U$$

where $\alpha \geq 0, n \in \mathbb{N}$*. We denote this class with* $US_n(\alpha)$*.*

We denote by $UT_n(\alpha)$ *the subclass* $T \bigcap US_n(\alpha)$*.*

Definition 2.5 *[3] Let* $f \in A$*, we say that* f *is n-uniform close to convex function of type* α *in respect to the function n-uniform starlike of type* α $g(z)$*, where* $\alpha \geq 0$*,* $n \in \mathbb{N}$*, if*

$$Re \left(\frac{D^{n+1} f(z)}{D^n g(z)} \right) \geq \alpha \cdot \left| \frac{D^{n+1} f(z)}{D^n g(z)} - 1 \right|, z \in U$$

where $\alpha \geq 0, n \in \mathbb{N}$*. We denote this class with* $UCC_n(\alpha)$*.*

Definition 2.6 *[5] Let $f \in A$, we say that f is n-uniform starlike function of order γ and type α if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| + \gamma, z \in U$$

where $\alpha \geq 0, \gamma \in [-1,1), \alpha + \gamma \geq 0$, $n \in \mathbb{N}$. We denote this class with $US_n(\alpha,\gamma)$.

We denote by $UT_n(\alpha,\gamma)$ the subclass $T \bigcap US_n(\alpha,\gamma)$.

Theorem 2.3 *[5] Let $\alpha \geq 0, \gamma \in [-1,1)$, $\alpha + \gamma \geq 0$ and $n \in \mathbb{N}$. The function f of the form (1) is in $UT_n(\alpha,\gamma)$ if and only if*

$$\sum_{j=2}^{\infty} j^n[(\alpha+1)j - (\alpha+\gamma)]a_j \leq 1-\gamma.$$

Definition 2.7 *[3] Let $f \in A$, we say that f is n-uniform close to convex function of order γ and type α in respect to the function n-uniform starlike of order γ and type α, $g(\alpha)$, where $\alpha \geq 0$, $\gamma \in [-1,1)$, $\alpha+\gamma \geq 0$, $n \in \mathbb{N}$, if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n g(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right| + \gamma, z \in U$$

where $\alpha \geq 0$, $\gamma \in [-1,1)$, $\alpha + \gamma \geq 0$, $n \in \mathbb{N}$. We denote this class with $UCC_n(\alpha,\gamma)$.

Definition 2.8 *[2] Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2,3,...$, $z \in U$. We say that f is in the class $UCCT_n(\alpha), \alpha \geq 0$, $n \in \mathbb{N}$, with respect to the function $g(z) \in UT_n(\alpha)$ $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2,3,...$, $z \in U$, if:*

$$Re\left(\frac{D^{n+1}f(z)}{D^n g(z)}\right) > \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right| z \in U.$$

Definition 2.9 *[2] Let* $f \in T, f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ..., z \in U$. *We say that f is in the class* $UCCT_n(\alpha, \gamma)$, $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0, n \in \mathbb{N}$, *with respect to the function* $g(z) \in UT_n(\alpha, \gamma)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, ..., z \in U$, *if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n g(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right| + \gamma,\ z \in U.$$

Theorem 2.4 *[2] Let* $n \in \mathbb{N}, \alpha \geq 0, \gamma \in [-1, 1)$, *with* $\alpha + \gamma \geq 0$. *The function* $f \in T$ *of the form (1) is in* $UCCT_n(\alpha, \gamma)$, *with respect to the function* $g \in UT_n(\alpha, \gamma)$, $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, ..., z \in U$, *if and only if*

$$\sum_{j=2}^{\infty} j^n[(\alpha + 1)|ja_j - b_j| + (1 - \gamma)b_j] \leq 1 - \gamma\,. \tag{4}$$

3 Main results

Theorem 3.1 *If* $F \in UT_n(\alpha, \gamma)$, $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, $n \in \mathbb{N}$ *and* $f = I^\lambda(F)$, *where* I^λ *is defined by (2), then* $f \in UT_n(\alpha, \gamma)$, $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, $n \in \mathbb{N}$.

Proof. From $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$ and $f(z) = I^\lambda(F)(z)$ we have:

$$f(z) = z - \sum_{j=2}^{\infty} \alpha_j z^j,\ \text{where } \alpha_j = \frac{1}{j^\lambda} a_j \geq 0,\ j = 2, 3, ...$$

From Theorem 2.3 we need only to show that:

$$\sum_{j=2}^{\infty} j^n[(\alpha + 1)j - (\alpha + \gamma)]\alpha_j \leq 1 - \gamma\,. \tag{5}$$

For $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $a_j \geq 0$ and $j = 2, 3, \dots$, we have:

$$\alpha_j = \frac{1}{j^\lambda} a_j \leq a_j$$

and thus

$$\sum_{j=2}^{\infty} j^n [(\alpha+1)j - (\alpha+\gamma)]\alpha_j \leq \sum_{j=2}^{\infty} j^n [(\alpha+1)j - (\alpha+\gamma)]a_j \tag{6}$$

where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$ and $n \in \mathbb{N}$.

From $F \in UT_n(\alpha, \gamma)$, using Theorem 2.3, we have

$$\sum_{j=2}^{\infty} j^n [(\alpha+1)j - (\alpha+\gamma)]a_j \leq 1 - \gamma$$

and thus from (6) we obtain the condition (5).

Theorem 3.2 *If $F \in UCCT_n(\alpha, \gamma)$, $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, $n \in \mathbb{N}$, with respect to the function $G \in UT_n(\alpha, \gamma)$ and $f = I^\lambda(F)$, $g = I^\lambda(G)$, where I^λ is defined by (2), then $f \in UCCT_n(\alpha, \gamma)$, $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, $n \in \mathbb{N}$, with respect to the function $g \in UT_n(\alpha, \gamma)$.*

Proof. From the above Theorem we have $g \in UT_n(\alpha, \gamma)$.

For $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \dots$ and $G(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $b_j \geq 0$, $j = 2, 3, \dots$ we have $f(z) = I^\lambda(F)(z) = z - \sum_{j=2}^{\infty} \alpha_j z^j$, where $\alpha_j = \frac{1}{j^\lambda} a_j \geq 0$, $j = 2, 3, \dots$ and $g(z) = I^\lambda(G)(z) = z - \sum_{j=2}^{\infty} \beta_j z^j$, where $\beta_j = \frac{1}{j^\lambda} b_j \geq 0$, $j = 2, 3, \dots$.

From Theorem 2.4 we need only to show that:

$$\sum_{j=2}^{\infty} j^n [(\alpha+1)\,|j\alpha_j - \beta_j| + (1-\gamma)\beta_j] \leq 1 - \gamma\,. \tag{7}$$

It is easy to see that

$$(8)\ (\alpha+1)\,|j\alpha_j-\beta_j|+(1-\gamma)\beta_j=\frac{1}{j^{\lambda}}\,[(\alpha+1)\,|ja_j-b_j|+(1-\gamma)b_j]\le$$

$$\le(\alpha+1)\,|ja_j-b_j|+(1-\gamma)b_j$$

for $\lambda\in\mathbb{R},\ \lambda\ge 0,\ a_j\ge 0,\ b_j\ge 0,\ j=2,3,\ldots,\ \alpha\ge 0,\ \gamma\in[-1,1)$ and $\alpha+\gamma\ge 0$.

From $F\in UCCT_n(\alpha,\gamma)$, with respect to the function $G\in UT_n(\alpha,\gamma)$, we have (see Theorem 2.4):

$$\sum_{j=2}^{\infty}j^n[(\alpha+1)\,|ja_j-b_j|+(1-\gamma)b_j]\le 1-\gamma$$

and thus from (8) we obtain the condition (7).

If we take $\gamma=0$ in Theorem 3.1 and Theorem 3.2 we obtain:

Theorem 3.3 *If $F\in UT_n(\alpha)$, $\alpha\ge 0$, $n\in\mathbb{N}$ and $f=I^{\lambda}(F)$, where I^{λ} is defined by (2), then $f\in UT_n(\alpha)$, $\alpha\ge 0$, $n\in\mathbb{N}$.*

Theorem 3.4 *If $F\in UCCT_n(\alpha)$, $\alpha\ge 0$, $n\in\mathbb{N}$, with respect to the function $G\in UT_n(\alpha)$ and $f=I^{\lambda}(F)$, $g=I^{\lambda}(G)$, where I^{λ} is defined by (2), then $f\in UCCT_n(\alpha)$, $\alpha\ge 0$, $n\in\mathbb{N}$, with respect to the function $g\in UT_n(\alpha)$.*

If we take $\gamma\in[0,1)$ and $\alpha=0$ in Definition 2.6 we have $UT_n(0,\gamma)=T_n(\gamma)$ and thus from Theorem 3.1, with $\gamma\in[0,1)$ and $\alpha=0$, we obtain:

Theorem 3.5 *If $F\in T_n(\gamma)$, $\gamma\in[0,1)$, $n\in\mathbb{N}$ and $f=I^{\lambda}(F)$, where I^{λ} is defined by (2), then $f\in T_n(\gamma)$, $\gamma\in[0,1)$, $n\in\mathbb{N}$.*

Remark 3.1 *From Theorem 3.2 with* $\gamma \in [0, 1)$ *and* $\alpha = 0$*, we obtain the preserving property of the generalized Alexander operator on the subclass* $UCCT_n(0, \gamma)$*,* $\gamma \in [0, 1)$*, which is not the same with the class* $CCT_n(\gamma)$ *.*

In a similarly way with the proof of the Theorem 3.2, using the condition (3) instead of the condition (4), we obtain:

Theorem 3.6 *If* $F \in CCT_n(\gamma)$*,* $\gamma \in [0, 1)$*,* $n \in \mathbb{N}$*, with respect to the function* $G \in T_n(0)$ *and* $f = I^\lambda(F)$*,* $g = I^\lambda(G)$*, where* I^λ *is defined by (2), then* $f \in CCT_n(\gamma)$*,* $\gamma \in [0, 1)$*,* $n \in \mathbb{N}$*, with respect to the function* $g \in T_n(0)$ *.*

References

[1] M. Acu, *On a subclass of functions with negative coefficients*, General Mathematics, Vol. 11(2003), Nr. 1 - 2, 57-66.

[2] M. Acu, *A subclass of uniformly close to convex functions with negative coefficients*, Analele Univ. Oradea, Fascicola Matematica, Tom X(2003), 57-68.

[3] D. Blezu, *On the n-uniform close to convex functions with respect to a convex domain*, General Mathematics, Vol. 9, Nr. 3 - 4, 3 - 14.

[4] I. Magdaş, *On* α*-type uniformly convex functions*, Studia Mathematica, Vol. XLIV, Nr. 1 (1999), 11 - 17.

[5] I. Magdaş, *A new subclass of uniformly convex functions with negative coefficients*, Studia Mathematica (to appear)

[6] G. S. Sălăgean, *Geometria Planului Complex*, Ed. Promedia Plus, Cluj-Napoca, 1997.

[7] G. S. Sălăgean, *On some classes of univalent functions*, Seminar of geometric function theory, Cluj-Napoca, 1983.

[8] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., 5 (1975), 109 - 116.

University "Lucian Blaga" of Sibiu
Department of Mathematics
Str. Dr. I. Raţiu, Nr. 5–7,
550012 - Sibiu, Romania.
E-mail address: *acu_mugur@yahoo.com*