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On the existence of periodic solutions to second order Hamiltonian systems

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Abstract. In this paper, the existence of periodic solutions to the second order Hamiltonian systems is investigated. By introducing a new growth condition which generalizes the Ambrosetti–Rabinowitz condition, we prove a existence result of nontrivial T-periodic solution via the variational methods. Our result is new because it can deal with not only the superquadratic case, but also the anisotropic case which allows the potential to be superquadratic growth in only one direction and asymptotically quadratic growth in other directions.

Keywords: second order Hamiltonian systems, periodic solutions, existence, variational method.

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1 Introduction and main result

Consider the following second order Hamiltonian systems

$$\begin{cases} -\ddot{u}(t) + L(t)u(t) = \nabla_x F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $u(t) = (u_1(t), u_2(t), \ldots, u_N(t))$, $N \geq 1$, $T > 0$, $L(t) := \big(l_{ij}(t)\big) \in C(0, T; \mathbb{R}^{N\times N})$ is a symmetric positive matrix and T-periodic in t, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in t and satisfies the following assumptions:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \qquad |\nabla_x F(t, x)| \leq a(|x|)b(t)$$

for $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $x := (x_1, \ldots, x_N)$, $\nabla_x F(t, x) := \big(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_N}\big)$.

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The periodic solutions to non-autonomous system (1.1) has an extensive history in the case of singular systems (cf., e.g., Ambrosetti–Coti Zelati [1]). The first to consider it for nonsingular potentials were Berger and Schechter [3] in 1977. Since then, the existence of periodic solutions to system (1.1) have been deeply studied by a large number of researchers. Many solvability conditions about the potentials have been obtained, we refer the readers to [4, 11, 12, 16, 17, 19–23, 26–28] and their references. In 1978, Rabinowitz [13] established the existence of a non-constant T-periodic solution when $L(t) \equiv 0$ by assuming that the potential F satisfies the following superquadratic condition

(AR) there exist constants $r_0 > 0$ and $\theta > 2$ such that

$$0 < \theta F(t,x) \leq (\nabla_x F(t,x), x)$$

for $|x| \geq r_0$ and a.e. $t \in [0,T]$, where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^N$.

This is the so-called Ambrosetti–Rabinowitz ((AR) for short) condition which plays a key role in verifying the mountain pass geometry and the compactness for the Euler–Lagrange functional associated to system (1.1). So (AR) condition has been widely used in follow-up research for the superquadratic problem, for example, see [5] and their references. If $F \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R})$, one can easily deduce from (AR) that

$$F(t,x) \geq a|x|^\theta - b$$

for $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$, where $a, b > 0$. This implies a more intrinsic superquadratic condition

(SQ) $\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} = +\infty$ uniformly in a.e. $t \in [0,T]$.

Under condition (SQ), one can also add some other conditions on F to guarantee the existence of T-periodic solutions. For example, Fei [7] assumed the nonquadratic condition

(NQ) $\liminf_{|x|\to\infty} \frac{(\nabla_x F(t,x),x) - 2F(t,x)}{|x|^\beta} > 0$ uniformly in a.e. $t \in [0,T]$,

where $\beta > 1$. Luan–Mao [10] supposed that F satisfied the following condition

(LM) there exist $c > 0$, $r_1 > 0$ and some $\sigma > 1$ such that

$$\frac{|\nabla_x F(t,x)|^\sigma}{|x|^\sigma} \leq cH(x,s)$$

for $|x| \geq r_1$ and a.e. $t \in [0,T]$, where $H(x,s) := (\nabla_x F(t,x), x) - 2F(t,x)$.

Wu and Tang [24] introduced a new superquadratic situation

(WT) there exist $c > 0$, $r_2 > 0$ such that

$$\frac{F(t,x)}{|x|^2} \leq cH(x,s)$$

for $|x| \geq r_2$ and a.e. $t \in [0,T]$.

Ye–Tang [30] and Li–Schechter [9] studied the situation that F satisfied the following monotonic condition

(M) there exist $D \geq 1$ and $C_* \in L^1(0,T;\mathbb{R}^+)$ such that

$$H(t,sx) \leq DH(t,x) + C_*(t), \qquad \forall s \in [0,1]$$

for $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$.

Schechter [18] assumed

(S1) $2F(t,x) \geq \lambda_{l-1}|x|^2$ for $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$, where λ_i is the *ith* eigenvalue of the operator $-\frac{d^2}{dt^2} + L(t)$,

(S2) there are constants $m > 0$ and $\vartheta > 0$ such that

$$2F(t,x) \leq \vartheta|x|^2$$

for $|x| \leq m$ and a.e. $t \in [0,T]$.

The readers are referred to [6,8,29] for more types of conditions under condition (SQ).
In addition, without condition (SQ), Schechter [14] assumed that

(S3) $F(t,x) \geq 0$ for $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$,

(S4) there are constants $m > 0$, $\alpha \leq 6m^2/T^2$ such that

$$F(t,x) \leq \alpha$$

for $|x| \leq m$ and a.e. $t \in [0,T]$,

(S5) there are $\mu > 2$, $r_3 > 0$ and $W \in L^1([0,T])$ such that

$$\begin{cases} (i) \ \ \frac{H_\mu(t,x)}{|x|^2} \leq W(t) \text{ for } |x| \geq r_3 \text{ and a.e. } t \in [0,T], \\ (ii) \ \ \limsup_{|x|\to+\infty} \frac{H_\mu(t,x)}{|x|^2} \leq 0, \end{cases}$$

where $H_\mu(t,x) := \mu F(t,x) - (\nabla_x F(t,x), x)$,

(S6) there is a subset $\Sigma \subset [0,T]$ of positive measure such that

$$\liminf_{|x|\to+\infty} \frac{F(t,x)}{|x|^2} > 0 \quad \text{uniformly in a.e. } t \in \Sigma.$$

In [15], the potentials F satisfy (S3)–(S5) and

(S7) there are constants $\beta > \frac{2\pi^2}{T^2}$ and $r_3 > 0$ such that

$$F(t,x) \geq \beta|x|^2$$

for $|x| > r_3$ and a.e. $t \in [0,T]$.

Wang–Zhang [25] assumed F satisfies (S3), (S4), (S6) and

(WZ) (i) there exist $M_1 > 0$, $\sigma > 1$ and $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to\infty} f(|x|) = +\infty$ and $\frac{f(|x|)}{|x|^\sigma}$ is non-increasing on $\mathbb{R}^+$ such that

$$(\nabla_x F(t,x), x) - 2F(t,x) \geq f(|x|)\frac{|\nabla F(t,x)|^\sigma}{|x|^\sigma}$$

for $|x| \geq M_1$ and a.e. $t \in [0,T]$, or
(ii) there exist $M_2 > 0$ and $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to\infty} g(|x|) = +\infty$ and $\frac{g(|x|)}{|x|^2}$ is non-increasing on $\mathbb{R}^+$ such that

$$(\nabla_x F(t,x), x) - 2F(t,x) \geq g(|x|)\frac{F(t,x)}{|x|^2}$$

for $|x| \geq M_2$ and a.e. $t \in [0,T]$.

In [31], Zhang–Tang assumed

(ZT) there exist constants $\mu > 2$, $0 < \beta < 2$, $L > 0$ and a function $a \in L^1(0,T;\mathbb{R}^+)$ such that

$$\mu F(t,x) \leq (\nabla_x F(t,x)x) + a(t)|x|^\beta$$

for $|x| \leq L$ and a.e. $t \in [0,T]$.

In this paper, we will give a new solvable condition. Our main result is the following theorem.

Theorem 1.1. *Assume that F satisfies assumptions* (A) *and*

(F_1) $\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = 0$ *uniformly in a.e.* $t \in [0,T]$,

(F_2) *there exist a constant $r_* > 0$ and a function θ such that*

$$0 < (2 + \theta(x))F(t,x) \leq (\nabla_x F(t,x), x)$$

for $|x| \geq r_*$ and a.e. $t \in [0,T]$, where $\theta : \{x \in \mathbb{R}^N : |x| \geq r_*\} \to \mathbb{R}$ is continuous and satisfies the following assumption

$$(\bigstar)\begin{cases} (i) \quad \theta(x) > 0,\ \forall |x| \geq r_*, \\ (ii) \quad \lim\limits_{|x|\to+\infty} \theta(x)|x|^2 = +\infty, \\ (iii) \quad \text{there is } x^0 \in \mathbb{R}^N \text{ with } |x^0| = 1 \text{ satisfying } \lim\limits_{r\to+\infty} \displaystyle\int_{r_*}^{r} \frac{\theta(sx^0)}{s} ds = +\infty, \end{cases}$$

then system (1.1) *has a nontrivial periodic solution.*

Remark 1.2. (1) Condition (F_2) is strictly weaker than the (AR) condition. In fact, we can derive from condition (F_2) that $\inf_{|x|\geq r_*} \theta(x) \geq 0$, and the (AR) condition is exactly equivalent to condition (F_2) when $\inf_{|x|\geq r_*} \theta(x) > 0$. On the one hand, the (AR) condition implies condition (F_2) with $\theta(x) \equiv \theta - 2 > 0$ and $r_* = r_0$. On the other hand, when $\inf_{|x|\geq r_*} \theta(x) > 0$, condition ($F_2$) implies the (AR) condition with $\theta := 2 + \inf_{|x|\geq r_*} \theta(x) > 2$ and $r_0 = r_*$. In addition, there are functions F satisfying condition (F_2) with $\inf_{|x|\geq r_*} \theta(x) = 0$, for example,

(Superlinear case) let

$$F(t,x)=\begin{cases}|x|^2\ln|x|-\frac{1}{2}e^2-\frac{1}{16}, & |x|\geq e;\\ \frac{1}{2}|x|^2-\frac{1}{16}, & \frac{1}{2}\leq |x|\leq e;\\ |x|^4, & |x|\leq \frac{1}{2}.\end{cases}$$

Then we have

$$\nabla_x F(t,x)=\begin{cases}2x\ln|x|+x, & |x|\geq e;\\ x, & \frac{1}{2}\leq |x|\leq e;\\ 4|x|^2x, & |x|\leq \frac{1}{2}\end{cases}$$

and

$$(\nabla_x F(t,x),x)-2F(t,x)=\begin{cases}|x|^2+e^2+\frac{1}{8}, & |x|\geq e;\\ \frac{1}{16}, & \frac{1}{2}\leq |x|\leq e;\\ 2|x|^4, & |x|\leq \frac{1}{2}.\end{cases}$$

It is easy to verify that F satisfies assumptions (A), (F_1), (F_2) with $\theta(x)=\frac{1}{\ln|x|}$ and $r_*=e$. However, $\inf_{|x|\geq e}\frac{1}{\ln|x|}=0$, so condition (AR) is not satisfied.

(2) It is particularly noteworthy that our Theorem 1.1 can deal with the potentials F without condition (SQ). In fact, there are functions with anisotropic growth satisfying condition (F_2), for example,

(Anisotropic case) let $N=2$, $x:=(x_1,x_2)\in\mathbb{R}^2$, and

$$F(t,x)=\begin{cases}x_1^4+\frac{5}{6}x_2^4, & x_2\leq 1;\\ x_1^4+x_2^2\cdot e^{-x_2^{-\frac{4}{3}}+1}-\frac{1}{6}, & x_2\geq 1.\end{cases}$$

Through simple calculation, we have

$$\nabla_x F(t,x):=\left(\frac{\partial F}{\partial x_1},\frac{\partial F}{\partial x_2}\right)=\begin{cases}4x_1^3+\frac{10}{3}x_2^3, & x_2\leq 1;\\ 4x_1^3+\left(2x_2+\frac{4}{3}x_2^{-\frac{1}{3}}\right)\cdot e^{-x_2^{-\frac{4}{3}}+1}, & x_2\geq 1\end{cases}$$

and

$$(\nabla_x F(t,x),x)-2F(t,x)=\begin{cases}2x_1^4+\frac{5}{3}x_2^4, & x_2\leq 1;\\ 2x_1^4+\frac{4}{3}x_2^{\frac{2}{3}}e^{-x_2^{-\frac{4}{3}}+1}+\frac{1}{3}, & x_2\geq 1.\end{cases}$$

Let

$$\theta(x)=\begin{cases}2, & x_2\leq 1;\\ \frac{2x_1^4+\frac{4}{3}x_2^{\frac{2}{3}}e^{-x_2^{-\frac{4}{3}}+1}+\frac{1}{3}}{x_1^4+x_2^2\cdot e^{-x_2^{-\frac{4}{3}}+1}-\frac{1}{6}}, & x_2\geq 1,\end{cases}$$

then for $|x|=\sqrt{x_1^2+x_2^2}$, $r_*=\sqrt{2}$ and $\mathbf{e}_1=(1,0)\in\mathbb{R}^2$, we can deduce that $\theta(x)>0$ for $|x|\geq\sqrt{2}$, $\lim_{|x|\to\infty}\theta(x)|x|^2=+\infty$, and

$$\lim_{r\to+\infty}\int_{\sqrt{2}}^{r}\frac{\theta(s\mathbf{e}_1)}{s}ds=\lim_{r\to+\infty}\int_{\sqrt{2}}^{r}\frac{2}{s}ds=+\infty,$$

which implies that F is superquadratic growth in direction $\mathbf{e}_1$. In addition, it is easy to verify that F satisfies assumptions (A), (F_1), (F_2). However, for $\mathbf{e}_2 = (0,1) \in \mathbb{R}^2$, we have

$$\lim_{x=s\mathbf{e}_2,\ |x|\to+\infty} \frac{F(t,x)}{|x|^2} = \lim_{|s|\to+\infty} \frac{F(t,s\mathbf{e}_2)}{|s|^2} = e,$$

which shows that F is asymptotically quadratic growth in direction $\mathbf{e}_2$. In conclusion, F satisfy condition (F_2) but not condition (SQ).

(3) Our Theorem 1.1 is different from all the results mentioned above. Firstly, condition (F_2) is strictly weaker than the (AR) condition. Secondly, we do not need the condition (SQ). More precisely, Theorem 1.1 can deal with not only the superquadratic case but also the the anisotropic case. Thirdly, we do not need more stringent and complex growth assumptions on F at 0.

2 Proof of the theorem

Let

$$H_T^1 = \left\{u \in L^2(0,T;\mathbb{R}^N) \mid u \text{ is weakly differentiable and } \dot{u} \in L^2(0,T;\mathbb{R}^N)\right\}$$

be a Hilbert space with the inner product and the induced norm respectively given by

$$\langle u,v\rangle_{H_T^1} = \int_0^T (\dot{u},\dot{v}) + (u(t),v(t))dt, \qquad \|u\|_{H_T^1} = \left(\int_0^T |\dot{u}(t)|^2 + |u(t)|^2 dt\right)^{\frac{1}{2}}.$$

Denoting by $\lambda_{\min}(t)$ and $\lambda_{\max}(t)$ respectively the smallest and the biggest eigenvalue of $L(t)$, then $\lambda_{\min}(t)$, $\lambda_{\max}(t) \in C(0,T;\mathbb{R}^+)$. Setting

$$\underline{\lambda} := \min_{t\in[0,T]} \lambda_{\min}(t), \qquad \overline{\lambda} := \max_{t\in[0,T]} \lambda_{\max}(t),$$

we have $0 < \underline{\lambda} \le \overline{\lambda}$ and

$$\underline{\lambda}|\xi|^2 \le \big(L(t)\xi,\xi\big) \le \overline{\lambda}|\xi|^2$$

for $\xi \in \mathbb{R}^N$ and $t \in [0,T]$. Thus, the following inner product and the corresponding induced norm on H_T^1 defined by

$$\langle u,v\rangle = \int_0^T (\dot{u},\dot{v}) + \big(L(t)u(t),v(t)\big)dt, \qquad \|u\| = \left(\int_0^T |\dot{u}(t)|^2 + \big(L(t)u(t),u(t)\big)dt\right)^{\frac{1}{2}}$$

are respectively equivalent to $\langle u,v\rangle_{H_T^1}$ and $\|u\|_{H_T^1}$. In fact, it is easy to verify that

$$\sqrt{\min\{1,\underline{\lambda}\}}\|u\|_{H_T^1} \le \|u\| \le \sqrt{\max\{1,\overline{\lambda}\}}\|u\|_{H_T^1}$$

for $u \in H_T^1$. By Sobolev's inequality, there is $M > 0$ such that

$$\|u\|_\infty \le M\|u\|, \qquad \forall u \in H_T^1,$$

where $\|u\|_\infty := \max_{t\in[0,T]}|u(t)|$. In addition, from the assumption (A) it follows that the functional Φ given by

$$\Phi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2}\int_0^T \big(L(t)u(t),u(t)\big)dt - \int_0^T F(t,u(t))dt$$

is continuously differentiable on H_T^1, and

$$\langle \Phi'(u), v\rangle = \int_0^T \left[\big(\dot u(t), \dot v(t)\big) + \big(L(t)u(t), v(t)\big) - \big(\nabla_x F(t, u(t)), v(t)\big) \right] dt.$$

Furthermore, the weak solutions to system (1.1) are exactly the critical points of Φ in H_T^1.

Lemma 2.1. *Assume that $\theta : \{x \in \mathbb{R}^N : |x| \geq r_*\} \to \mathbb{R}$ is continuous and satisfies condition $(\bigstar)(i)$, and suppose that there is a sequence $\{y_n\} \subset \{x \in \mathbb{R}^N : |x| \geq r_*\}$ such that*

$$\theta(y_n) \to 0 \quad \text{as } n \to \infty,$$

then $|y_n| \to +\infty$ as $n \to \infty$.

Proof. By negation, there exists a subsequence, still denoted by $\{y_n\}$, is bounded. After passing to a subsequence, we may assume that there is $y_0 \in \{x \in \mathbb{R}^N : |x| \geq r_*\}$ such that

$$y_n \to y_0 \quad \text{as } n \to \infty,$$

from this, $(\bigstar)(i)$ and the continuity of θ it follows that

$$0 = \lim_{n\to\infty} \theta(y_n) = \theta(y_0) > 0,$$

a contradiction. The proof of Lemma 2.1 is completed. □

Lemma 2.2. *Assume that F satisfies (F_1), then there are $\rho > 0$ and $\alpha > 0$ such that $\Phi(u) \geq \alpha$ for $u \in H_T^1$ with $\|u\| = \rho$.*

Proof. From (F_1), for $\varepsilon \in \big(0, \frac{1}{4TM^2}\big)$, there exists a constant $\delta > 0$ such that

$$|F(t,x)| \leq \varepsilon |x|^2$$

for $|x| < \delta$ and a.e. $t \in [0, T]$. Arbitrarily taking $\rho \in \big(0, \frac{\delta}{M}\big)$, we have

$$\|u\|_\infty \leq M\|u\| \leq M\rho < \delta$$

for $u \in H_T^1$ with $\|u\| = \rho$, this leads to

$$\Phi(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon \int_0^T |u(t)|^2 dt \geq \left(\frac{1}{2} - \varepsilon M^2 T\right) \|u\|^2 \geq \frac{\rho^2}{4}$$

for $u \in H_T^1$ with $\|u\| = \rho$. Setting $\alpha := \frac{\rho^2}{4} > 0$, then the proof Lemma 2.2 is completed. □

Lemma 2.3. *Assume that F satisfies assumptions (A) and (F_2), then there is $u_0 \in H_T^1$ with $\|u_0\| > \rho$ such that $\Phi(u_0) < 0$.*

Proof. From assumptions (A) and (F_2) it follows that

$$F(t, sx^0) \geq \frac{F(t, r_* x^0)}{r_*^2} \cdot e^{\int_{r_*}^s \frac{\theta(\tau x^0)}{\tau} d\tau} \cdot s^2$$

for $s \geq r_*$ and a.e. $t \in [0, T]$, then we have

$$\Phi(sx^0) = \frac{1}{2}\|sx^0\|^2 - \int_0^T F(t, sx^0) dt \leq \left(\frac{\overline{\lambda}^2}{2} - \int_0^T \frac{F(t, r_* x^0)}{r_*^2} dt \cdot e^{\int_{r_*}^s \frac{\theta(\tau x^0)}{\tau} d\tau} \right) s^2$$

for $s \geq r_*$ and a.e. $t \in [0, T]$, which implies $\Phi(u_0) < 0$ with $u_0 = sx^0$ for large s. This completes the proof of Lemma 2.3 □

Lemma 2.4. *Assume that F satisfies assumptions* (A), (F_1) *and* (F_2), *then Φ satisfies the* (C) *condition, that is, for any $c \in \mathbb{R}$ and every sequence $\{u_n\}$ such that*

$$\|\Phi'(u_n)\|(1+\|u_n\|) \to 0 \quad \textit{and} \quad \Phi(u_n) \to c \quad \textit{as } n \to \infty \tag{2.1}$$

has a convergent subsequence.

Proof. It suffices to prove that $\{u_n\}$ is bounded. Moreover, the proof is trivial when $\inf_{|x|\geq r_*}\theta(x) > 0$, so we just need to prove this lemma when $\inf_{|x|\geq r_*}\theta(x) = 0$.

We argue by contradiction. If $\{u_n\}$ is unbounded, then after passing to a subsequence, we may assume that

$$\lambda_n := \|u_n\| \to +\infty \quad \text{as } n \to \infty. \tag{2.2}$$

Setting $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$ and $u_n = \lambda_n w_n$. Thus, we deduce

$$\|w_n\|_\infty \leq M\|w_n\| \leq M.$$

Fixing $x_{\lambda_n} \in \left\{x \in \mathbb{R}^N : r_* \leq |x| \leq \lambda_n M\right\}$ to be such that

$$\theta(x_{\lambda_n}) = \min_{r_* \leq |x| \leq \lambda_n M} \theta(x), \tag{2.3}$$

then we have $\lambda_n \geq \frac{|x_{\lambda_n}|}{M}$, $0 < \theta(x_{\lambda_n}) \leq \theta^* := \min_{|x|=r_*}\theta(x)$,

$$0 < (2+\theta(x_{\lambda_n}))F(t,x) \leq (2+\theta(x))F(t,x) \leq (\nabla_x F(t,x), x) \tag{2.4}$$

for $r_* \leq |x| \leq \lambda_n M$ and a.e. $t \in [0,T]$. Moreover, from (2.2), (2.3) and $\inf_{|x|\geq r_*}\theta(x) = 0$ it follows that

$$\theta(x_{\lambda_n}) \to 0 \quad \text{as } n \to \infty.$$

Thus, from Lemma 2.1, we obtain

$$|x_{\lambda_n}| \to +\infty \quad \text{as } n \to \infty. \tag{2.5}$$

Setting

$$E_n^- = \{t \in [0,T] : |u_n(t)| < r_*\}, \qquad E_n^+ = \{t \in [0,T] : |u_n(t)| \geq r_*\},$$

then from (2.1) it follows that

$$\begin{aligned} o(1) &= \left|\langle \Phi'(u_n), u_n\rangle\right| \\ &= \left|\lambda_n^2 - \int_{E_n^-}(u_n(t), \nabla_x F(t,u_n(t))dt - \int_{E_n^+}(u_n(t), \nabla_x F(t,u_n(t))dt\right|, \end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$, this implies that

$$\int_{E_n^+}(u_n(t), \nabla_x F(t,u_n(t))dt \leq \lambda_n^2 + \left|\int_{E_n^-}(u_n(t), \nabla_x F(t,u_n(t))dt\right| + o(1). \tag{2.6}$$

In addition, it follows from assumption (A) that there is $\beta > 0$ such that

$$\left|\int_{E_n^-}(u_n(t), \nabla_x F(t,u_n(t))dt\right|, \quad \left|\int_{E_n^-} F(t,u_n(t))dt\right| \leq \beta,$$

which together with (2.1), (2.4) and (2.6) gives

$$\begin{aligned}
c + o(1) &= \Phi(u_n) \\
&= \frac{\lambda_n^2}{2} - \int_{E_n^-} F(t, u_n(t))dt - \int_{E_n^+} F(t, u_n(t))dt \\
&\geq \frac{\lambda_n^2}{2} - \beta - \frac{1}{2+\theta(x_{\lambda_n})} \int_{E_n^+} (u_n(t), \nabla_x F(t, u_n(t))dt \\
&\geq \frac{\lambda_n^2}{2} - \beta - \frac{1}{2+\theta(x_{\lambda_n})} \left(\lambda_n^2 + \left| \int_{E_n^-} (u_n(t), \nabla_x F(t, u_n(t))dt \right| + o(1) \right) \\
&\geq \frac{\theta(x_{\lambda_n})\lambda_n^2}{2(2+\theta(x_{\lambda_n}))} - \beta - \frac{\beta + o(1)}{2+\theta(x_{\lambda_n})} \\
&\geq \frac{\theta(x_{\lambda_n})|x_{\lambda_n}|^2}{2(2+\theta^*)M^2} - \frac{3\beta + o(1)}{2},
\end{aligned}$$

which is in contradiction with (2.5) and the assumption $\theta(x)|x|^2 \to +\infty$ as $|x| \to \infty$. Hence, $\{u_n\}$ is bounded, the proof of Lemma 2.4 is completed. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemmas 2.2–2.4, we obtain a nontrivial solution to system (1.1) via the Mountain Pass Theorem under the (C) condition which the readers can refer to [2]. □

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