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Some preserving properties of the generalized Alexander operator

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

The aim of this note is to show that the class of uniform starlike functions with negative coefficients, the class of uniform convex functions with negative coefficients and the class of uniform convex functions of type α and order γ with negative coefficients are preserved by the generalized Alexander integral operator. More, using some results from a previously paper, we show that the Alexander integral operator preserve the class of n-uniform starlike functions of order γ and type α and the class of n-uniform close to convex functions of order γ and type α.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of all analytic functions in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1\}$ and $S = \{f \in A : f \in \mathcal{H}_u(U)\}$ where $\mathcal{H}_u(U)$ is the set of analytic and univalent functions in the unit disc U.

Let define the Alexander operator $I^p : A \to \mathcal{H}(U)$

$$I^0 f(z) = f(z)$$

(1)
$$I^1 f(z) = If(z) = \int_0^z \frac{f(t)}{t} dt$$

$$I^p f(z) = I(I^{p-1} f(z)), \quad p = 1, 2, 3, ...$$

We have for $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$,

$$I^p f(z) = z + \sum_{k=2}^{\infty} \frac{1}{k^p} a_k z^k, p = 1, 2, 3, ...$$

Now we can define the generalized Alexander operator

(2)
$$I^\lambda : A \to \mathcal{H}(U), \quad I^\lambda f(z) = z + \sum_{k=2}^{\infty} \frac{1}{k^\lambda} a_k z^k,$$

with $\lambda \in \mathbb{R},\ \lambda \geq 0$, where $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$.

The purpose of this paper is to show that the class of uniform starlike functions with negative coefficients, the class of uniform convex functions with negative coefficients and the class of uniform convex functions of type α and order γ with negative coefficients are preserved by the generalized Alexander integral operator. More, using some results from a previously paper, we show that the Alexander integral operator preserve the class of n-uniform starlike functions of order γ and type α and the class of n-uniform close to convex functions of order γ and type α.

2 Preliminary results:

Definition 2.1. *Let T be the set of all functions $f \in S$ having the form:*

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0, \ k = 2, 3, ..., \quad z \in U.$$

Theorem 2.1. ([8]). *If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0, \ k = 2, 3, ..., \quad z \in U$ then the next assertions are equivalent:*

(i) $\sum_{k=2}^{\infty} k \cdot a_k \leq 1$;

(ii) $f \in T$.

(iii) $f \in T^*$, where $T^* = T \bigcap S^*$ and S^* is the well known class of starlike functions.

Definition 2.2. *A function f is said to be uniformly starlike in the unit disc U if f is starlike and has the property that for every circular arc γ contained in U, with center α also in U, the arc f(γ) is starlike with respect to $f(\alpha)$. We let US^* denote the class of all such functions. By taking $UT^* = T \cap US^*$ we define the class of uniformly starlike functions with negative coefficients.*

Remark 2.1. *An arc f(γ) is starlike with respect to a point $w_0 = f(\alpha)$ if* $\arg(f(z) - w_0)$ *is nondecreasing as z traces γ in the positive direction.*

Theorem 2.2. ([5]). *Let $f \in S, f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. If $\sum_{k=2}^{\infty} \sqrt{2} \cdot k \cdot |a_k| \leq 1$ then $f \in US^*$.*

Definition 2.3. *A function f is uniformly convex in the unit disc U if $f \in S^C$ (convex) and for every circular arc γ contained in U, with center α also in U, the arc $f(\gamma)$ is also convex. We let US^C denote the class of all such functions. By taking $UT^C = T \cap US^C$ we define the class of uniformly convex functions with negative coefficients.*

Remark 2.2. *The arc Γ (t), $a < t < b$ is convex if the argument of the tangent to Γ (t) is nondecreasing with respect to t.*

Theorem 2.3. **([10]).** *Let $f \in T$, $f(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$, $a_k \geq 0, k \geq 2$. Then $f \in UT^C$ if and only if:*

$$\sum_{k=2}^{\infty} k(2k-1) \cdot a_k \leq 1.$$

Theorem 2.4. **([10]).** *$UT^* \subset T^*$ where $T^* = T \cap S^*$ and S^* is the class of functions $f \in S$ which are starlike.*

Remark 2.3. *In [6] is showed that the Libera integral operator $L : A \to A$ defined as $Lf(z) = \frac{2}{z} \int\limits_0^z f(t)dt, z \in U$ preserve the class UT^*. In [1] is showed that the Alexander operator $If(z) = \int\limits_0^z \frac{f(t)}{t} dt$, $z \in U$ preserve both the classes UT^* and UT^C, and the Bernardy integral operator $B_a : A \to \mathcal{H}(U)$ defined as $B_a f(z) = \frac{1+a}{z^a} \int\limits_0^z f(t) \cdot t^{a-1} dt$, $a = 1, 2, 3, ...,$ $z \in U$ preserve the class UT^C.*

Definition 2.4. *A function f is said to be uniformly convex of type α and order γ, $\alpha \geq 0, \gamma \in [-1, 1), \alpha + \gamma \geq 0$ if:*

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \alpha \left| \frac{zf''(z)}{f'(z)} \right| + \gamma, \quad z \in U.$$

We denote by $US^C(\alpha, \gamma)$ the class of all such functions.

By taking $UT^C(\alpha,\gamma) = T \cap US^C(\alpha,\gamma)$ *we define the class of uniformly convex functions of type* α *and order* γ *with negative coefficients.*

Theorem 2.5. ([7]). *Let* $f \in T, f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j, a_j \geq 0, j = 2, 3, ...$ $\alpha \geq 0, \gamma \in [-1,1), \alpha+\gamma \geq 0$. *Then* $f \in UT^C(\alpha,\gamma)$ *if and only if:*

$$\sum_{j=2}^{\infty} j[(\alpha+1)j - (\alpha+\gamma)]a_j \leq 1-\gamma.$$

Definition 2.5. *Let* D^n *be the Sălăgean differential operator defined as:*

$$D^n : A \to A,\ n \in \mathbb{N} \text{ and}$$

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)).$$

Definition 2.6. *Let* $f \in A$, *we say that f is n-uniform starlike function of order* γ *and type* α *if:*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| + \gamma,\ z \in U$$

where $\alpha \geq 0, \gamma \in [-1,1), \alpha+\gamma \geq 0, n \in \mathbb{N}$. *We denote this class with* $US_n(\alpha,\gamma)$.

Remark 2.4. *Geometric interpretation:* $f \in US_n(\alpha,\gamma)$ *if and only if* $\dfrac{D^{n+1}f(z)}{D^n f(z)}$ *take all values in the convex domain included in right half plane* $\triangle\alpha,\gamma$, *where* $\triangle\alpha,\gamma$ *is an eliptic region for* $\alpha > 1$, *a parabolic region for* $\alpha = 1$, *a hiperbolic region for* $0 < \alpha < 1$, *the half plane* $u > \gamma$ *for* $\alpha = 0$.

Definition 2.7. *Let $f \in A$, we say that f is n-uniform close to convex function of order γ and type α with respect to the function n-uniform starlike of order γ and type α $g(z)$, where $\alpha \geq 0, \gamma \in [-1, 1), \alpha + \gamma \geq 0$ if:*

$$Re \left(\frac{D^{n+1} f(z)}{D^n g(z)} \right) \geq \alpha \cdot \left| \frac{D^{n+1} f(z)}{D^n g(z)} - 1 \right| + \gamma, \ \ z \in U$$

where $\alpha \geq 0, \gamma \in [-1, 1), \alpha + \gamma \geq 0, n \in \mathbb{N}$. We denote this class with $UCC_n(\alpha, \gamma)$.

Remark 2.5. *Geometric interpretation: $f \in UCC_n(\alpha, \gamma)$ if and only if $\dfrac{D^{n+1} f(z)}{D^n g(z)}$ take all values in the convex domain included in right half plane $\triangle\alpha, \gamma$, where $\triangle\alpha, \gamma$ is a eliptic region for $\alpha > 1$, a parabolic region for $\alpha = 1$, a hiperbolic region for $0 < \alpha < 1$, the half plane $u > \gamma$ for $\alpha = 0$.*

Definition 2.8. *Let define the integral operator $L_a : A \to \mathcal{H}(U)$ as: $f(z) = L_a F(z) = \frac{1+a}{z^a} \int\limits_0^z F(t) t^{a-1} dt, a \in \mathbb{C}, Re\ a \geq 0$.*

Theorem 2.6. ([2]). *If $F(z) \in US_n(\alpha, \gamma)$, with $\alpha \geq 0$ and $\gamma > 0$, then $f(z) = L_a F(z) \in US_n(\alpha, \gamma)$ with $\alpha \geq 0$ and $\gamma > 0$.*

Theorem 2.7. ([2]). *If $F(z) \in UCC_n(\alpha, \gamma)$, with respect to the function n-uniform starlike of order γ and type α $G(z)$, with $\alpha \geq 0$ and $\gamma > 0$, then $f(z) = L_a F(z) \in UCC_n(\alpha, \gamma)$ with respect to the function n-uniform starlike of order γ and type α, $g(z) = L_a G(z)$ with $\alpha \geq 0$ and $\gamma > 0$.*

3 Main results

Theorem 3.1. *If $F(z) \in UT^*$ and I^λ is the generalized Alexander operator defined by (2) with $\lambda \geq 1$ then $f(z) = I^\lambda F(z) \in UT^*$.*

Proof. From $F(z) \in T, F(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$, $a_k \geq 0$, $k \geq 2$, we have:

$$\sum_{k=2}^{\infty} k \cdot a_k \leq 1. \tag{3}$$

From (2) we have:

$$f(z) = I^{\lambda} F(z) = z - \sum_{k=2}^{\infty} b_k z^k$$

where $b_k = \dfrac{1}{k^{\lambda}} a_k$, $k \geq 2$, $\lambda \in \mathbb{R}$, $\lambda \geq 1$.

We observe that $\dfrac{1}{k^{\lambda}} a_k \geq 0$, $k \geq 2$ and thus $f(z) \in T$.

Now, we have to prove that $f(z)$ is in US^*.

From Theorem 2.2 we obtain that is sufficient to prove that $\sum\limits_{k=2}^{\infty} \sqrt{2} k |b_k| \leq 1$.

We have $\sqrt{2} k |b_k| = \sqrt{2} k \dfrac{1}{k^{\lambda}} a_k = \dfrac{\sqrt{2}}{k} k \left(\dfrac{1}{k}\right)^{\lambda - 1} a_k$.

Because $\dfrac{\sqrt{2}}{k} \in (0, 1)$ and $\left(\dfrac{1}{k}\right)^{\lambda - 1} \in (0, 1]$, $k \geq 2$, $\lambda \in \mathbb{R}$, $\lambda \geq 1$, we have: $\sqrt{2} k |b_k| \leq k a_k$.

Using (3) we obtain: $\sum\limits_{k=2}^{\infty} \sqrt{2} k |b_k| \leq 1$ and thus $f(z) \in US^*$.

Theorem 3.2. *If $F(z) \in UT^c$ and I^{λ} is the generalized Alexander operator defined by (2) then $f(z) = I^{\lambda} F(z) \in UT^c$.*

Proof. From $F(z) \in UT^c, F(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$, $a_k \geq 0$, $k \geq 2$ we have (see Theorem 2.3):

$$\sum_{k=2}^{\infty} k(2k - 1) a_k \leq 1. \tag{4}$$

From (2) we have:

$$f(z) = I^{\lambda} F(z) = z - \sum_{k=2}^{\infty} b_k z^k,$$

where $b_k = \dfrac{1}{k^\lambda} a_k, k \geq 2, \lambda \in \mathbb{R}, \lambda > 0$.

We observe that $\left(\dfrac{1}{k}\right)^\lambda \cdot a_k \geq 0$, $k \geq 2$ and thus $f(z) \in T$.

By using Theorem 2.3 it is sufficient to prove that $\sum\limits_{k=2}^{\infty} k(2k-1) \cdot |b_k| \leq 1$.

Because $\left(\dfrac{1}{k}\right)^\lambda \in (0,1]$, $k = 2, 3, ...,$ $\lambda \in \mathbb{R}$, $\lambda \geq 0$ we have $k(2k-1)\left(\frac{1}{k}\right)^\lambda \cdot |a_k| \leq k \cdot (2k-1)|a_k|$ and from (4) we obtain $\sum\limits_{k=2}^{\infty} k(2k-1)|b_k| \leq 1$, and thus $f \in UT^C$.

In a similar way we show the next theorem:

Theorem 3.3. *The generalized Alexander integral operator defined by (2) preserves the class $UT^C(\alpha, \gamma)$, with $\alpha \geq 0, \gamma \in [-1, 1), \alpha + \gamma \geq 0$, that is:*

If $F(z) \in UT^C(\alpha, \gamma)$, then $f(z) = I^\lambda F(z) \in UT^C(\alpha, \gamma)$.

If we take a= 0 in theorem 2.6 and theorem 2.7 we obtain the next two theorem:

Theorem 3.4. *If $F(z) \in US_n(\alpha, \gamma)$, with $\alpha \geq 0$ and $\gamma > 0$ then $f(z) = IF(z) \in US_n(\alpha, \gamma)$ with $\alpha \geq 0$ and $\gamma > 0$, where I is defined in (1).*

Theorem 3.5. *If $F(z) \in UCC_n(\alpha, \gamma)$, with respect to the function n-uniform starlike of order γ and type α $G(z)$, with $\alpha \geq 0$ and $\gamma > 0$, then $f(z) = I\ F(z) \in UCC_n(\alpha, \gamma)$ with respect to the function n-uniform starlike of order γ and type α, $g(z) = IG(z)$ with $\alpha \geq 0$ and $\gamma > 0$, where I is defined in (1).*

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