The Dirac Complex on Abstract Vector Variables: Megaforms

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CONTENTS

In this paper, we propose a method to obtain the syzygies of the Dirac complex defined on abstract vector variables. We propose a generalized theory of differential forms which acts as a de Rham-like sequence for the Dirac complex and we show that closure in this complex is equivalent to the syzygies for the Dirac complex.

1. INTRODUCTION

Our interest in complexes of systems of hypercomplex operators goes back to our first paper on the topic [Adams et al. 99], when we began studying nonhomogeneous Cauchy-Fueter systems on quaternions (see [Berenstein et al. 96] for the first announcement of the results). The lack of commutativity typical of the quaternionic situation implied that the compatibility conditions necessary and sufficient for the solutions of such systems were unusual and somewhat unexpected. The simplest analogue which was originally guiding our intuition was the study of the Cauchy-Riemann system. In that case, the Dolbeault sequence for the $\bar{\partial}$ operator is an exact sequence of spaces of differential forms, whose length is the dimension of the base space, and which is well known at every step. As a consequence, the resolution of the Cauchy-Riemann system in n complex variables is also well known: It is composed of linear maps, has length n, and, at each step, it is explicitly given in terms of the operators $\bar{\partial}$ in the various variables, and it leads naturally to the well-known Koszul complex.

When one studies the quaternionic analogue, the first difficulty is that no such thing as a Dolbeault sequence exists (unless, of course, one wishes to resort to the BGG sequence as indicated in [Somberg 01], where our quaternionic theory is reinterpreted in light of such a sequence). Nevertheless, we were able to employ the theory of Gröbner bases to compute the resolution for the

2000 AMS Subject Classification: Primary 30G35; Secondary 16E05

Keywords: Complexes of hypercomplex operators, Dirac operator, differential forms, radial algebra, syzygies

Cauchy-Fueter system, and we discovered that the resolution employs quadratic entries, has length $2n-1$, if n is the number of independent Cauchy-Fueter operators, and, most important, cannot be expressed in terms of the original Cauchy-Fueter operators themselves.

The syzygies which cannot be expressed in terms of the original operators eventually became known as "exceptional syzygies" and for a long time their meaning escaped us. We were finally able to provide a full description of the relevant syzygies in a series of papers that were inspired by some lengthy computations which we did employing the CoCoA package[1] (see, for example, [Sabadini and Struppa 02], for a discussion of the experiments we have done in this direction). The results which were obtained can be found in [Adams and Loustaunau 01] and [Adams et al. 97]. In [Sabadini et al. 02], we extended our study to the case of Dirac operators on Clifford algebras. The analysis turned out to be very useful as it allowed us to understand that the reason for the presence of exceptional syzygies was related to a dimensionality issue, which could be avoided by raising the dimension of the underlying vector space. In particular, this showed that the case of quaternions was somehow an exception. At the same time, the study of the resolutions introduced even more unexpected phenomena, such as syzygies of mixed degree. The lack of an appropriate theory of differential forms forced our work to be more algebraic in nature, and less apt to exploit the specific shape of Clifford algebras. Once again, CoCoA proved to be a fundamental tool in assisting us to experiment with systems of increasingly large dimensions and complexity.

In this paper, we propose a self-consistent theory for systems of Dirac operators. This is achieved through the construction of an appropriate abstract theory of generalized differential forms for complexes of Dirac operators. The theory is based on the assumption that all the possible relations among Dirac operators must satisfy the axioms of the so-called radial algebra, as introduced in [Sommen 97], and requires the introduction of new abstract objects which we call "megaforms" and which can be thought of as "forms of mixed degree 1 and 2," and a suitable complex $\{F_k,\, d^k:\ F_k \to F_{k+1}\}$. We do not yet have the entire theory of such megaforms (that we have completed only in the case of two operators) but, based on a series of calculations (some of which are included in this paper), we conjecture that such a complex is, in fact, sufficient to compute all the syzygies for the Dirac complex. Our expectation is that such a complex has length $2n-1$ where n is the number of Dirac operators involved, and that, like the Dolbeault complex, the last step is the dual of the first step itself. Working with megaforms is quite complicated, especially because of the amount of computations, but the advantage of this approach is that the theory of megaforms provides a method to produce all the syzygies of the Dirac complex in the radial case. In fact, the reader must note that the computations with CoCoA can only be done with specific values of the dimension of the Clifford algebra and for a given number of operators. Moreover, CoCoA can only give the number and the degree of the syzygies, while the theory of megaforms allows the explicit computation of the maps in the complex and it is independent of the dimension. Most of the results which we present or conjecture in this paper can only be imagined through a series of complex computations. We have executed these computations by hand since we have to take into account additional relationships among the variables. One of our students is currently working toward the creation of a package that will allow performance of such symbolic calculations.

[1] CoCoA is a special computer system for doing computations in commutative algebra. It is freely available by anonymous FTP from http://cocoa.dima.unige.it.

2. STANDARD DIRAC COMPLEX

Let $\mathbb{R}^m$ be the real Euclidean space and define $\mathbb{R}_m$ as the real Clifford algebra generated by the orthogonal basis elements $\{e_1,\ldots,e_m\}$ together with the defining relations of the form

$$e_j e_k + e_k e_j = -2\delta_{jk}.$$

Let $x_1,\ldots,x_m$ denote the standard commuting variables and let $f(x)$ be an $\mathbb{R}_m$-valued function. One may consider the action of the Dirac operator, or vector derivative,

$$\partial_x:\ f(x) \to \partial_x f(x) = \sum_{j=1}^{m} e_j \partial_{x_j} f(x)$$

and the solutions of the homogeneous equation $\partial_x f(x) = 0$, that are called monogenic functions; one may also consider the inhomogeneous equation $\partial_x f(x) = g(x)$ whose study is a standard topic in Clifford analysis. Without claiming completeness, we refer the reader to [Brackx et al. 82], [Delanghe et al. 92], and [Gilbert and Murray 90].

Less well developed is the theory of Dirac systems in several vector variables (see [Berenstein et al. 96], [Constales 89], [Laville 83], [Sommen 87]) which generalizes the better known theory of several quaternion variables

([Adams et al. 97], [Adams et al. 99], [Adams and Loustaunau 01], [Baston 92], [Colombo et al. 03], [Palamodov 99], [Pertici 88], [Sabadini and Struppa 97]). To this purpose, consider several vector variables

$$\underline{x}_1, \ldots, \underline{x}_\ell, \qquad \underline{x}_j = \sum_{k=1}^{m} x_{jk} e_k,$$

and functions $f(\underline{x}_1, \ldots, \underline{x}_\ell)$ with values in $\mathbb{R}_m$. One of the natural problems deals with the solvability of the system

$$\left\{ \begin{array}{l} \partial_{\underline{x}_1} f = g_1 \\ \quad\cdots \\ \partial_{\underline{x}_\ell} f = g_\ell \end{array} \right. \tag{2–1}$$

for given $\vec{g} = (g_1, \ldots, g_\ell)$, $g_j : (\mathbb{R}^m)^\ell \to \mathbb{R}_m$. Namely, one wants to find the necessary and sufficient conditions on the g_i for the system (2–1) to be solvable. This is in reality just the first step of a much more general problem, i.e., the determination of a resolution for the module associated with the system (2–1). The reader is referred to [Struppa 98] or to our forthcoming book [Colombo et al. 03] for the details on this theory.

The compatibility relations for the solvability of system (2–1), the so-called first syzygies, give a homogeneous system of differential equations $P_1(D)\vec{g} = 0$. One then goes on to study the new inhomogeneous system $P_1(D)\vec{g} = \vec{h}$ and determines the so-called second syzygies which are the compatibility relations for the solvability of this system and so on; this is the required resolution for the Dirac complex. We originally thought that the syzygies should always be expressible in terms of the vector derivatives $\partial_{\underline{x}_1}, \ldots, \partial_{\underline{x}_\ell}$. However, the Cauchy-Fueter system requires "exceptional syzygies" which cannot be expressed in terms of $\partial_{\underline{x}_1}, \ldots, \partial_{\underline{x}_\ell}$. In [Sabadini et al. 02], we proved that these syzygies arise because of the invalidity of the Fischer-decomposition theorem in low dimensions. In fact, this theorem is only valid in case the dimension m of the space is such that $m \geq 2\ell - 1$ and it was established in [Sabadini et al. 02] that for dimensions $m \geq 2\ell - 1$ the syzygies are all expressible within the real algebra generated by $\partial_{\underline{x}_1}, \ldots, \partial_{\underline{x}_\ell}$ and in particular, in that case, the first syzygies are all quadratic.

Before we proceed any further, we wish to give an idea of the complexity of the problem by discussing in detail the case of three operators acting on the algebra $\mathbb{R}_m$, with $m \geq 5$. The complete computations are given in [Sabadini et al. 02], but we want to provide the reader with the flavor of the nature of the direct computations.

Let us consider the system

$$\left\{ \begin{array}{ll} \partial_{\underline{x}_1} f & = g_1 \\ \partial_{\underline{x}_2} f & = g_2 \\ \partial_{\underline{x}_3} f & = g_3, \end{array} \right. \tag{2–2}$$

where $f : (\mathbb{R}^m)^3 \to \mathbb{R}_m$, $m \geq 5$. Let $R = \mathbb{C}[x_{11}, x_{12}, \ldots, x_{1m}, \ldots\, x_{31}, x_{32}, \ldots, x_{3m}]$ and denote by $\mathcal{M}_3$ the module $R^{2^m}/\langle P^t \rangle$, P being the $3 \cdot 2^m \times 2^m$ polynomial matrix which represents the Fourier transform of the differential operators which appear in (2–2) after we rewrite the system in real coordinates. In the case of $m = 2$, for example, P would look like

$$\begin{bmatrix} 0 & -x_{11} & -x_{12} & 0 \\ x_{11} & 0 & 0 & x_{12} \\ x_{12} & 0 & 0 & -x_{11} \\ 0 & -x_{12} & x_{11} & 0 \\ 0 & -x_{21} & -x_{22} & 0 \\ x_{21} & 0 & 0 & x_{22} \\ x_{22} & 0 & 0 & -x_{21} \\ 0 & -x_{22} & x_{21} & 0 \\ 0 & -x_{31} & -x_{32} & 0 \\ x_{31} & 0 & 0 & x_{32} \\ x_{32} & 0 & 0 & -x_{31} \\ 0 & -x_{32} & x_{31} & 0 \end{bmatrix}.$$

According to [Sabadini et al. 02], if $m \geq 5$, $\mathcal{M}_3$ has the resolution

$$\begin{aligned} &0 \longrightarrow R^{2^m}(-9) \longrightarrow R^{3\cdot 2^m}(-8) \longrightarrow R^{8\cdot 2^m}(-6) \\ &\longrightarrow R^{6\cdot 2^m}(-4) \oplus R^{6\cdot 2^m}(-5) \\ &\longrightarrow R^{8\cdot 2^m}(-3) \longrightarrow R^{3\cdot 2^m}(-1) \longrightarrow R^{2^m} \longrightarrow \mathcal{M}_3 \longrightarrow 0. \end{aligned} \tag{2–3}$$

Note that the exponents of R denote the number of real syzygies which appear at every stage, while the number in parentheses indicates the degree of such syzygies (or better the differences of the numbers in parentheses represent the degrees). So, for example, the first syzygies are quadratic $(2 = 3 - 1)$, while the second syzygies are linear (six of them) and quadratic (six of them), etc. Note that the number of real syzygies is always a multiple of the real dimension 2^m of $\mathbb{R}_m$. When we write the syzygies in term of $\mathbb{R}_m$-variables, the effective number does not contain 2^m. For example, one sees that the system $P_1(D)\vec{g} = 0$ entails 8 compatibility conditions on the g_i. Six of them are given by the relations:

$$\partial_{\underline{x}_i} \partial_{\underline{x}_j} g_j - \partial^2_{\underline{x}_j} g_i = 0$$

for each of the ordered pairs of indices $1 \leq i, j \leq 3$, while the remaining two are given by the 3 relations

$$\{\partial_{\underline{x}_i}, \partial_{\underline{x}_j}\} g_k = \partial_{\underline{x}_k} \partial_{\underline{x}_i} g_j + \partial_{\underline{x}_k} \partial_{\underline{x}_j} g_i$$

of which only two are independent. Those 8 syzygies lead to the new nonhomogeneous system $P_1(D)\vec{g} = \vec{h}$

$$\left\{\begin{array}{l} \partial_{\underline{x}_2}\partial_{\underline{x}_1}g_1 - \partial^2_{\underline{x}_1}g_2 = h_{12} \\ \partial_{\underline{x}_3}\partial_{\underline{x}_1}g_1 - \partial^2_{\underline{x}_1}g_3 = h_{13} \\ \partial_{\underline{x}_1}\partial_{\underline{x}_2}g_2 - \partial^2_{\underline{x}_2}g_1 = h_{21} \\ \partial_{\underline{x}_3}\partial_{\underline{x}_2}g_2 - \partial^2_{\underline{x}_2}g_3 = h_{23} \\ \partial_{\underline{x}_1}\partial_{\underline{x}_3}g_3 - \partial^2_{\underline{x}_3}g_1 = h_{31} \\ \partial_{\underline{x}_2}\partial_{\underline{x}_3}g_3 - \partial^2_{\underline{x}_3}g_2 = h_{32} \\ \{\partial_{\underline{x}_2},\partial_{\underline{x}_3}\}g_1 - \partial_{\underline{x}_1}\partial_{\underline{x}_2}g_3 - \partial_{\underline{x}_1}\partial_{\underline{x}_3}g_2 = a_1 \\ \{\partial_{\underline{x}_1},\partial_{\underline{x}_3}\}g_2 - \partial_{\underline{x}_2}\partial_{\underline{x}_1}g_3 - \partial_{\underline{x}_2}\partial_{\underline{x}_3}g_1 = a_2 \\ \{\partial_{\underline{x}_1},\partial_{\underline{x}_2}\}g_3 - \partial_{\underline{x}_3}\partial_{\underline{x}_1}g_2 - \partial_{\underline{x}_3}\partial_{\underline{x}_2}g_1 = a_3, \end{array}\right. \tag{2–4}$$

with the constraint $a_1 + a_2 + a_3 = 0$. Resolution (2–3) shows that the compatibility conditions for $P_1(D)\vec{g} = \vec{h}$ will contain 6 linear syzygies and 6 quadratic syzygies.

The 6 linear compatibility conditions of system (2–4), that give the syzygies at the second step, are given by

$$\left\{\begin{array}{l} \partial_{\underline{x}_2}h_{12} + \partial_{\underline{x}_1}h_{21} = 0 \\ \partial_{\underline{x}_3}h_{13} + \partial_{\underline{x}_1}h_{31} = 0 \\ \partial_{\underline{x}_3}h_{23} + \partial_{\underline{x}_2}h_{32} = 0 \\ \partial_{\underline{x}_3}h_{12} + \partial_{\underline{x}_2}h_{13} = \partial_{\underline{x}_1}a_1 \\ \partial_{\underline{x}_1}h_{23} + \partial_{\underline{x}_3}h_{21} = \partial_{\underline{x}_2}a_2 \\ \partial_{\underline{x}_2}h_{31} + \partial_{\underline{x}_1}h_{32} = \partial_{\underline{x}_3}a_3, \end{array}\right.$$

while the 6 quadratic compatibility conditions are given by the cyclic permutations of $(1,2,3)$ in

$$\{\partial_{\underline{x}_1},\partial_{\underline{x}_2}\}h_{23} + \partial^2_{\underline{x}_2}a_3 = \partial_{\underline{x}_3}\partial_{\underline{x}_2}h_{21},$$

of which only 3 are independent. Finally, we have the permutations of the condition

$$\partial^2_{\underline{x}_1}h_{23} - \partial^2_{\underline{x}_2}h_{13} = \partial_{\underline{x}_3}\partial_{\underline{x}_1}h_{21},$$

which gives 3 more quadratic relations. The syzygies found at the second step give the nonhomogeneous system formed by the 3 equations

$$\partial_{\underline{x}_i}h_{ji} + \partial_{\underline{x}_j}h_{ij} = R_k,$$

for $(i,j,k) = (2,1,3)$, $(2,3,1)$, or $(3,1,2)$ together with the 3 equations

$$\partial_{\underline{x}_i}h_{jk} + \partial_{\underline{x}_k}h_{ji} - \partial_{\underline{x}_j}a_k = S_k$$

$(i,j,k) = (1,2,3)$, $(2,3,1)$, or $(3,1,2)$, then the 6 equations given by the permutations of $(i,j,k) = (1,2,3)$ in

$$\{\partial_{\underline{x}_i},\partial_{\underline{x}_j}\}h_{jk} + \partial^2_{\underline{x}_j}a_k - \partial_{\underline{x}_k}\partial_{\underline{x}_j}h_{ji} = T_{ki}$$

and, finally, by the 6 permutations of $(i,j,k) = (1,2,3)$ in

$$\partial_{\underline{x}_i}\partial_{\underline{x}_j}h_{kj} + \partial^2_{\underline{x}_k}h_{ji} - \partial^2_{\underline{x}_j}h_{ki} = U_{kj}.$$

We have the following constraints on T_{ij} and U_{ij}:

$$\begin{aligned} &T_{23} + T_{32} = \partial_{\underline{x}_1}S_1, \quad T_{31} + T_{13} = \partial_{\underline{x}_2}S_2, \\ &T_{12} + T_{21} = \partial_{\underline{x}_3}S_3 \\ &U_{23} + U_{32} = \partial_{\underline{x}_1}R_1, \quad U_{31} + U_{13} = \partial_{\underline{x}_2}R_2, \\ &U_{12} + U_{21} = \partial_{\underline{x}_3}R_3. \end{aligned}$$

These constraints reduce the total number of equations in the system to 12.

The 8 compatibility conditions of the previous system are given by

1. the 3 conditions obtained by the cyclic permutations of $(1,2,3)$ in the formula
$$\partial_{\underline{x}_2}U_{13} - \partial_{\underline{x}_1}U_{32} + \partial^2_{\underline{x}_2}R_2 - \partial^2_{\underline{x}_3}R_3 = 0$$
of which 2 are independent;
2. the 6 permutations of the relation
$$\partial^2_{\underline{x}_1}S_2 + \partial_{\underline{x}_2}T_{23} - \partial_{\underline{x}_3}\partial_{\underline{x}_1}R_3 - \partial_{\underline{x}_1}U_{12} = 0.$$

We now have the nonhomogeneous system consisting of

$$\partial^2_{\underline{x}_1}S_2 + \partial_{\underline{x}_2}T_{23} - \partial_{\underline{x}_3}\partial_{\underline{x}_1}R_3 - \partial_{\underline{x}_1}U_{12} = B_{12}$$

together with its permutations, and the 3 relations obtained by the cyclic permutations of $(1,2,3)$ in

$$\partial_{\underline{x}_1}U_{32} - \partial_{\underline{x}_3}U_{21} + \partial^2_{\underline{x}_2}R_2 - \partial^2_{\underline{x}_1}R_1 = C_1,$$

with the constraint $C_1 + C_2 + C_3 = 0$. We have the following constraints: $T_{12} + T_{21} = \partial_{\underline{x}_3}S_3$, $U_{12} + U_{21} = \partial_{\underline{x}_3}R_3$ together with their cyclic permutations. For this system, we have that the 3 compatibility conditions are given by

$$\begin{aligned} \{\partial_{\underline{x}_1},\partial_{\underline{x}_2}\}C_3 + \partial_{\underline{x}_2}\partial_{\underline{x}_1}C_2 &= \partial_{\underline{x}_3}\partial_{\underline{x}_2}B_{12} + \partial^2_{\underline{x}_2}B_{13} \\ &\quad - \partial_{\underline{x}_3}\partial_{\underline{x}_1}B_{21} - \partial^2_{\underline{x}_1}B_{23} \end{aligned}$$

and its cyclic permutations.

The closing of the complex requires just one more relation. Considering the equations

$$\begin{aligned} E_1 &= \{\partial_{\underline{x}_2},\partial_{\underline{x}_3}\}C_1 + \partial_{\underline{x}_3}\partial_{\underline{x}_2}C_3 - \partial_{\underline{x}_1}\partial_{\underline{x}_3}B_{23} - \partial^2_{\underline{x}_3}B_{21} \\ &\quad + \partial_{\underline{x}_1}\partial_{\underline{x}_2}B_{32} + \partial^2_{\underline{x}_2}B_{31} \\ E_2 &= \{\partial_{\underline{x}_3},\partial_{\underline{x}_1}\}C_2 + \partial_{\underline{x}_1}\partial_{\underline{x}_3}C_1 - \partial_{\underline{x}_2}\partial_{\underline{x}_1}B_{31} - \partial^2_{\underline{x}_1}B_{32} \\ &\quad + \partial_{\underline{x}_2}\partial_{\underline{x}_3}B_{13} + \partial^2_{\underline{x}_3}B_{12} \\ E_3 &= \{\partial_{\underline{x}_1},\partial_{\underline{x}_2}\}C_3 + \partial_{\underline{x}_2}\partial_{\underline{x}_1}C_2 - \partial_{\underline{x}_3}\partial_{\underline{x}_2}B_{12} - \partial^2_{\underline{x}_2}B_{13} \\ &\quad + \partial_{\underline{x}_3}\partial_{\underline{x}_1}B_{21} + \partial^2_{\underline{x}_1}B_{23} \end{aligned}$$

and using $C_1 + C_2 + C_3 = 0$, it follows that

$$\partial_{\underline{x}_1} E_1 + \partial_{\underline{x}_2} E_2 + \partial_{\underline{x}_3} E_3 = 0,$$

which closes the complex.

These computations suggest the consideration of the algebra of abstract variables, called "radial algebra" (see [Sommen 97]) generated by a set S of abstract variables $x_1, \ldots, x_l, \ldots$ with defining relations $[x_i, \{x_j, x_k\}] = x_i(x_j x_k + x_k x_j) - (x_j x_k + x_k x_j)x_i = 0$. The idea of such algebra leads to a mathematical foundation of the "Geometric Calculus" introduced in [Hestenes and Sobczyk 85], by which it was inspired, and one may define abstract Dirac operators or vector derivatives $\partial_{x_1}, ..., \partial_{x_l}, ...$ as endomorphisms on it. It was shown in [Sommen 01] that, in the setting of abstract vector variables, the Fischer decomposition is always valid and therefore one may define the Dirac complex for the abstract vector derivatives $\partial_{x_1}, ..., \partial_{x_l}, ...$ and then determine the corresponding syzygies.

In order to gain the appropriate perspective, let us look back at the more classical case of differential forms of type $(0,k)$ of $\mathcal{C}^\infty$ class on $\mathbb{C}^n = \mathbb{R}^{2n}$, i.e., differential forms having local representation of the type

$$\sum f_{i_1 \ldots i_k} d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_k},$$

where $f_{i_1 \ldots i_k}$ are infinitely differentiable functions. The space of forms of this type will be denoted by $\mathcal{C}^{(0,k)}$. Let us denote by $\bar{\partial}_k$ the antiholomorphic exterior differentiation of degree k, i.e.,

$$\bar{\partial}_k\left(\sum f_{i_1 \ldots i_{k-1}} d\bar{z_{i_1}} \wedge \ldots \wedge d\bar{z}_{i_{k-1}}\right) = \sum \partial_{\bar{z}_k} f_{i_1 \ldots i_{k-1}} d\bar{z}_k \wedge d\bar{z_{i_1}} \wedge \ldots \wedge d\bar{z}_{i_{k-1}};$$

we have the following sequence of sheaves (the so-called Dolbeault complex):

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{C}^{(0,0)} \xrightarrow{\bar{\partial}_0} \mathcal{C}^{(0,1)} \xrightarrow{\bar{\partial}_1} \ldots \xrightarrow{\bar{\partial}_{n-1}} \mathcal{C}^{(0,n)} \longrightarrow 0.$$

Now consider the system of n Cauchy-Riemann operators $P(D) = [\partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n}]$ and consider its resolution built using the Koszul complex (see [Krantz 83]),

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{C}^\infty \xrightarrow{P(D)} (\mathcal{C}^\infty)^{\binom{n}{1}} \xrightarrow{P_1(D)} \ldots \xrightarrow{P_{n-1}(D)} (\mathcal{C}^\infty)^{\binom{n}{n}} \longrightarrow 0.$$

The key point in the theory of holomorphic differential forms is the fact that a k-form $\omega = \sum \alpha_{i_1 \ldots i_k} d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_k}$ is closed if and only if $\alpha = [\alpha_{i_1 \ldots i_k}]$ satisfies the k-th compatibility conditions $P_k(D)\alpha = 0$, i.e., if and only if α satisfies the k-th syzygies of the Cauchy-Riemann system. To clarify this discussion, let us examine in detail the case of two Cauchy-Riemann operators. The Dolbeault complex in $\mathbb{C}^2$ is

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{C}^{(0,0)} \xrightarrow{\bar{\partial}_0} \mathcal{C}^{(0,1)} \xrightarrow{\bar{\partial}_1} \mathcal{C}^{(0,2)} \longrightarrow 0,$$

where $\bar{\partial}_0 f = \bar{\partial} f = \partial_{\bar{z}_1} d\bar{z}_1 + \partial_{\bar{z}_2} f d\bar{z}_2$ for $f \in \mathcal{C}^{(0,0)}$, and $\bar{\partial}_1 \omega = \bar{\partial}_1(\alpha_1 d\bar{z}_1 + \alpha_2 d\bar{z}_2) = (\partial_{\bar{z}_1}\alpha_2 - \partial_{\bar{z}_2}\alpha_1) d\bar{z}_1 \wedge d\bar{z}_2$ for any $\omega \in \mathcal{C}^{(0,1)}$. The Koszul complex is

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{C}^\infty \xrightarrow{[\partial_{\bar{z}_1}\ \partial_{\bar{z}_2}]} (\mathcal{C}^\infty)^2 \xrightarrow{[\partial_{\bar{z}_2}\ -\partial_{\bar{z}_1}]^t} \mathcal{C}^\infty \longrightarrow 0.$$

Note that $\bar{\partial}_1 \omega = 0$ if and only if $\alpha = [\alpha_1\ \alpha_2]$ satisfies $\partial_{\bar{z}_1}\alpha_2 - \partial_{\bar{z}_2}\alpha_1 = 0$, i.e., it satisfies the system $P_1(D)\alpha = 0$. The fundamental conclusion we draw is that the complex of $(0,k)$ differential forms and the resolution of $(\partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n})$ yield exactly the same operators. If one were to try to mimic this construction in the Clifford case with a naïve definition, one would probably introduce a basis for first order differential forms $D_1, \ldots, D_\ell$ and would define a differential df such that

$$df = \partial_1 f D_1 + \ldots \partial_\ell f D_\ell.$$

The condition $d^2 = 0$ would immediately give (in view of the noncommutativity of the differentiation operators ∂_i) that all products $D_i D_j$ vanish, and the complex provides no useful information on the syzygies of the operators. This is not surprising, in view of the fact that we know that any first syzygy must include quadratic terms and therefore could not be obtained from the closure condition $d\omega = 0$ for $\omega = \omega_1 D_1 + \ldots + \omega_\ell D_\ell$, a 1-form. If one could develop an appropriate theory of forms and an appropriate notion of differential for this system, one could expect to derive its syzygies as a closure condition for suitable differential forms. In the next section, we will develop the theory of megaforms which, in fact, is inspired by this idea.

3. MEGAFORMS ON RADIAL ALGEBRAS

Definition 3.1. Let S be a set of objects which we will consider as "abstract vector variables." The radial algebra $R(S)$ is defined to be the associative algebra generated by S over $\mathbb{R}$ with the defining relations

$$[\{x,y\},z] = xyz + yxz - zxy - zyx = 0, \quad \text{for} \quad x,y,z \in S. \tag{3–1}$$

Note that such an algebra could be constructed over any suitable ring, though in this paper, we will confine ourselves to $\mathbb{R}$. Let $T(S)$ be the tensor algebra generated by the elements of S, and let $I(S)$ be the two-sided ideal generated by the polynomials

$$[\{x, y\}, z].$$

Then

$$R(S) = T(S)/I(S).$$

We denote by $r(S)$ the subalgebra of $R(S)$ generated by the so-called *scalar elements* $\{x, y\} = xy + yx$, x, $y \in S$. We call $r(S)$ the *scalar subalgebra* of $R(S)$. The reason for this terminology lies in the fact that when x and y are not just abstract variables, but in fact represent Clifford variables, then $\{x, y\}$ is real. By $R_k(S)$, we denote the subspace of k-vectors in $R(S)$ which is generated over $r(S)$ by "wedge products of length k" of the form

$$x_1 \wedge ... \wedge x_k := \frac{1}{k!} \sum_{\pi} sgn(\pi) x_{\pi(1)} ... x_{\pi(k)},$$

where $\pi = (\pi(1), \ldots, \pi(k))$ denotes a permutation of $(1, \ldots, k)$, and the sum is taken over all possible permutations π. We have the direct sum decomposition

$$R(S) = R_0(S) \oplus R_1(S) \oplus \ldots \oplus R_k(S) \oplus \ldots,$$

where $R_0(S) = r(S)$ are the scalar elements. The vector derivatives $\partial_x, \partial_y, \partial_z, ...,$ associated to elements $x, y, z, \ldots \in S$ are defined as the endomorphisms on $R(S)$ which satisfy the following axioms:

for $x \in S$, $f \in r(S)$, $F \in R(S)$, and $G \in R(S \setminus \{x\})$, it is

$$\begin{aligned}
&(D_1) \quad \partial_x[fF] = \partial_x[f]F + f\partial_x[F], \\
& [fF]\partial_x = F\partial_x[f] + f[F]\partial_x \\
&(D_2) \quad \partial_x[FG] = \partial_x[F]G, \qquad [GF]\partial_x = G[F]\partial_x \\
&(D_3) \quad \partial_x[F\partial_y] = [\partial_x F]\partial_y \\
&(D_4) \quad \partial_x x^2 = 2x, \qquad \partial_x\{x, y\} = 2y.
\end{aligned}$$

It may be shown ([Sommen 01]) that the objects $\partial_x[x]$ are independent of the choice $x \in S$, that $\partial_x[x] = [x]\partial_x$, and that $\partial_x[x]$ is a commutative object $M = \partial_x[x]$ which we call the abstract dimension of S.

Next, let $x \in S$ be a chosen vector variable and consider the other variables of S as parameters; then, an element $F \in R(S)$ may be written as a formal function $F(x)$ with respect to the variable x. $F(x)$ is called left-monogenic if it solves the equation $\partial_x F(x) = 0$. Similarly, let us select a finite subset $\{x_1, ..., x_n\}$ of S called the vector variables and let us call the remaining vectors in S "parameter vectors." Then $F \in R(S)$ may be written into the form $F(x_1, ..., x_n)$ and it is called monogenic if it satisfies the system

$$\partial_{x_j} F(x_1, ..., x_n) = 0, \qquad j = 1, ..., n.$$

One may consider the inhomogeneous system of the form

$$\begin{cases} \partial_{x_1} f = g_1 \\ \quad \cdots \\ \partial_{x_n} f = g_n \end{cases}$$

in the radial algebra setting, and look for its syzygies, i.e., the compatibility equations to be satisfied by $g_1, ..., g_n$ for this system to be solvable. Necessary conditions for this are certainly the relations coming from the fact that the algebra generated by $\{\partial_{x_1}, ..., \partial_{x_n}\}$ is a radial algebra itself with relations

$$\partial_{x_j}\{\partial_{x_k}, \partial_{x_l}\} = \{\partial_{x_k}, \partial_{x_l}\}\partial_{x_j},$$

namely

$$\partial_{x_j}(\partial_{x_k} g_l + \partial_{x_l} g_k) = \{\partial_{x_k}, \partial_{x_l}\} g_j.$$

One can embed the problem in Clifford analysis using a Clifford algebra representation

$$x \to \underline{x} = \sum e_j x_j,$$

setting the dimension m of the algebra equal to M, and identifying

$$\partial_{x_j} \to -\partial_{\underline{x_j}} = -\sum e_k \partial_{x_{jk}}.$$

This can in turn be written in terms of real coordinates on which computer calculations with CoCoA can be used.

We will now introduce megaforms as the analogue of classical differential forms in case the real coordinates $x_1, ..., x_m$ are replaced by abstract vector variables and the Dirac complex is what should result from what corresponds to the generalized de Rham complex. This replacement of scalar by vector variables is in fact the key idea behind many discoveries in Clifford analysis. To define the classical basic differential forms, one starts from the operator $d = \sum \partial_{x_j} dx_j$ acting on the algebra generated by $x_1, ..., x_m$, $dx_1, ..., dx_m$ together with the properties

1. $d(\omega) = d(\sum F_j dg_j) = \sum dF_j \wedge dg_j$,
2. $d(f\omega) = df \wedge \omega + f d\omega$,
3. $dx_j \wedge dx_k = -dx_k \wedge dx_j$.

In case one replaces $x_1, ..., x_m$ by the generators of a radial algebra, which we will denote in the same way, it is natural to replace the partial derivatives by the vector derivatives $\partial_{x_1}, ..., \partial_{x_n}$, but it is not so clear how to generalize the properties above. Instead, we try to generalize the formula "$d = \sum \partial_{x_j} d_{x_j}$" in the case of vector derivatives $\partial_{x_1}, ..., \partial_{x_n}$, which themselves satisfy the defining relations for a radial algebra. To be able to make use of the radial algebra defining relations, we will construct spaces F_k of forms, and differentials $d^k : F_k \to F_{k+1}$ such that $d^{k+1}d^k = 0$. However, we will recognize that the differentials d^k may consist of two pieces: a "degree one" piece d_1^k of the form $\sum_{j=1}^m D_j^k \partial_j$ and a "degree two" piece d_2^k of the form $\sum_{i,j=1}^m D_{ij}^k \partial_i \partial_j$. The symbols of degree one, D_j^k, are the Dirac analogue of the dx_j which are used in the classical de Rham complex, while the symbols of degree two, D_{ij}^k, are new symbols which will be necessary to reflect the existence of quadratic syzygies. As such, these symbols will have to satisfy some axioms in order to guarantee $d^{k+1}d^k = 0$. Note also that the symbol D_j^k and the symbols D_{ij}^k may, a priori, depend on k, though we will see the full theory can be developed with only a minimal dependence on k.

Definition 3.2. Let $x_1, ..., x_n \in S$; then the algebra $M(x_1, ..., x_n; S)$ of free megaforms in the variables $x_1, ..., x_n$ with coefficients in $R(S)$ is the associative algebra which is generated over the algebra $R(S)$ by the set of "basic megaforms" $\{D_i^k, D_{j\ell}^k : i, j, \ell = 1, ..., n\}$ together with the identities derived by $d^{k+1}d^k = 0$.

To our purposes, the position of the various symbols D_i^k, $D_{j\ell}^k$ in the relations we obtain makes clear that we can omit the top label. It suffices to keep in mind that the relations obtained for $D_{j\ell} = D_{j\ell}^k$ at the k-th step cannot be used for $D_{j\ell} = D_{j\ell}^{k+1}$. From now on, we will write D_i, $D_{j\ell}$, and we will leave it to the reader to distinguish among the different levels (in fact, the order in which they appear identifies their level immediately).

A full description of this theory is beyond our grasp at this point, but in the next two sections, we will provide the treatment for the case of two and three Dirac operators, as well as some conjectures for the general case.

4. THE CASE OF TWO DIRAC OPERATORS

Let $\mathcal{R}$ be the space of monogenic functions in two variables x_1, x_2. Let F_0 be the space of $\mathcal{C}^\infty$ functions in x_1, x_2 and let F_1 be the space of "1-forms" whose elements are written as $g = \sum_{i=1}^2 D_i g_i$, $g_i \in F_0$, so that if $d^0 = \sum_{i=1}^2 D_i \partial_i$, where $\partial_i = \partial_{x_i}$, we have an exact sequence

$$0 \to \mathcal{R} \hookrightarrow F_0 \xrightarrow{d^0} F_1.$$

The next step in the construction of the complex consists in defining a space F_2 of "2-forms" and a suitable "differential" $d^1 : F_1 \to F_2$ such that $d^1 d^0 = 0$. As we explained in Section 3, we postulate that d^1 be made of two components d_1^1 and d_2^1 of degrees, respectively, one and two. Thus, we assume, for $g \in F_1$,

$$d^1 g = d_1^1 g + d_2^1 g = \sum_{j,k=1}^2 D_k D_j \partial_k g_j + \sum_{i,j,k=1}^2 D_{ki} D_j \partial_k \partial_i g_j.$$

The condition $d^1 d^0 = 0$ implies that $\sum_{j,k=1}^2 D_k D_j \partial_k \partial_j f = 0$, i.e., that for any k and j, $D_k D_j = 0$. But then, because of the form of elements in F_1, one has that $d_1^1 \equiv 0$, and therefore

$$d^1 g = d_2^1 g = \sum_{i,j,k=1}^2 D_{ki} D_j \partial_k \partial_i g_j.$$

Proposition 4.1. *Let $d^0 = D_1\partial_1 + D_2\partial_2$ and $d^1 = D_{11}\partial_1^2 + D_{12}\partial_1\partial_2 + D_{21}\partial_2\partial_1 + D_{22}\partial_2^2$. Then $d^1 d^0 = 0$ implies the following:*

$$\begin{aligned} &D_{ij} D_i = 0 && i, j = 1, 2 \\ &D_{ii} D_j + D_{ji} D_i = 0 && i, j = 1, 2, \quad i \neq j. \end{aligned} \tag{4–1}$$

Proof: The condition $d_2^1 d^0 = 0$ implies that for any $f \in F_0$, it is

$$\sum_{i,j,k=1}^2 D_{ki} D_j \partial_k \partial_i \partial_j f = 0.$$

By writing explicitly the right-hand term, we obtain

$$\begin{aligned} (&D_{11}D_1\partial_1^3 + D_{12}D_1\partial_1\partial_2\partial_1 + D_{21}D_1\partial_2\partial_1^2 \\ &+ D_{22}D_1\partial_2^2\partial_1 + D_{11}D_2\partial_1^2\partial_2 + D_{12}D_2\partial_1\partial_2^2 \\ &+ D_{21}D_2\partial_2\partial_1\partial_2 + D_{22}D_2\partial_2^3)f = 0, \end{aligned}$$

so that, using the radial algebra defining relations $\partial_i^2 \partial_j = \partial_j \partial_i^2$, $i \neq j$, and grouping the various terms, we get the statement. □

Remark 4.2. The relations (4–1) are the analogues of the complex relations $d\bar{z}_1 \wedge d\bar{z}_2 = -d\bar{z}_2 \wedge d\bar{z}_1$, $d\bar{z}_i \wedge d\bar{z}_i = 0$.

We now study the kernel of the map $d^1 : F_1 \to F_2$ and we show that a 1-form g is d^1-closed if and only if its components g_j satisfy the compatibility conditions of the system $d^0 f = g$.

Proposition 4.3. *Let $g = D_1 g_1 + D_2 g_2$ be an element of F_1. Then $d^1 g = 0$ if and only if*

$$\partial_i^2 g_j - \partial_j \partial_i g_i = 0, \quad i = 1, 2, \tag{4–2}$$

i.e., $d^1 g = 0$ if and only if (g_1, g_2) satisfy the compatibility conditions for the solvability of the system

$$\begin{cases} \partial_1 f = g_1 \\ \partial_2 f = g_2. \end{cases}$$

Proof: By the definition of g and d^1, we have that $d^1 g = 0$ can be written as

$$\begin{aligned} &D_{11} D_1 \partial_1^2 g_1 + D_{11} D_2 \partial_1^2 g_2 + D_{12} D_1 \partial_1 \partial_2 g_1 \\ &\quad + D_{12} D_2 \partial_1 \partial_2 g_2 + D_{21} D_1 \partial_2 \partial_1 g_1 + D_{21} D_2 \partial_2 \partial_1 g_2 \\ &\quad + D_{22} D_1 \partial_2^2 g_1 + D_{22} D_2 \partial_2^2 g_2 = 0. \end{aligned}$$

In view of (4–1), this can be rewritten as

$$D_{11} D_2 (\partial_1^2 g_2 - \partial_2 \partial_1 g_1) + D_{22} D_1 (\partial_2^2 g_1 - \partial_1 \partial_2 g_2) = 0,$$

which completes the proof. □

We know, from the general theory, that the complex closes with one more linear condition that is the compatibility condition for the solvability of the system

$$\begin{cases} \partial_1^2 g_2 - \partial_2 \partial_1 g_1 = h_{12} \\ \partial_2^2 g_1 - \partial_1 \partial_2 g_2 = h_{21}. \end{cases} \tag{4–3}$$

We wish to show how this condition can be derived using megaforms and their closure.

Proposition 4.4. *Let $d^1 = D_{11} \partial_1^2 + D_{12} \partial_1 \partial_2 2 + D_{21} \partial_2 \partial_1 + D_{22} \partial_2^2$ and let $d^2 = D_1 \partial_1 + D_2 \partial_2 + D_{11} \partial_1^2 + D_{12} \partial_1 \partial_2 2 + D_{21} \partial_2 \partial_1 + D_{22} \partial_2^2$; then $d^2 d^1 = 0$ implies for $i, j = 1, 2,\ i \neq j$*

$$D_i D_{ii} D_j = 0 \quad D_{ii}^2 D_j = 0 \quad D_{ii} D_{jj} D_i - D_{ij} D_{ii} D_j = 0,$$

$$D_{ij} D_{jj} D_i = 0,$$

and

$$D_2 D_{11} D_2 + D_1 D_{22} D_1 = 0.$$

Proof: Because of the previous computations, we know that for any $g \in F_1$,

$$d^2 d^1 g = d^2 [D_{11} D_2 (\partial_1^2 g_2 - \partial_2 \partial_1 g_1) + D_{22} D_1 (\partial_2^2 g_1 - \partial_1 \partial_2 g_2)].$$

This expression can be split into two parts, the first containing degree 3 derivatives in g:

$$D_1 D_{11} D_2 (\partial_1^3 g_2 - \partial_1 \partial_2 \partial_1 g_1) + D_2 D_{11} D_2 (\partial_2 \partial_1^2 g_2 - \partial_2^2 \partial_1 g_1)$$

$$+ D_1 D_{22} D_1 (\partial_1 \partial_2^2 g_1 - \partial_1^2 \partial_2 g_2) + D_2 D_{22} D_1 (\partial_2^3 g_1 - \partial_2 \partial_1 \partial_2 g_2);$$

and the second in which the degree 4 derivatives appear:

$$\begin{aligned} &D_{11}^2 D_2 (\partial_1^4 g_2 - \partial_1^2 \partial_2 \partial_1 g_1) + D_{11} D_{22} D_1 (\partial_1^2 \partial_2^2 g_1 - \partial_1^3 \partial_2 g_2) \\ &\quad + D_{12} D_{11} D_2 (\partial_1 \partial_2 \partial_1^2 g_2 - \partial_1 \partial_2^2 \partial_1 g_1) \\ &\quad + D_{12} D_{22} D_1 (\partial_1 \partial_2^3 g_1 - \partial_1 \partial_2 \partial_1 \partial_2 g_2) \\ &\quad + D_{21} D_{11} D_2 (\partial_2 \partial_1^3 g_2 - \partial_2 \partial_1 \partial_2 \partial_1 g_1) \\ &\quad + D_{21} D_{22} D_1 (\partial_2 \partial_1 \partial_2^2 g_1 - \partial_2 \partial_1^2 \partial_2 g_2) \\ &\quad + D_{22} D_{11} D_2 (\partial_2^2 \partial_1^2 g_2 - \partial_2^3 \partial_1 g_1) \\ &\quad + D_{22}^2 D_1 (\partial_2^4 g_1 - \partial_2^2 \partial_1 \partial_2 g_2). \end{aligned}$$

By setting each of them equal to zero, by using the radial algebra relations on the derivatives of g (essentially these relations reduce to the commutativity of ∂_i^2 with any ∂_j, $i, j = 1, 2$), and by setting equal to zero the coefficients of all the independent derivatives, we finally obtain the desired result. □

Proposition 4.5. *Let $h = D_{11} D_2 h_{12} + D_{22} D_1 h_{21}$ be a generic element in F_2. Then $d^2 h = 0$ if and only if*

$$\partial_1 h_{21} + \partial_2 h_{12} = 0,$$

i.e., $d^2 h = 0$ if and only if $h = (h_{12}, h_{21})$ satisfies the compatibility condition for the system (4–3).

Proof: By the definition of $h \in F_2$ and d^2, and by using the monomial relations from the previous proposition, we have that $d^2 h = 0$ can be written as

$$\begin{aligned} &D_1 D_{22} D_1 \partial_1 h_{21} + D_2 D_{11} D_2 \partial_2 h_{12} + D_{11} D_{22} D_1 \partial_1^2 h_{21} \\ &\quad + D_{12} D_{11} D_2 \partial_1 \partial_2 h_{12} + D_{21} D_{22} D_1 \partial_2 \partial_1 h_{21} \\ &\quad + D_{22} D_{11} D_2 \partial_2^2 h_{12} = 0. \end{aligned}$$

If we now use the binomial relations from the previous proposition, we obtain

$$\begin{aligned} &D_1 D_{22} D_1 (\partial_1 h_{21} + \partial_2 h_{12}) + D_{11} D_{22} D_1 (\partial_1^2 h_{21} + \partial_1 \partial_2 h_{12}) \\ &\quad + D_{21} D_{22} D_1 (\partial_2 \partial_1 h_{21} + \partial_2^2 h_{12}) = 0. \end{aligned}$$

Noting the independence of the symbols for the 3-forms and the fact that the coefficients of the last two forms depend on the coefficient of the first form, we obtain the result. □

On the basis of what we know from the syzygies of the Dirac complex in two operators, we would now expect that the complex of differential forms should be naturally close to zero. In fact, we can easily establish that the differential operator d^3 is identically zero on F_3 and therefore we have the following result that concludes the description in the case of two variables:

Theorem 4.6. *Monogenic functions in two variables can be embedded in the following de Rham-like complex:*

$$0 \to \mathcal{R} \hookrightarrow F_0 \xrightarrow{d^0} F_1 \xrightarrow{d^1} F_2 \xrightarrow{d^2} F_3 \xrightarrow{d^3} 0.$$

Proof: As in the case of lower degree forms, we begin by establishing what relations are implied by $d^3 d^2 = 0$. Define $d^3 = D_1\partial_1 + D_2\partial_2 + D_{11}\partial_1^2 + D_{12}\partial_1\partial_2 2 + D_{21}\partial_2\partial_1 + D_{22}\partial_2^2$. Since

$$\begin{aligned} d^2 h = {} & D_1 D_{22} D_1(\partial_1 h_{21} + \partial_2 h_{12}) \\ & + D_{11} D_{22} D_1(\partial_1^2 h_{21} \\ & + \partial_1\partial_2 h_{12}) + D_{21} D_{22} D_1(\partial_2\partial_1 h_{21} + \partial_2^2 h_{12}) \end{aligned}$$

and using the relations obtained from the previous proposition, we argue as in the previous proofs and obtain the following relations on the coefficients for degree 3 derivatives,

$$\begin{aligned} & D_i D_{11} D_{22} D_1 + D_{i1} D_1 D_{22} D_1 = 0 \\ & D_{i2} D_1 D_{22} D_1 + D_i D_{21} D_{22} D_1 = 0 \quad i = 1, 2, \end{aligned}$$

and for the degree 4 derivatives,

$$\begin{aligned} & D_{11}^2 D_{22} D_1 = 0 \\ & D_{22} D_{21} D_{22} D_1 = 0 \\ & D_{12} D_{11} D_{22} D_1 = 0 \\ & D_{21}^2 D_{22} D_1 = 0 \\ & D_{1i} D_{21} D_{22} D_1 + D_{2i} D_{11} D_{22} D_1 = 0 \quad i = 1, 2. \end{aligned}$$

Let ω be a form in F_3. By the computations done earlier, ω can be written as

$$\omega = D_1 D_{22} D_1 \omega_1 + D_{11} D_{22} D_1 \omega_2 + D_{21} D_{22} D_1 \omega_3.$$

We now use the relations we have just found and easily get $d^3\omega \equiv 0$. □

Remark 4.7. The de Rham-like complex which we have constructed is self dual in the sense that d^2 is the transpose of d^0. This is not surprising because the same structure occurs in the resolution of the Dirac system in two variables,

$$\begin{aligned} 0 \longrightarrow R^{2^m}(-4) \longrightarrow R^{2\cdot 2^m}(-3) \longrightarrow R^{2\cdot 2^m}(-1) \\ \longrightarrow R^{2^m} \longrightarrow \mathcal{M}_2 \longrightarrow 0. \end{aligned}$$

5. THE CASE OF THREE VARIABLES

We now deal with the complex of three Dirac operators whose complete description, originally given in [Sabadini et al. 02], has been recalled in Section 2. Let $\mathcal{R}$ be the space of monogenic functions in three variables x_1, x_2, x_3. Let F_0 be the space of $\mathcal{C}^\infty$ functions in x_1, x_2, x_3 and let F_1 be the space of "1-forms" whose elements are written as $g = \sum_{i=1}^3 D_i g_i$, $g_i \in F_0$, so that if $d^0 = \sum_{i=1}^3 D_i\partial_i$, where $\partial_i = \partial_{x_i}$, we have an exact sequence,

$$0 \to \mathcal{R} \hookrightarrow F_0 \xrightarrow{d^0} F_1.$$

The next step in the construction of the complex consists in defining a space F_2 of "2-forms" and a suitable "differential" $d^1 : F_1 \to F_2$ such that $d^1 d^0 = 0$. As in the case of two operators, one immediately sees that the first order differential d^1 does not have the degree one component d^1_1, so that

$$d^1 g = d^1_2 g = \sum_{i,j,k=1}^{3} D_{ki} D_j \partial_k \partial_i g_j.$$

We have the following proposition:

Proposition 5.1. *Let $d^0 = \sum_{i=1}^3 D_i\partial_i$ and $d^1 = \sum_{i,j=1}^3 D_{ij}\partial_i\partial_j$. Then $d^1 d^0 = 0$ implies the following:*

$$\begin{aligned} & D_{ij} D_i = 0, && i, j \in \{1, 2, 3\} \\ & D_{ii} D_j + D_{ji} D_i = 0, && i, j \in \{1, 2, 3\} \quad i \neq j \\ & D_{ik} D_j + D_{jk} D_i = 0, && (i, j, k) = (1, 3, 2), (3, 2, 1), (2, 1, 3) \\ & D_{23} D_1 + D_{31} D_2 + D_{12} D_3 = 0. \end{aligned} \tag{5–1}$$

Proof: The condition $d^1_2 d^0 = 0$ implies that for any $f \in F_0$, it is

$$\sum_{i,j,k=1}^{3} D_{ki} D_j \partial_k \partial_i \partial_j f = 0.$$

By writing the right-hand term explicitly, by using the radial algebra defining relations $\partial_i^2 \partial_j = \partial_j \partial_i^2$, $i \neq j$, and grouping the various terms, we get the statement. □

We now study the kernel of the map $d^1 : F_1 \to F_2$ and we show that a 1-form g is d^1-closed if and only if its components g_j satisfy the compatibility conditions of the system $d^0 f = g$.

Proposition 5.2. *Let $g = \sum_{i=1}^3 D_i g_i$ be an element of F_1. Then $d^1 g = 0$ if and only if*

$$\begin{aligned} &\partial_j\partial_i g_i - \partial_i^2 g_j = 0, \quad i,j = 1,2,3 \quad i \neq j \\ &\{\partial_2, \partial_3\} g_1 - \partial_1\partial_2 g_3 - \partial_1\partial_3 g_2 = 0 \\ &\{\partial_1, \partial_3\} g_2 - \partial_2\partial_1 g_3 - \partial_2\partial_3 g_1 = 0, \end{aligned} \tag{5–2}$$

i.e., $d^1 g = 0$ if and only if (g_1, g_2, g_3) satisfy the compatibility conditions for the solvability of the system

$$\begin{cases} \partial_1 f = g_1 \\ \partial_2 f = g_2 \\ \partial_3 f = g_3. \end{cases}$$

Proof: By the definition of g and d^1, we have that $d^1 g = 0$ can be written as

$$\sum_{i,j,k=1}^{3} D_{ij} D_k \partial_i \partial_j g_k = 0.$$

In view of (5–1), this can be rewritten in the form

$$\begin{aligned} &\sum_{i,j=1, i\neq j}^{3} D_{ii} D_j (\partial_j \partial_i g_i - \partial_i^2 g_j) \\ &\quad + D_{12} D_3 (\{\partial_2, \partial_3\} g_1 - \partial_1 \partial_2 g_3 - \partial_1 \partial_3 g_2) \\ &\quad + D_{31} D_2 (\{\partial_1, \partial_3\} g_2 - \partial_2 \partial_1 g_3 - \partial_2 \partial_3 g_1) = 0. \end{aligned}$$

Setting the coefficients of the independent symbols of the forms equal to zero, we get the statement. □

Remark 5.3. The computations which we have just concluded show that the space F_2 is generated by eight different symbols of the type $D_{ij} D_k$. More precisely, a 2-form $h \in F_2$ can be written as

$$\begin{aligned} h = {}& D_{33} D_1 h_{331} + D_{22} D_1 h_{221} + D_{11} D_2 h_{112} + D_{11} D_3 h_{113} \\ &+ D_{22} D_3 h_{223} + D_{33} D_2 h_{332} + D_{31} D_2 h_{312} + D_{12} D_3 h_{123}. \end{aligned}$$

From [Sabadini et al. 02], we know that the next stage of the complex should yield six linear syzygies and six quadratic syzygies. These are the syzygies of the system

$$\begin{cases} \partial_j \partial_i g_i - \partial_i^2 g_j = h_{iij} \qquad i \neq j \\ \{\partial_2, \partial_3\} g_1 - \partial_1 \partial_2 g_3 - \partial_1 \partial_3 g_2 = h_{123} \\ \{\partial_1, \partial_3\} g_2 - \partial_2 \partial_1 g_3 - \partial_2 \partial_3 g_1 = h_{213}. \end{cases} \tag{5–3}$$

The same result can be proved using megaforms, once we have computed the relations among the symbols D_i, $D_{j\ell}$. This is done in the next proposition:

Proposition 5.4. *Let d^1 be as above and let $d^2 = \sum_{i=1}^3 D_i \partial_i + \sum_{i,j=1}^3 D_{ij} \partial_i \partial_j$. Then $d^2 d^1 = 0$ implies*

$$\begin{aligned} &D_i D_{ii} D_j = 0 \qquad i \neq j \\ &D_i D_{jj} D_i - D_j D_{ii} D_j = 0 \qquad i \neq j \\ &D_1 D_{22} D_3 = D_3 D_{22} D_1 = D_2 D_{31} D_2, \quad D_2 D_{12} D_3 = 0 \\ &D_1 D_{31} D_2 = 0, \\ &D_3 D_{11} D_2 = D_2 D_{11} D_3 = -D_1 D_{12} D_3 \\ &D_3 D_{12} D_3 = D_1 D_{33} D_2 = D_2 D_{33} D_1 = -D_3 D_{31} D_2 \end{aligned} \tag{5–4}$$

and

$$\begin{aligned} &D_{ij} D_{jj} D_i = 0 \qquad D_{ii}^2 D_j = 0 \quad i \neq j, \quad i,j \in \{1,2,3\} \\ &D_{ii} D_{jj} D_i = D_{ij} D_{ii} D_j \qquad i \neq j, \quad i,j \in \{1,2,3\} \end{aligned} \tag{5–5}$$

$$\begin{aligned} &D_{ji} D_{jj} D_k = -D_{kj} D_{jj} D_i = D_{ij} D_{jj} D_k, \\ &\quad D_{jk} D_{jj} D_i = 0, \quad (i,j,k) = (1,2,3), (2,1,3), (2,3,1) \\ &D_{ii} D_{31} D_2 = -D_{ii} D_{12} D_3 = D_{21} D_{ii} D_3 \quad i = 1,2 \\ &D_{32} D_{33} D_1 = D_{33} D_{12} D_3, \quad D_{33} D_{31} D_2 = 0 \end{aligned} \tag{5–6}$$

$$\begin{aligned} &D_{31}^2 D_2 = D_{12}^2 D_3 = D_{32} D_{12} D_3 = D_{23} D_{11} D_2 = 0 \\ &D_{21} D_{12} D_3 = D_{12} D_{31} D_2 = D_{13} D_{22} D_1 = D_{21} D_{31} D_2 \\ &\qquad = D_{32} D_{11} D_2 = 0 \\ &D_{31} D_{22} D_3 = D_{32} D_{31} D_2, \\ &D_{32} D_{11} D_3 = -D_{31} D_{12} D_3 \\ &D_{21} D_{33} D_2 = D_{23} D_{12} D_3 = -D_{23} D_{31} D_2 \\ &D_{31} D_{22} D_1 = D_{22} D_{11} D_3 = -D_{11} D_{22} D_3, \\ &D_{13} D_{12} D_3 = D_{12} D_{33} D_1 = -D_{13} D_{31} D_2 \\ &D_{22} D_{33} D_1 + D_{33} D_{22} D_1 - D_{23} D_{12} D_3 - D_{32} D_{31} D_2 = 0 \\ &D_{33} D_{11} D_2 + D_{31} D_{12} D_3 - D_{13} D_{12} D_3 + D_{11} D_{33} D_2 = 0 \\ &D_{ji} D_{kk} D_i + D_{ii} D_{kk} D_j - D_{jk} D_{ii} D_k - D_{ik} D_{ij} D_k = 0 \\ &\qquad (i,j,k) = (1,2,3), (3,1,2). \end{aligned} \tag{5–7}$$

Proof: Let us compute $d^2 d^1 g$ for any $g \in F_1$. We have

$$\begin{aligned} d^2 d^1 g = d^2 [&D_{11} D_2 (\partial_2 \partial_1 g_1 - \partial_1^2 g_2) \\ &+ D_{11} D_3 (\partial_3 \partial_1 g_1 - \partial_1^2 g_3) \\ &+ D_{22} D_1 (\partial_1 \partial_2 g_2 - \partial_2^2 g_1) + D_{22} D_3 (\partial_3 \partial_2 g_2 - \partial_2^2 g_3) \\ &+ D_{33} D_1 (\partial_1 \partial_3 g_3 - \partial_3^2 g_1) + D_{33} D_2 (\partial_2 \partial_3 g_3 - \partial_3^2 g_2) \\ &+ D_{12} D_3 (\partial_2 \partial_3 g_1 + \partial_3 \partial_2 g_1 - \partial_1 \partial_2 g_3 - \partial_1 \partial_3 g_2) \\ &+ D_{31} D_2 (\partial_2 \partial_1 g_3 + \partial_2 \partial_3 g_1 - \partial_1 \partial_3 g_2 - \partial_3 \partial_1 g_2)]. \end{aligned}$$

We may rewrite this expression by grouping the terms containing degree 3 derivatives of the g_i. The terms containing two different indices give the first two relations, while the terms containing all the indices $1, 2, 3$ can be grouped according to the number of indices 1, 2, and 3 appearing. This gives the remaining degree 3 relations. Analogously, one can compute $d^2 d^1 g$ for any $g \in F_1$. To treat this case, one needs to write and group 180 summands. For the sake of brevity, we will write explicitly only two examples of computations: one containing two different indices and one containing three different indices. The other cases, that can be more complicated, are treated in a similar way.

Let us begin with the case in which only two different indices appear and that will give the relations of the form (5–5): We look at the summands containing $1, 1, 2, 2, 2$. Using the radial algebra relations, one gets

$$\begin{aligned}&(D_{21}D_{22}D_1 - D_{22}D_{11}D_2)\partial_2^3\partial_1 g_1 - D_{12}D_{22}D_1\partial_1\partial_2\partial_1\partial_2 g_2\\ &\quad + D_{12}D_{22}D_1\partial_1\partial_2^3 g_1 + (D_{22}D_{11}D_2 - D_{21}D_{22}D_1)\partial_1^2\partial_2^2 g_2\\ &\quad = 0.\end{aligned}$$

By setting the coefficients of the independent derivatives equal to zero, one gets relations of the form (5–5). Now we look at the summands containing the indices $1, 3, 2, 2, 2$:

$$\begin{aligned}&(D_{23}D_{22}D_1 - D_{22}D_{31}D_2 - D_{22}D_{12}D_3)\partial_2^3\partial_3 g_1\\ &\quad + (D_{32}D_{22}D_1 - D_{22}D_{12}D_3)\partial_3\partial_2^3 g_1\\ &\quad + (-D_{22}D_{31}D_2 + D_{21}D_{22}D_3)\partial_2^3\partial_1 g_3\\ &\quad + (D_{22}D_{12}D_3 + D_{12}D_{22}D_3)\partial_1\partial_2^3 g_3\\ &\quad + D_{22}D_{31}D_2\partial_2^2\partial_3\partial_1 g_2\\ &\quad + (D_{22}D_{31}D_2 + D_{22}D_{12}D_3)\partial_2^2\partial_1\partial_3 g_2\\ &\quad - D_{21}D_{22}D_3\partial_2\partial_1\partial_3\partial_2 g_2 - D_{32}D_{22}D_1\partial_3\partial_2\partial_1\partial_2 g_2\\ &\quad - D_{23}D_{22}D_1\partial_2\partial_3\partial_1\partial_2 g_2 - D_{12}D_{22}D_3\partial_1\partial_2\partial_3\partial_2 g_2.\end{aligned}$$

The derivatives of g_1 and g_2 are independent and their coefficients can be set equal to zero, while for the derivatives of g_2, we can use the radial algebra relations. For example, by replacing $\partial_3\partial_2\partial_1$ with $\partial_2\partial_1\partial_3 + \partial_1\partial_2\partial_3 - \partial_3\partial_1\partial_2$, we get

$$\begin{aligned}&(D_{22}D_{31}D_2 + D_{32}D_{22}D_1)\partial_2^2\partial_3\partial_1 g_2\\ &\quad + (D_{22}D_{31}D_2 + D_{22}D_{12}D_3)\partial_2^2\partial_1\partial_3 g_2\\ &\quad - (D_{21}D_{22}D_3 + D_{32}D_{22}D_1)\partial_2\partial_1\partial_3\partial_2 g_2\\ &\quad - D_{23}D_{22}D_1\partial_2\partial_3\partial_1\partial_2 g_2\\ &\quad - (D_{12}D_{22}D_3 + D_{32}D_{22}D_1)\partial_1\partial_2\partial_3\partial_2 g_2.\end{aligned}$$

Setting the coefficients equal to zero, we get

$$\begin{aligned}&D_{21}D_{22}D_3 = -D_{32}D_{22}D_1 = D_{12}D_{22}D_3 = D_{22}D_{31}D_2\\ &\qquad = -D_{22}D_{12}D_3,\\ &D_{23}D_{22}D_1 = 0\end{aligned}$$

that are relations of the form (5–6). All the other possible alignments of five indices chosen in $\{1, 2, 3\}$ of the form i, i, i, j, k give similar relations while the case of form i, i, j, j, k is more complicated, but can be treated in a similar way and give the relations of the form (5–7). □

Proposition 5.5. *Let* $h = \sum_{i,j=1, i\neq j}^{3} D_{ii}D_j h_{iij} + D_{12}D_3 h_{123} + D_{31}D_2 h_{312}$ *be a generic element in* F_2. *Then* $d^2 h = 0$ *if and only if*

$$\begin{aligned}&\partial_j h_{iij} + \partial_i h_{jji} = 0 \qquad i, j = 1, 2, 3\\ &\partial_1 h_{123} - \partial_2 h_{113} - \partial_3 h_{112} = 0\\ &\partial_2 h_{312} + \partial_1 h_{223} + \partial_3 h_{221} = 0\\ &\partial_1 h_{332} + \partial_2 h_{331} + \partial_3 h_{123} - \partial_3 h_{312} = 0\end{aligned} \tag{5–8}$$

and

$$\begin{aligned}&\partial_i^2 h_{jji} + \partial_i\partial_j h_{iij} = 0, \qquad i, j = 1, 2, 3\\ &\partial_i^2 h_{312} - \partial_i^2 h_{123} + \{\partial_i, \partial_j\} h_{ii3} - \partial_3\partial_1 h_{iij} = 0,\\ &\qquad\qquad i, j = 1, 2, \quad i \neq j\\ &\partial_3^2 h_{123} + \{\partial_2, \partial_3\} h_{331} - \partial_1\partial_3 h_{332} = 0\\ &\partial_i^2 h_{jjk} - \partial_j^2 h_{iik} - \partial_k\partial_i h_{jji} = 0, \quad (i, j, k) = (1, 2, 3)\\ &\partial_i^2 h_{33j} - \partial_3^2 h_{iij} - \partial_j\partial_i h_{33i} = 0, \quad i, j = 1, 2 \quad i \neq j\\ &\partial_i\partial_j h_{33j} + \partial_i\partial_3 h_{jj3} = 0, \qquad i, j = 1, 2 \quad i \neq j\\ &\partial_3\partial_1 h_{123} - \partial_3\partial_2 h_{113} - \partial_3^2 h_{112} = 0\\ &\partial_2^2 h_{331} + \partial_3\partial_1 h_{223} + \partial_3\partial_2 h_{312} - \partial_1\partial_2 h_{332} = 0\\ &\partial_2^2 h_{331} + \partial_2\partial_1 h_{332} + \partial_2\partial_3 h_{123} - \partial_2\partial_3 h_{312} = 0\\ &\{\partial_1, \partial_2\} h_{331} + \partial_1\partial_3(h_{123} - h_{312}) + \partial_3^2 h_{112} = 0.\end{aligned} \tag{5–9}$$

Proof: By the definition of $h \in F_2$, of d^2, and by using the monomial relations (5–4) in the previous proposition, we can write the relation $d^2 h$ grouping the various terms, according to the number of indices. We get

$$\begin{aligned}&(D_1\partial_1 + D_2\partial_2 + D_3\partial_3)h = D_1D_{33}D_1(\partial_1 h_{331} + \partial_3 h_{113})\\ &\quad + D_1D_{22}D_1(\partial_1 h_{221} + \partial_2 h_{112})\\ &\quad + D_2D_{33}D_2(\partial_2 h_{332} + \partial_3 h_{223})\\ &\quad + D_1D_{22}D_3(\partial_1 h_{223} + \partial_3 h_{221} + \partial_2 h_{312})\\ &\quad + D_1D_{33}D_2(\partial_1 h_{332} + \partial_2 h_{331} + \partial_3 h_{123} - \partial_3 h_{312})\\ &\quad + D_1D_{12}D_3(\partial_1 h_{123} - \partial_2 h_{113} - \partial_3 h_{112})\end{aligned}$$

that gives the relations (5–8). We now consider $\sum_{i,j=1}^{3} D_{ij}\partial_i\partial_j h = 0$: the left-hand side consists of 72 summands that we will consider separately, according to the number of indices 1, 2, 3 in it. The summands containing two different indices i, j are of the form

$$\begin{aligned}&D_{ii}D_{jj}D_i\partial_i^2 h_{jji} + D_{ii}^2 D_j\partial_i^2 h_{iij} + D_{ij}D_{jj}D_i\partial_i\partial_j h_{jji}\\ &\quad + D_{ij}D_{ii}D_j\partial_i\partial_j h_{iij} + D_{jj}^2 D_i\partial_j^2 h_{jji} + D_{jj}D_{ii}D_j\partial_j^2 h_{iij}\\ &\quad + D_{ji}D_{jj}D_i\partial_j\partial_i h_{jji} + D_{ji}D_{ii}D_j\partial_j\partial_i h_{iij} = 0,\end{aligned}$$

which, using relation (5–5), gives

$$\begin{aligned}&D_{ii}D_{jj}D_i(\partial_i^2 h_{jji} + \partial_i\partial_j h_{iij})\\ &\quad + D_{jj}D_{ii}D_j(\partial_j^2 h_{iij} + \partial_j\partial_i h_{jji}) = 0,\end{aligned}$$

i.e., a relation of the type $\partial_i^2 h_{jji} + \partial_i\partial_j h_{iij} = 0$. Now we look at the summands containing 3 different indices of the type (i,i,i,j,k): Since the three different cases that can appear can be treated in a similar way, we only treat in detail the case $(1,1,1,2,3)$. We have

$$\begin{aligned} &D_{11}D_{31}D_2\partial_1^2h_{312} + D_{11}D_{12}D_3\partial_1^2h_{123} \\ &\quad + D_{12}D_{11}D_3\partial_1\partial_2h_{113} + D_{21}D_{11}D_3\partial_2\partial_1h_{113} \\ &\quad + D_{31}D_{11}D_2\partial_3\partial_1h_{112} + D_{13}D_{11}D_2\partial_1\partial_3h_{112} = 0 \end{aligned}$$

that, using relation (5–6), becomes

$$\begin{aligned} &D_{11}D_{31}D_2(\partial_1^2h_{312} - \partial_1^2h_{123} + \partial_1\partial_2h_{113} \\ &\quad + \partial_2\partial_1h_{113} - \partial_3\partial_1h_{112}) = 0. \end{aligned}$$

The case $(2,2,2,1,3)$ gives a similar relation (it suffices to exchange the role of 1 and 2 in the previous relation) while the case $(3,3,3,1,2)$ gives

$$\partial_3^2h_{123} + \{\partial_2,\partial_3\}h_{331} - \partial_1\partial_3h_{332} = 0.$$

Finally, we look at summands containing (i,i,j,j,k). The case $(1,1,2,2,3)$ is the simplest one: We get

$$\begin{aligned} &D_{11}D_{22}D_3\partial_1^2h_{223} + D_{12}D_{31}D_2\partial_1\partial_2h_{312} \\ &\quad + D_{12}^2D_3\partial_1\partial_2h_{123} + D_{22}D_{11}D_3\partial_2^2h_{113} \\ &\quad + D_{21}D_{31}D_2\partial_2\partial_1h_{312} + D_{21}D_{12}D_3\partial_2\partial_1h_{123} \\ &\quad + D_{31}D_{22}D_1\partial_3\partial_1h_{221} + D_{13}D_{22}D_1\partial_1\partial_3h_{221} \\ &\quad + D_{32}D_{11}D_2\partial_3\partial_2h_{112} + D_{23}D_{11}D_2\partial_2\partial_3h_{112} = 0, \end{aligned}$$

which, using (5–6), becomes

$$D_{11}D_{22}D_3(\partial_1^2h_{223} - \partial_2^2h_{113} - \partial_3\partial_1h_{221}) = 0.$$

We look at $(1,2,2,3,3)$ (the remaining case, $(1,1,2,3,3)$, can be treated in a similar way) and we get

$$\begin{aligned} &D_{12}D_{33}D_2\partial_1\partial_2h_{332} + D_{22}D_{33}D_1\partial_2^2h_{331} \\ &\quad + D_{21}D_{33}D_2\partial_2\partial_1h_{332} + D_{31}D_{22}D_3\partial_3\partial_1h_{223} \\ &\quad + D_{13}D_{22}D_3\partial_1\partial_3h_{223} + D_{32}D_{12}D_3\partial_3\partial_2h_{123} \\ &\quad + D_{23}D_{31}D_2\partial_2\partial_3h_{312} + D_{23}D_{12}D_3\partial_2\partial_3h_{123} \\ &\quad + D_{32}D_{31}D_2\partial_3\partial_2h_{312} + D_{33}D_{22}D_1\partial_3^2h_{21} = 0, \end{aligned}$$

from which we get

$$\begin{aligned} &D_{33}D_{22}D_1(\partial_1\partial_2h_{332} - \partial_2^2h_{331} + \partial_3^2h_{221}) \\ &\quad + D_{13}D_{22}D_3(\partial_1\partial_2h_{332} + \partial_1\partial_3h_{223}) \\ &\quad + D_{32}D_{31}D_2(-\partial_1\partial_2h_{332} + \partial_2^2h_{331} \\ &\qquad\qquad + \partial_3\partial_1h_{223} + \partial_3\partial_2h_{312}) \\ &\quad + D_{23}D_{12}D_3(\partial_2^2h_{331} + \partial_2\partial_1h_{332} \\ &\qquad\qquad - \partial_2\partial_3h_{312} + \partial_2\partial_3h_{123}) = 0, \end{aligned}$$

i.e., the conditions (5–9). □

Proposition 5.6. *Let h be a generic element in F_2. Then $d^2h = 0$ if and only if h satisfies the compatibility conditions (2–4) for the system (5–3).*

Proof: We have shown in the previous proposition that the condition $d^2h = 0$ is equivalent to (5–8) and (5–9). By setting $h_{iij} = h_{ij}$, $h_{123} = a_1$, $h_{312} = -a_2$, and $a_3 = -a_1 - a_2$, one can see that relation (5–8) coincides with the linear syzygies at the second step given in (2–4), while (5–9) contains the quadratic syzygies at the second step plus some redundant relations that are implied by the linear syzygies (like $\partial_i^2h_{jji} + \partial_i\partial_jh_{iij} = 0$) or that are dependent on the quadratic ones. □

In principle, it is possible to continue following the same procedure. At the next step, one can select a basis of the space F_3: For example, one writes a generic element in F_3 as

$$\begin{aligned} &D_1D_{22}D_1R_3 + D_1D_{33}D_1R_2 + D_2D_{33}D_2R_1 \\ &\quad + D_1D_{33}D_2S_3 + D_1D_{22}D_3S_2 + D_1D_{12}D_3S_1 \\ &\quad + D_{12}D_{22}D_3T_{13} + D_{11}D_{31}D_2T_{32} + D_{31}D_{33}D_2T_{12} \\ &\quad + D_{33}D_{22}D_1U_{23}D_{11}D_{33}D_2 + U_{13} + D_{11}D_{22}D_3U_{12}. \end{aligned}$$

Then one finds the relations among the symbols of megaforms and verifies that the closure conditions of d^3 correspond to the compatibility conditions of the inhomogeneous system at this step, and then one continues in the same way. We do not perform all those computations here, but we only give an example by selecting a special set of indices. If we choose to consider the alignment $(1,1,2,2,3,3)$, the condition $d^3d^2 = 0$ implies

$$\begin{aligned} D_{12}D_1D_{33}D_2 &= D_{21}D_1D_{33}D_2 = D_{13}D_1D_{22}D_3 \\ &= D_{31}D_1D_{22}D_3 = 0 \\ D_{11}D_2D_{33}D_2 &= -D_{22}D_1D_{33}D_1 = -D_3D_{11}D_{22}D_3 \\ D_{33}D_1D_{22}D_1 &= D_2D_{11}D_{33}D_2 \\ &= D_3D_{11}D_{22}D_3 - D_1D_{33}D_{22}D_1. \end{aligned}$$

The usual procedure shows that the relations imposed on an element in F_3, for the given alignment of the indices, are:

$$\begin{aligned} &\partial_3^2R_3 - \partial_2^2R_2 - \partial_1^2R_1 + \partial_2U_{13} + \partial_3U_{12} = 0 \\ &-\partial_3^2R_3 + \partial_1U_{23} - \partial_2U_{13} = 0. \end{aligned}$$

Using the identity $U_{ij} + U_{ji} = \partial_kR_k$ given in Section 2, we get the analogue of the syzygy $\partial_3^2R_3 - \partial_1^2R_1 - \partial_2U_{31} + \partial_3U_{12} = 0$.

These computations allow us to conjecture that this procedure can be applied at every step producing the following De Rham-like complex:

$$0 \to \mathcal{R} \hookrightarrow F_0 \xrightarrow{d^0} F_1 \xrightarrow{d^1} F_2 \xrightarrow{d^2} F_3 \xrightarrow{d^3} F_4 \xrightarrow{d^4} F_5 \xrightarrow{d^5} 0.$$

More generally, it is natural to conjecture that the theory of megaforms is the suitable tool to explicitly write all the maps appearing in the Dirac complex, at least when the dimension of the algebra is at least $2\ell - 1$ where ℓ is the number of operators considered. These computations show the difficulty of obtaining the syzygies. However, the megaforms which we have introduced provide a direct constructive process. This process, differently from CoCoA, uses the Clifford structure directly and provides the syzygies in terms of Clifford algebras and not of real numbers. We believe that further experimentation along these lines will allow us to conclusively establish this conjecture.

ACKNOWLEDGMENTS

The first and third authors wish to thank the State University of Ghent for the kind hospitality in the period during which this paper was written. The authors wish to thank Fabrizio Colombo for useful discussions. This research was sponsored by FWO-Krediet aan Navorsers 1.5.106.02.

REFERENCES

[Adams et al. 99] W. W. Adams, C. A. Berenstein, P. Loustaunau, I. Sabadini, and D. C. Struppa. "Regular Functions of Several Quaternionic Variables and the Cauchy-Fueter Complex." *J. Geom. Anal.* 9 (1999), 1–16.

[Adams and Loustaunau 94] W. W. Adams and P. Loustaunau. *An Introduction to Gröbner Basis*, Graduate Studies in Mathematics, Volume 3. Providence, RI: Amer. Math. Soc., 1994.

[Adams and Loustaunau 01] W. W. Adams and P. Loustaunau. "Analysis of the Module Determining the Properties of Regular Functions of Several Quaternionic Variables." *Pacific J.* 196 (2001) 1–15.

[Adams et al. 97] W. W. Adams, P. Loustaunau, V. Palamodov, and D. C. Struppa. "Hartog's Phenomenon for Polyregular Functions and Projective Dimension of Related Modules." *Ann. Inst. Fourier* 47 (1997), 623–640.

[Baston 92] R. J. Baston. "Quaternionic Complexes." *J. Geom. Phys.* 8:1–4 (1992), 29–52.

[Berenstein et al. 96] C. A. Berenstein, I. Sabadini, and D. C. Struppa. *Boundary Values of Regular Functions of Quaternionic Variables*, Pitman Res. Notes in Math. Ser., Volume 347. Harlow: Longman Scientific and Technical, 1996.

[Brackx et al. 82] F. Brackx, R. Delanghe, and F. Sommen. *Clifford Analysis*, Pitman Res. Notes in Math., Volume 76. Harlow: Longman Scientific and Technical, 1982.

[Colombo et al. 03] F. Colombo, I. Sabadini, F. Sommen, and D. C. Struppa. *Analysis of Dirac Systems and Computational Algebra*, Progress in Mathematical Physics. Cambridge, MA: Birkhäuser Verlag, 2003 (in preparation).

[Constales 89] D. Constales. "The Relative Position of L^2-Domains in Complex and Clifford Analysis." PhD thesis, State Univ. Ghent, 1989.

[Delanghe et al. 92] R. Delanghe, F. Sommen, and V. Souček. *Clifford Algebra and Spinor Valued Functions: A Function Theory for the Dirac Operator*, Math. and its Appl., Volume 53. Dordrecht: Kluwer Acad. Publ., 1992.

[Ehrenpreis 70] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*. New York: Wiley-Interscience Publ., 1970.

[Gilbert and Murray 90] J. Gilbert and M. Murray. *Clifford Algebras and Dirac Operators in Harmonic Analysis*. Cambridge, UK: Cambridge Univ. Press, 1990.

[Hestenes and Sobczyk 85] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus*. Dordrecht: D. Reidel, 1985.

[Komatsu 73] H. Komatsu. "Relative Cohomology of Sheaves of Solutions of Differential Equations." In *Hyperfunctions and Pseudo-Differential Equations*, pp. 196–261, Lecture Notes in Mathematics, Volume 287, edited by H. Komatsu. Berlin-Heidelberg: Springer-Verlag, 1973.

[Krantz 83] S. Krantz. *Function Theory of Several Complex Variables*. Belmont, CA: Wadsworth and Brooks/Cole, 1992.

[Laville 83] G. Laville. "Une famille de solutions de l'equation de Dirac avec champ electromagnetique quelconque." C.R. Acad. Sci. Paris 296 (1983), 1029–1032.

[Palamodov 70] V. P. Palamodov. *Linear Differential Operators with Constant Coefficients*. New York: Springer Verlag, 1970.

[Palamodov 99] V. V. Palamodov. "Holomorphic Synthesis of Monogenic Functions of Several Quaternionic Variables." *J. Anal. Math.* 78 (1999), 177–204.

[Pertici 88] D. Pertici. "Funzioni regolari di piú variabili quaternioniche." *Ann. Mat. Pura Appl., Serie IV, CLI* (1988), 39–65.

[Sabadini and Struppa 97] I. Sabadini and D. C. Struppa. "Some Open Problems on the Analysis of the Cauchy-Fueter System in Several Variables." *Suk. RIMS Kyoto Univ.* 1001 (1997), 1–21.

[Sabadini and Struppa 02] I. Sabadini amd D. C. Struppa. "Computational Algebraic Analysis Methods in Clifford Analysis." *Math. Methods Appl. Sci.* 25 (2002), 1415–1427.

[Sabadini et al. 02] I. Sabadini, F. Sommen, D. C. Struppa, and P. Van Lancker. "Complexes of Dirac Operators in Clifford Algebras." *Math. Z.* 239:2 (2002) 293–320.

[Somberg 01] P. Somberg. "Quaternionic Complexes in Clifford Analysis." In *Clifford Analysis and its Applications*, pp. 293–301, NATO ARW Series, edited by F. Brackx, J. S. R. Chisholm, and V. Souček. Dordrecht: Kluwer Acad. Publ., 2001.

[Sommen 87] F. Sommen. "Martinelli-Bochner Type Formulae in Complex Clifford Analysis." *Z.A.A.* 6:1 (1987), 75–82.

[Sommen 97] F. Sommen. "An Algebra of Abstract Vector Variables." *Portugaliae Mathematica* 54:3 (1997), 287–310.

[Sommen 99] F. Sommen. "Clifford Analysis in Two and Several Vector Variables." *Applicable Analysis* 73 (1999), 225–253.

[Sommen 01] F. Sommen. "Clifford Analysis on the Level of Abstract Vector Variables." In *Clifford Analysis and its Applications*, pp. 303–322, NATO ARW Series, edited by F. Brackx, J. S. R. Chisholm, and V. Souček. Dordrecht: Kluwer Acad. Publ., 2001.

[Struppa 98] D. C. Struppa. "Gröbner Bases in Partial Differential Equations." In *Gröbner Bases and Applications (Linz, 1998),* pp. 235–245, London Math. Soc. Lecture Note Ser., Volume 251. Cambridge, UK: Cambridge Univ. Press, 1998.

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Received August 21, 2002; accepted in revised form September 2, 2003.