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A variational approach to spline functions theory

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

Spline functions have proved to be very useful in numerical analysis, in numerical treatment of differential, integral and partial differential equations, in statistics, and have found applications in science, engineering, economics, biology, medicine, etc. It is well known that interpolating polynomial splines can be derived as the solution of certain variational problems. This paper presents a variational approach to spline interpolation. By considering quite general variational problems in abstract Hilbert spaces setting, we derive the concept of "**abstract splines**". The aim of this paper is to present a sequence of theorems and results starting with Holladay's classical results concerning the variational property of natural cubic splines and culminating in some general variational approach in abstract splines results.

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1 Introduction

It is more than 50 years since I. J. Schoenberg ([45], 1946) introduced "spline functions" to the mathematical literature. Since then, splines, have proved to be enormously important in brance of mathematics such as approximation theory, numerical analysis, numerical treatment of differential, integral and partial differential equations, and statistics. Also, they have become useful tools in field of applications, especially CAGD in manufacturing, in animation, in tomography, even in surgery.

Our aim is to draw attention to a variational approach to spline functions and to underline how a beautiful theory has evolved from a simple classical interpolation problem. As we will show, the variational approach gives a new way of thinking about splines and opens up directions for theoretical developments and new applications.

Despite of so many results, this topics is not mentioned in many relevant texts on numerical analysis or approximation theory: even books on splines tend to mention the variational approach only tangentially or not at all.

Even though, there are recently published a few papers which underline the variational aspects of splines, and we mention the papers of Champion, Lenard and Mills ([17], 2000, [16], 1996) and of Beshaev and Vasilenko ([11], 1993).

The plan of this paper contains the following sections:

1. Preliminaries, definitions and usual notations.
2. Development of variational approach to splines.
3. Abstract splines.
4. Conclusion and comments.

The theorems and results of increasing generality or complexity which culminate in some general and elegant abstract results are not necessarily chronological.

2 Preliminaries

Notations:

$\mathbb{R}$ – the set of real numbers

$I : [a, b] \subset \mathbb{R}$

$\mathcal{P}_m := \{p \in \mathbb{R} \to \mathbb{R},\ p$ is real polynomial of degree $\leq m,\ m \in \mathbb{N}\}$

$H^m(I) := \{x : I \to \mathbb{R},\ x^{(m-1)}$ abs. cont. on $I,\ x^{(m)} \in L^2(I)$, $m \in \mathbb{N},$ given$\}$

If we define an inner product on $H^m(I)$ by

$$(x_1, x_2) := \int_I \sum_{j=0}^{m} x_1^{(j)}(t) x_2^{(j)}(t) dt$$

then $H^m(I)$ becomes a Hilbert space.

If X is a linear space, then θ_X will denote the zero element of X.

Definition 1. *Let $a = t_0 < t_1 < \ldots < t_n < t_{n+1} = b$ be a partition of I. The function $s : I \to \mathbb{R}$ is a polynomial spline of degree m with respect to this partition if*

- $s \in C^{m-1}(I)$
- *for each $i \in \{0, 1, \ldots, n\}$, $s|_{[t_i, t_{i+j}]} \in \mathcal{P}_m$*

The interior points $\{t_1, t_2, \ldots, t_n\}$ are known as "knots".

Natural cubic splines

Suppose that $t_1 < t_2 < \ldots < t_n$ and $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$ are given. The classical problem of interpolation is to find a "nice" function Φ which interpolates the data point (t_i, z_i), $1 \leq i \leq n$, that is:

$$\Phi(t_i) = z_i, \quad 1 \leq i \leq n$$

Classical approaches developed by Lagrange, Hermite, Cauchy and others rely on choosing Φ to be some suitable **polynomial**. But are there better functions for solving this interpolation problem? The first answer to this question can be found in a result which was proved by Holladay [27] in 1957.

Theorem 1. (Holladay, 1957) *If*

- $X := H^2(I)$,
- $a \leq t_1 < \ldots < t_n \leq b$; $n \geq 2$,
- $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$, *and*
- $I_n := \{x \in X : \; x(t_i) = z_i, \; 1 \leq i \leq n\}$,

then exists a unique $\sigma \in I_n$ *such that*

(1) $$\int_I [\sigma^{(2)}(t)]^2 dt = \min \left\{ \int_I [x^{(2)}(t)]^2 dt : \; x \in I_n \right\}$$

Furthermore,

- $\sigma \in C^2(I)$,
- $\sigma|_{[t_i, t_{i+1}]} \in \mathcal{P}_3$ *for* $1 \leq i \leq n-1$,

- $\sigma|_{[a,t_1]} \in \mathcal{P}_1$ *and* $\sigma|_{[t_n,b]} \in \mathcal{P}_1$.

From (1) we conclude that σ is an optimal interpolating function – "optimal", in the sense that it minimize the functional $\int_I [x^{(2)}(t)]^2 dt$ over all functions in I_n. The theorem goes on to state that σ *is a cubic spline function in the meaning of Schoenberg definition (1946).* As σ is linear outside $[t_1, t_n]$ it is called "*natural cubic spline*".

So, in a technical sense, we have found functions which are better than polynomials for solving the interpolation problem. Holladay's theorem is most surprising not only because its proof is quite elementary, relying on nothing more complicated than integration by parts, but it shows the intrinsec aspect of splines as solution of a variational problem (1) that has been a starting point to develop a variational approach to splines.

It is natural to ask: " *Why would one choose to minimize* $\int_I [x^{(2)}(t)]^2 dt$*?*"

For three reasons:

i) The curvature of function σ is $\sigma^{(2)}/(1+\sigma'^2)^{3/2}$ and so the natural cubic spline is the best in the sense that it approximates the interpolating function with minimum total curvature if σ' is small.

ii) The second justification is that the natural cubic spline approximates the solution of a problem in physics, in which a uniform, thin, elastic, linear bar is deformed to interpolate the knots specified in absence of external forces. This shape of such a bar is governed by a minimum energy in this case minimum elastic potential energy. The first order approximation to this energy is proportional to the functional (1). Hence the term natural spline is borrowed the term "spline" from the drafting instrument also known as a spline.

iii) When presented with a set of data points (t_i, z_i), $1 \le i \le n$, a statistician can find a regression line which is the line of best fit in the least squares sense. This line is close to the data points Holladay's theorem shows that σ minimizes $\int_I [x^{(2)}(t)]^2 dt$ while still interpolating the data. We could say that σ is an interpolating function which is "close to a straight lines" in that it minimizes this integral.

Thus, linear regression gives us

a straight line passing close to the points

whereas Holladay's result gives a curve σ which is

close to a straight line but passing through the points.

3 More splines

As we shall see, the Holladay's theorem was the starting point in developing the variational approach to splines. In what follows we shall describe a few of the many important generalizations and extensions of Holladay's theorem.

D^m*-splines*

The next step was taken in 1963 by Carl de Boor [13] with the following result.

Theorem 2. (C. de Boor, 1963) *If*

- $X := H^m(I)$,

- $a \leq t_1 < t_2 < \ldots < t_n \leq b;\ n \geq m,$
- $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$ *and*
- $I_n := \{x \in X :\ x(t_i) = z_i,\ 1 \leq i \leq n\}$

then exists a unique $\sigma \in I_n$ *such that*

$$\int_I [\sigma^{(m)}(t)]^2 dt = \min \left\{ \int_I [x^{(m)}(t)]^2 dt :\ x \in I_n \right\}$$

Furthermore,

- $\sigma \in C^{2m-2}(I),$
- $\sigma|_{[t_i, t_{i+1}]} \in \mathcal{P}_{2m-1},\ 1 \leq i \leq n-1,$ *and*
- $\sigma|_{[a,t_1]} \in \mathcal{P}_{m-1}$ *and* $\sigma|_{[t_n,b]} \in \mathcal{P}_{m-1}.$

The function σ was called D^m**-spline** because it minimizes $\int_I (D^m x)^2 dt$, as x varies over I_n. The function σ is called the interpolating **natural spline function of odd degree**.

Clearly if we let $m = 2$ in de Boor result, then we obtain Holladay result. For the even degree splines, such result was given by P. Blaga and G. Micula in 1993 [38].

Trigonometric splines

In 1964, Schoenberg [46] changed the setting of the interpolation problem from the interval $[a, b]$ to the unit circle: that is, from a non-periodic setting to a periodic setting.

Similarly, let $H^k_{2\pi}([0,2\pi))$ denote the following space of 2π-periodic functions:

$$H^k_{2\pi}([0,2\pi)) := \{x : [0,2\pi) \to \mathbb{R} : \ x - 2\pi - \text{periodic},$$

$$x^{(k-1)} \text{ abs. cont. on } [0,2\pi), \ x^{(k)} \in L^2_{2\pi}([0,2\pi))\}.$$

Theorem 3. (Schoenberg, 1964) *If*

- $X := H^{2m+1}_{2\pi}([0,2\pi))$
- $0 \le t_1 < t_2 < \ldots < t_n < 2\pi, \ n > 2m+1$
- $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$ *and*
- $T : X \to L^2_{2\pi}([0,2\pi))$, *where* $T := D(D^2+1^2)\ldots(D^2+m^2)$,

then exists a unique $\sigma \in I_n$ *such that*

$$\int_0^{2\pi} [T(\sigma)(t)]^2 dt = \min\left\{ \int_0^{2\pi} [T(x)(t)]^2 dt : \ x \in I_n \right\}.$$

The optimal interpolating function σ is called the **trigonometric spline**. Schoenberg defined a trigonometric spline as a smooth function which in a particular piecewise trigonometric polynomial manner. He shows that trigonometric splines, so defined, provide the solution of this variational problem.

Note that the differential operator T has as $KerT$ all the trigonometric polynomials of order m, that is, of the form:

$$x(t) = a_0 + \sum_{j=1}^{m} (a_j \cos jt + b_j \sin jt).$$

g-splines

Just over 200 years ago in 1870 Lagrange has constructed the polynomial of minimal degree such that the polynomial assumed prescribed values at given nodes and the derivatives of certain orders of the polynomial also assumed prescribed values at the nodes.

In 1968, Schoenberg [47] extended the idea of Hermite for splines. To specify that the orders of the derivatives specified may vary from node to node we introduce an **incidence matrix** E. As usual, let $I := [a, b]$ be an interval partitioned by the nodes $a \leq t_1 < t_2 < \ldots < t_n \leq b$. Let l be the maximum of the orders of the derivatives to be specified at the nodes. The incidence matrix E is defined by:

$$E := (e(i,j) : \ 1 \leq i \leq n, \ 0 \leq j \leq l) =: (e(i,j))$$

where each $e(i,j)$ is 0 or 1. Assume also that each row of E and the last column of E contain a 1.

Definition 2. *If $m \geq 1$ is an integer, we will say that the incidence matrix $E = (e(i,j))$ is m-poised with respect to $t_1 < t_2 < \ldots < t_n$ if*

- $P \in \mathcal{P}_{m-1}$ *and*
- $e(i,j) = 1 \Rightarrow P^{(j)}(t_i) = 0$

together imply that $P \equiv 0$.

Now we can state Schoenberg's result.

Theorem 4. (Schoenberg, 1968) *If*

- $X := H^m(I)$

- $a \le t_1 < t_2 < \ldots < t_n \le b$
- *E is an m-poised incidence matrix of dimension* $n \times (l+1)$
- $l < m \le \sum_i \sum_j e(i,j)$
- $\{z_{ij}: \ e(i,j) = 1\} \subset \mathbb{R}$ *and*
- $I_n := \{x \in X: \ x^{(j)}(t_i) = z_{ij} \ \textit{if} \ e(i,j) = 1\}$

then exists a unique $\sigma \in I_n$ such that

$$\int_I [\sigma^{(m)}(t)]^2 dt = \min\left\{\int_I [x^{(m)}(t)]^2 dt : \ x \in I_n\right\}$$

Schoenberg called the function σ as g-spline from "generalized-splines". Better may have been H-splines after Hermite or HB-splines after Hermite and Birkhoff.

Again, Schoenberg has defined g-splines as smooth piecewise polynomials where the smoothness is governed by E and then he proved that g-splines solves the above variational problem.

L-Splines

In 1967, Schultz and Varga [48] gave a major extension of the D^m-splines. Instead of the m-order derivative, operator D^m they considered a linear differential operator L creating a theory of so called L-splines. We shall state only one simple consequence of the many results of Schultz and Varga.

Theorem 5. (Schultz and Varga, 1967)

- $X := H^m(I)$
- $a \le t_1 < t_2 < \ldots < t_n \le b;\ n \ge m$
- $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$
- $I_n := \{x \in X :\ x(t_i) = z_i,\ 1 \le i \le n\}$
- $L : X \to L^2(I)$, *so that* $L[x](t) := \sum_{j=0}^{m} a_j(t) D^j x(t)$, *where* $a_j \in C^j(I)$, $0 \le j \le m$, *and exists* $\omega > 0$ *such that* $a_m(t) \ge \omega > 0$ *on* I *and*
- L *has Pólya's property* W *on* I

then exists a unique $\sigma \in I_n$ *such that*

$$\int_I [L[\sigma](t)]^2 dt = \min\left\{ \int_I [L[x](t)]^2 dt :\ x \in I_n \right\}$$

Clearly complexity is increasing with generality.

We note that L has Pólya's property W on I if $L[x] = 0$ has m solutions $x_1, x_2, \ldots, x_m$ such that, for all $t \in I$ and for all $k \in \{1, 2, \ldots, m\}$

$$\det \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_m(t) \\ Dx_1(t) & Dx_2(t) & \ldots & Dx_m(t) \\ \ldots & \ldots & \ldots & \ldots \\ D^{k-1}x_1(t) & D^{k-1}x_2(t) & \ldots & D^{k-1}x_m(t) \end{bmatrix} \ne 0.$$

The relevance of Pólya's property W is contained in the following sentence. To say that L has Pólya's property W on I implies that, if $L[x] = 0$ and x has m or more zeros on I, then $x \equiv 0$.

The optimal function σ is known as an L-**spline**.

If $L \equiv D^m$ we obtain the D^m-spline: so this is a major extension of previously stated results.

Schultz and Varga have defined an L-spline to be a smooth function constructed in a piecewise manner, where each piece is a solution of the differential equation $L^*Lx = 0$ where L^* is the formal adjoint of the operator L.

A consequence of their paper is that L-spline provide a solution of the above variational problem.

Remark.

- The result of Schultz and Varga was proved in 1964 by Ahlberg, Nilson and Walsh [2]. They called σ a "generalized splines".

- The above result also follows from a paper of de Boor and Lynch [15] published in 1966.

- Perhaps the first paper along these lines of replacing the operator D^m by a more general differential operator was given by Greville [25] also in 1964. Unfortunately this often cited technical report was never published. Greville illustrates his method with an application to the classical numerical problem of interpolating mortality tables. Schultz and Varga applied their ideas to the numerical analysis of nonlinear two-point boundary value problems.

- Prenter [42] and Micula [39] are two of the few text books which touch this topic.

Lg-Splines

Schoenberg extended the concept of D^m-splines to allow interpolation conditions of the Hermite type: this leads to g-splines. Schultz and Varga (and others) extended the concept of D^m-spline in a different direction by replacing the differential operator D^m by a more general operator: this leads to L-splines. The question is if one could combine both these extensions. In 1969 Jerome and Schumaker [31] combined these two extensions together in a very effective manner. One of their results is the following:

Theorem 6. (Jerome and Schumaker, 1969) *If*

- $X := H^m(I)$
- $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is a set of linearly independent, continuous linear functionals on* X
- $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$
- $I_n := \{x \in X : \ \lambda_i(x) = z_i, \ 1 \leq i \leq n\}$
- $L : X \to L^2(I)$ *so that*

  $$L[x](t) = \sum_{j=0}^{m} a_j(t) D^j x(t),$$

  $a_j \in C^j(I), \ 0 \leq j \leq m,$ *and exists* $\omega > 0$ *such that* $a_m(t) \geq \omega > 0$ *on* I *and*
- $ker L \cap \{x \in X : \ \lambda_i(x) = 0, \ 1 \leq i \leq n\} = \{\theta_X\}$

then exists a unique $\sigma \in I_n$ *such that*

$$\int_I [L[\sigma](t)]^2 dt = \min \left\{ \int_I [L[x](t)]^2 dt : \ x \in I_n \right\}.$$

The optimal function σ is called the Lg-**spline**. The hypothesis about Pólya's property W in Theorem 5 has with the more functional-analytic flavour. Jerome and Schumaker allow interpolation conditions for the more general form $\lambda_i(x) = z_i$, $1 \le i \le n$, where λ_i $(1 \le i \le n)$ are continuous linear functionals on X. This idea could cover also others conditions like $\int_{t_i}^{t_{i+1}} x(t)dt = z_i$, $1 \le i \le n$. We note also that they replace the conditions $\lambda_i(x) = z_i$ by $\underline{z}_i \le \lambda_i(x) \le \overline{z}_i$, where $\underline{z}_i$ and $\overline{z}_i$ $(i = 1, 2, \ldots, n)$ are given real numbers with $\underline{z}_i \le \overline{z}_i$.

pLg-Splines

For $1 < p < \infty$ we define the space $H^m(I^p)$ of functions by:

$$H^{m,p}(I) := \{x : I \to \mathbb{R} :\ x^{(m-1)} \text{ abs. cont.}, x^{(m)} \in L^p(I)\}$$

With a norm on $H^{m,p}(I)$ defined by:

$$\|x\|_{m,p} := \sum_{j=0}^{m} |x^{(j)}(a)| + \left(\int_I |x^{(m)}(t)|^p dt \right)^{1/p}$$

the $H^{m,p}(I)$ is a Hilbert space.

In 1978 Copley and Schumaker [12] established the following result:

Theorem 7. (Copley and Schumaker, 1978) *If*

- $X := H^{m,p}(I),\ p > 1$
- $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is a set of linearly independent continuous linear functionals on* X
- $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$

- $I_n := \{x \in X : \ \lambda_i(x) = z_i, \ 1 \leq i \leq n\} \neq \emptyset$

- $L : X \to L^p(I)$ *so that*

$$L[x](t) = \sum_{j=0}^{m} a_j(t) D^j x(t),$$

  $a_j \in C^j(I)$, $0 \leq j \leq m$ *and exists* $\omega > 0$ *such that* $a_m(t) \geq \omega > 0$ *on* I, *and*

- $kerL \cap \{x \in X : \ \lambda_i(x) = 0, \ 1 \leq i \leq n\} = \{\theta_X\}$

then exists a unique $\sigma \in I_n$ *such that:*

$$\int_I |L[\sigma](t)|^p dt = \min \left\{ \int_I |L[x](t)|^p dt : \ x \in I_n \right\}.$$

The optimal function σ is called a **pLg-spline**. For the first time, in this paper Copley and Schumaker have defined a pLg-spline to be a solution of the variational interpolation problem. One of the main problems that they investigated is to determine the structure of such splines. Can they be constructed in a piecewise manner? The complexity of their answer compensates the simplicity of their definition on a pLg-spline. In fact, Copley and Schumaker investigated more general interpolation problems. For example, they consider sets of linear functionals $\{\lambda_\alpha : \ \alpha \in A\}$ where the index set A may be infinite, and also many extremly important examples.

Vector-valued Lg-Splines

The following extension have come from researches in electrical engineering. In 1979 Sidhu and Weinert [49] consider the problem of simultaneous interpolation, that is, a method by which one could interpolate several functions at once.

Theorem 8. (Sidhu and Weinert, 1979)

- $r \geq 1,\ n_1 \geq 0, \ldots, n_r \geq 0$ *are fixed integers*
- $X := H^{n_1}(I) \times H^{n_2}(I) \times \ldots \times H^{n_r}(I)$
- $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is a set of linearly independent continuous linear functionals on* X
- $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$
- $I_n := \{x \in X :\ \lambda_i(x) = z_i,\ 1 \leq i \leq n\}$
- $L : X \to L^2(I) \times \ldots \times L^2(I)$ *(an* r*-fold product), where*

$$L[x](t) := \left(\sum_{j=1}^{r} L_{ij}[x_j](t) :\ i = 1, 2, \ldots, r \right)',$$

$$L_{ij} := \sum_{k=0}^{n_j} a_{ijk}(t) D^k;\ a_{ijn_j} = \delta_{ij};\ a_{ijk} \in C^k(I),\ 0 \leq k \leq n_j,\ \text{and}$$

- $ker L \cap \{x \in X :\ \lambda_i(x) = 0,\ 1 \leq i \leq n\} = \{\theta_X\}$

then exists a unique $\sigma \in X$ *such that:*

$$\int_I (L[\sigma](t))' L[\sigma](t) dt = \min \left\{ \int_I (L[x](t))' L[x](t) dt :\ x \in I_n \right\}.$$

(Here A' *indicates the transpose of the matrix or vector* A*.)*

The optimal interpolating vector σ is known as a **vector-valued Lg-spline**. The authors have defined a vector-valued Lg-spline to be the solution of a variational interpolation problem, proved the existence-uniqueness theorem and then discussed an algorithm for calculating such splines in the special case that the functional λ_i are of extended Hermite-Birkhoff type.

Thin plate splines

So far we have been considering the problem of interpolating functions of a single variable. In 1976, Jean Duchon [20] developed a variational approach to interpolating functions of several variables. We will state his result only for functions of two variables. We denote an arbitrary element of $\mathbb{R}^2$ by $t = (\xi_1, \xi_2)$, $\|t\|^2 := \xi_1^2 + \xi_2^2$ and the set of linear polynomials by:

$$\mathcal{P}_1 := \{p_1(t) = a_0 + a_1\xi_1 + a_2\xi_2 : \ \{a_0, a_1, a_2\} \subset \mathbb{R}\}$$

Theorem 9. (Duchon, 1976) *If*

- $X := H^2(\mathbb{R}^2)$,
- $\{t_1, t_2, \ldots, t_n\} \subset \mathbb{R}^2$ *such that if* $p_1 \in \mathcal{P}_1$ *and* $p_1(t_1) = \ldots = p_1(t_n) = 0$, *then* $p_1 \equiv 0$,
- $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{R}$,
- $I_n := \{x \in X : \ x(t_i) = z_i, \ 1 \le i \le n\}$ *and*
- $J : X \to \mathbb{R}$ *such that*

$$J(x) := \iint\limits_{\mathbb{R}^2} \left[\left(\frac{\partial^2 x}{\partial \xi_1^2} \right) + 2 \left(\frac{\partial^2 x}{\partial \xi_1 \partial \xi_2} \right)^2 + \left(\frac{\partial^2 x}{\partial \xi_2^2} \right) \right] d\xi_1 d\xi_2$$

then exists a unique $\sigma \in I_n$ *such that*

$$J(\sigma) = \min\{J(x) : \ x \in I_n\}.$$

Furthermore, for all $t \in \mathbb{R}^2$

$$\sigma(t) = \sum_{j=1}^{n} \mu_i \|t - t_i\|^2 \ln \|t - t_i\| + p_1(t)$$

where $p_1 \in \mathcal{P}_1$ *and for all* $q \in \mathcal{P}_1$), $\left(\sum_{i=1}^{n} \mu_i q(t_i) = 0 \right)$.

The optimal function σ is known as a "**thin plate spline**". The dramatic aspect of this result is the form of the spline σ: it is a piecewise polynomial function.

This two-dimensional result appeared almost 20 years after Holladay's one-dimensional result. The delay is not so surprising. Holladay's proof involves nothing more complicated than integration by parts whereas Duchon's paper uses tempered distribution, Radon measure and other tools from functional analysis.

Remarks.

i) A more elementary approach to Duchon's result is outlined in Powell [41].

ii) Duchon was not the first person to investigate the multivariate problem. In 1972 the work of two aircraft engineers Harder and Desmarais [26] approached this problem from an applied point of view. In 1974 Fisher and Jerome [21] addressed the multivariate problem. In 1970, J. Thoman [51] in his doctoral thesis considered a variational approach to interpolation on a rectangle or on a disk in $\mathbb{R}^2$. The book by Ahlberg, Nilson and Walsh [3] also deals with multivariate problems, but from a point of view which is essentially univariate.

Yet more splines

The overture of splines could be continued. There are other many splines associated with some variational interpolation problems and for each case we could state a theorem similar to those above. We shall only nominate they:

Λ-*splines* (1972, Jerome and Pierce [30])

LMg-splines (1979, R. J. P. de Figueiredo [18])

ARMA-splines (1979, Weinert, Sesai and Sidhu [56])

Spherical splines (1981, Freeden, Scheiner and Franke [22])

PDLg-splines (1990, R. J. P. de Figueiredo and Chen [19])

Polyharmonic splines (1990, C. Rabut [43])

Vector splines (1991, Amodei and Benbourhin [5])

Hyperspherical splines (1994, Taijeron, Gibson and Chandler [50]).

4 Abstract splines

The statements of the above theorems were becoming quite long and complicated. But, there is a general abstract result which captures the essence of most of them. The following result is attributed to M. Atteia [8], [9] and it relates to following diagram:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "A"'] & Y \\ Z & \end{tikzcd}$$

Theorem 10. (Atteia, 1992) *If*

- X, Y, Z *are Hilbert spaces,*
- T, A *are continuous linear surjections,*
- $z \in Z$
- $kerT + kerA$ *is closed in* X*,*
- $kerT \cap kerA = \{\theta_X\}$ *and*
- $I(z) = \{x \in X : \ Ax = z\}$

then exists a unique $\sigma \in I(z)$ *such that:*

$$\|T\sigma\|_Y = \min\{\|Tx\|_Y : \ x \in I(z)\}$$

The optimal σ is known as a **variational interpolating spline**.

To illustrate that this theorem reflects the essence of the most above results, let us see how it generalizes Theorem 1 of Holladay. Put $X = H^2(I)$, $Y = L^2(I)$, $Z = \mathbb{R}^n$, $Tx := x^{(2)}$, $Ax := (x(t_1), x(t_2), \ldots, x(t_n))$. All the hypotheses of Atteia's theorem are satisfied. Atteia's theorem does not cover all the above results, e.g. Theorem 7 which deals with pLg-splines.

- An equivalent result to Atteia's theorem is found in the often cited, but unfortunately never published, report by Golomb [23] in 1967.
- The essential ideas also can be found in Anselone and Laurent [6] in 1968 and in the classic book by Laurent [33], entitled Approximation et Optimisation (Herman, Paris, 1972).

There are important remarks to be made about this theorem.

1. The role of the condition about $kerT + kerA$ is to ensure the existence of σ whereas the role of the condition $kerT \cap kerA$ is to ensure the uniqueness of σ. This separation was made clear by Jerome and Schumaker [31] in 1969.

2. The challenge of any abstract theory is to generalize a wide variety of particular cases, and simultaneously, preserve as much of the detail as possible. To a large extent, Atteia and others have, over many years, being doing this in the case that X is a reproducing kernel Hilbert space. Details of this theory can be found in the excellent monographs of Atteia ([9], 1992) and Bezhaev and Vasilenko ([11], 1993). The origins of this program can be found in 1959 paper by Golomb and Weinberger [24], in Ph. Thesis of Atteia ([8], 1966) and in 1966 paper by de Boor and Lynch [15].

3. The above general theorem can itself be generalized in many directions.

One generalization enables us to consider constrained interpolation problem which are very important in contemporary mathematics. It is due to Utreras, [52] in 1987 and relates to the following diagram

$$\begin{tikzcd} C \subset X \arrow[r, "T"] \arrow[d, "A"'] & Y \\ z \in Z & \end{tikzcd}$$

Theorem 11. (Utreras, 1987) *If*

- X, Y, Z *are Hilbert spaces,*
- C *is a closed, convex subset of* X*,*
- $z \in Z$
- A, T *are continuous, linear surjections,*
- $w \in I(C, z) := \{x \in C : \ Ax = z\}$
- $kerT + (kerA \cap (C - w))$ *is closed in* X *and*
- $kerA \cap kerT = \{\theta_X\}$

then exists a unique $\sigma \in I(C, z)$ *such that*

$$\|T\sigma\|_Y = \min\{\|Tx\|_Y : \ x \in I(C, z)\}.$$

If we put $C = X$ then we obtain Theorem 10 of Atteia. Utreras' theorem is useful if, for example, we want to interpolate positive data by positive functions. In this case we have $X = H^m(I)$ and C is the set of positive function in X.

Other generalizations have extended Atteia's theorem to **Banach spaces** settings, rather than Hilbert spaces. So that are known the following new splines in Banach spaces:

R-splines (1972, Holmes [29])

M-splines (1972, Lucas [36], 1985 Abraham [1])

Lf-splines (1983, Pai [40])

Tf-splines (1993, Benbourhim and Gaches [10]).

A key work in the Banach space setting is the 1975 paper of Fischer and Jerome [21], where the perfect splines are very important in this contex.

5 Conclusions and comments

The book of Laurent ([33], 1972) was perhaps the first book which emphasized the variational approach to splines.

Atteia's book ([9], 1992) is the key work in this area, especially for those interested in functional analysis.

Whaba ([55], 1990) is the first book describing applications of these ideas (in smoothing rather the interpolation) to statistics.

Bezhaev and Vasilenko ([11], 1993) published in Novosibirsk entitled "Variational Spline Theory" contains the most abstracts and rigorous results in this field, but difficult to obtain.

To close this presentation there are three conclusions to be underlined.

1. Splines may be defined as solution of variational problems rather than functions constructed in some piecewise manner. We have seen that these variational problems have become increasingly abstract and hence the concept of "splines" has became increasingly abstract. This may not be everyone's liking, at least, initially. For example, in 1966 in [15] de Boor and Lynch have written: "*in order not to dilute the notion of spline functions too much, we prefer to follow Greville's definition of a general spline function*" – which is based on a piecewise, constructive approach. In any case, the variational theory gives us a

new appreciation of the concept of a "**spline**".

2. The variational approach facilitates a natural, attractive way to extend the classical theory of interpolating splines, especially to multivariate situations. The works of Duchon [20] in 1976 and Whaba [54] in 1981 illustrate this conclusion. More recently, in 1993, de Boor [14] changing his earlier opinion wrote: "*I am convinced that the variational approach to splines will play a much greater role in multivariate spline theory that it did or should have in univariate theory*".

3. The theory of variational splines demonstrates the power of functional analysis to yield a unified approach to computational problems in interpolation. As S. Sobolev [34] in 1997, one year before his dead has been quoted: "*It is impossible to image the theory of computations with no Banach spaces*".

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