Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 10, 1–17; https://doi.org/10.14232/ejqtde.2021.1.10 www.math.u-szeged.hu/ejqtde/

Coupled nonautonomous inclusion systems with spatially variable exponents

Peter E. Kloeden[1] and **Jacson Simsen**[✉2]

[1]Mathematisches Institut, Universität Tübingen, D-72076 Tübingen, Germany
[2]IMC, Universidade Federal de Itajubá, 37500-903 Itajubá - MG, Brazil

Received 21 November 2020, appeared 12 February 2021
Communicated by Christian Pötzsche

Abstract. A family of nonautonomous coupled inclusions governed by $p(x)$-Laplacian operators with large diffusion is investigated. The existence of solutions and pullback attractors as well as the generation of a generalized process are established. It is shown that the asymptotic dynamics is determined by a two dimensional ordinary nonautonomous coupled inclusion when the exponents converge to constants provided the absorption coefficients are independent of the spatial variable. The pullback attractor and forward attracting set of this limiting system is investigated.

Keywords: nonautonomous parabolic problems, variable exponents, pullback attractors, omega limit sets, upper semicontinuity.

2020 Mathematics Subject Classification: 35K55, 35K92, 35A16, 35B40, 35B41, 37B55.

1 Introduction

It is a well-known fact that many models of chemical, biological and ecological problems involve reaction-diffusion systems. For example, Fisher's equation:

$$w_t - D\frac{\partial^2 w}{\partial x^2} = aw(1-w).$$

A general reaction-diffusion system has the form

$$u_t - \mathcal{D}\Delta u = \mathbf{f}(u) \tag{RD}$$

where u is a vector representing chemical concentrations and $\mathcal{D}$ is a matrix of diffusion coefficients, assumed constant, and the second term represents chemical reactions. The form of $\mathbf{f}$ depends on the system being studied (it is typically nonlinear). Large diffusion phenomena many times appears in these systems. A shadow system, as a limiting system of reaction-diffusion model for algal bloom in which the diffusion rate tends to infinity, has been proposed in [27] to study whether or not stable nonconstant equilibrium solutions of the

✉Corresponding author. Email: jacson@unifei.edu.br

system exist. Large diffusion phenomena also appear in applications of chemical fluid flows [30].

When the diffusion does not follow a linear or a uniform structure the problem (RD) becomes

$$u_t - \mathcal{D}\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mathbf{f}(u).$$

Partial differential problems with variable exponents have application in electrorheological fluids (see [19,31,32]) and image processing (see [13,22]). Another important application is modelling of flow in porous media [1,2]. Some other applications of equations with variable exponent growth conditions are magnetostatics [12] and capillarity phenomena [5].

Sometimes it is necessary to consider a multivalued right-hand side when uncertainties or discontinuities appear in the reaction term, while coupled systems occur when different phenomena interact. In these cases we have to work with differential inclusions instead of differential equations (see, for example, [3,9,14,15,20,23,28,29,42] and the references therein). Such inclusions have been used for modelling processes of combustion in porous media [20] and the surface temperature on Earth [9,15]. Moreover, differential inclusions appear in numerous applications such as the control of forest fires [7], conduction of electrical impulses in nerve axons [40,41]. In climatology, the energy balance models may lead to evolution differential inclusions which involve the p-Laplacian [16,17]. A degenerate parabolic-hyperbolic problem with a differential inclusion appears in a glaciology model [18].

We will consider the following nonautonomous coupled inclusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + C_1(t)|u|^{p(x)-2}u \in F(u,v), & t > \tau, \\ \dfrac{\partial v}{\partial t} - D\,\mathrm{div}(|\nabla v|^{q(x)-2}\nabla v) + C_2(t)|v|^{q(x)-2}v \in G(u,v), & t > \tau, \\ (u(\tau), v(\tau)) \text{ in } L^2(\Omega) \times L^2(\Omega), \end{cases} \tag{S}$$

on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with homogeneous Neumann boundary conditions. Here $D \in [1, \infty)$, F and G are bounded upper semicontinuous and positively sublinear multivalued maps, and the exponents $p(\cdot)$, $q(\cdot) \in C(\overline{\Omega})$ satisfy

$$p^+ := \max_{x \in \overline{\Omega}} p(x) \; > \; p^- := \min_{x \in \overline{\Omega}} p(x) > 2, \; q^+ > q^- > 2.$$

In addition, the absorption coefficients C_1, $C_2 : [\tau, T] \times \Omega \to \mathbb{R}$ are functions in $L^\infty([\tau, T] \times \Omega)$ satisfying

(C1) there is a positive constant, γ such that $0 < \gamma \leq C_i(t,x)$ for almost all $(t,x) \in [\tau, T] \times \Omega$, $i = 1, 2$.

(C2) $C_i(t,x) \geq C_i(s,x)$ for a.a. $x \in \Omega$ and $t \leq s$ in $[\tau, T]$, $i = 1, 2$.

The authors of [21] considered this problem for only one equation with the external function globally Lipschitz, while those of [35] considered the autonomous version of this problem with $C_i(t,x) \equiv 1$. Nonautonomous equations of p-Laplacian type were previously considered in [24,38].

We will prove existence of strong global solutions for problem (S) and that these multivalued problems define exact generalized processes. The main tool used is a compactness result established in [36], which is a generalization of Baras' Theorem for the case that the main operator is time-dependent. In addition, we prove the existence of a pullback attractor and, when considering large diffusion and letting the exponents go to constants, we explore the robustness of the family of pullback attractors with respect to its limit problem which governs the whole asymptotic dynamics of the system.

The paper is organized as follows. In Section 2 we present some preliminaries. Section 3 is devoted to prove existence of global solutions for the system and in Section 4 we prove that problem (S) defines an exact generalized process which possess a pullback attractor. Finally, in Section 5 we consider the case when $D \to +\infty$ and the exponents converge to constants and investigate the dynamics of the limiting two dimensional ordinary nonautonomous coupled inclusion.

2 Preliminaries

Definition 2.1 ([43]). A subset K in $L^1(a,b;X)$ is uniformly integrable if, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\int_E \|f(t)\|_X dt \leq \varepsilon$ uniformly for $f \in K$ for each measurable subset E in $[a,b]$ with Lebesgue measure less than $\delta(\varepsilon)$.

Remark 2.2 ([8]). Since $[a,b]$ is compact, each uniformly integrable subset in $L^1(a,b;X)$ is bounded with respect to the norm of $L^1(a,b;X)$.

Consider the following IVP:

$$\begin{cases} \dfrac{du^n}{dt}(t) + A(t)u^n(t) \ni f_n(t), \quad t > \tau, \\ u^n(\tau) = u_{0_n}, \end{cases} \tag{$P_{t,n}$}$$

where for each $t > \tau$, $A(t)$ is maximal monotone in a Hilbert space H, $f_n \in K \subset L^1(\tau,T;H)$ and $u_{0_n} \in H$. In addition, suppose $\mathcal{D}(A(t)) = \mathcal{D}(A(\tau))$, $\forall\, t,\tau \in \mathbb{R}$, and $\overline{\mathcal{D}(A(t))} = H$, for all $t \in \mathbb{R}$.

Definition 2.3. A function $u^n : [\tau,T] \to H$ is called a strong solution of ($P_{t,n}$) on $[\tau,T]$ if

(i) $u^n \in \mathcal{C}([\tau,T];H)$;

(ii) u^n is absolutely continuous on any compact subset of (τ,T);

(iii) $u^n(t)$ is in $\mathcal{D}(A(t))$ for a.e. $t \in [\tau,T]$, $u^n(\tau) = u_{0_n}$, and satisfies the inclusion in ($P_{t,n}$) for a.e. $t \in [\tau,T]$.

We now present abstract conditions on the family of the operators $\{A(t)\}_{t>0}$ and f_n such that problem ($P_{t,n}$) has, for each $n \in \mathbb{N}$, a unique strong solution u^n on $[\tau,T]$. We are interested in the case where $A(t) = \partial\phi^t$, i.e., the evolution problem of the form

$$\frac{du}{dt}(t) + \partial\phi^t(u(t)) \ni f(t), \qquad \tau \leq t \leq T, \tag{E}$$

in a real Hilbert space H, where, for almost every $t \in [0,T]$, $A(t) := \partial\phi^t$ is the subdifferential of a lower semicontinuous, proper and convex function ϕ^t from H into $(-\infty,\infty]$. In this case, $A(t)$ is a maximal monotone operator.

Condition A: Let $T > \tau$ be fixed.

(I) There is a set $Z \subset\,]\tau,T]$ of zero measure such that ϕ^t is a lower semicontinuous proper convex function from H into $(-\infty,\infty]$ with a non-empty effective domain for each $t \in [\tau,T] - Z$.

(II) For any positive integer r there exist a constant $K_r > 0$, an absolutely continuous function $g_r : [\tau, T] \to \mathbb{R}$ with $g_r' \in L^{\beta}(\tau, T)$ and a function of bounded variation $h_r : [\tau, T] \to \mathbb{R}$ such that if $t \in [\tau, T] - Z$, $w \in D(\phi^t)$ with $|w| \le r$ and $s \in [t, T] - Z$, then there exists an element $\tilde{w} \in D(\phi^s)$ satisfying

$$|\tilde{w} - w| \le |g_r(s) - g_r(t)|(\phi^t(w) + K_r)^{\alpha},$$
$$\phi^s(\tilde{w}) \le \phi^t(w) + |h_r(s) - h_r(t)|(\phi^t(w) + K_r),$$

where α is some fixed constant with $0 \le \alpha \le 1$ and

$$\beta := \begin{cases} 2 & \text{if } 0 \le \alpha \le \frac{1}{2}, \\ \dfrac{1}{1-\alpha} & \text{if } \frac{1}{2} \le \alpha \le 1. \end{cases}$$

Proposition 2.4 ([44]). *Suppose that **Condition A** is satisfied. Then for each $f \in L^2(\tau, T; H)$ and $u_0 \in \overline{D(\phi^{\tau})}$, the equation (E) has a unique strong solution u on $[\tau, T]$ with $u(\tau) = u_0$.*

Moreover, u has the following properties:

(i) For all $t \in (\tau, T] - Z$ $u(t)$ is in $D(\phi^t)$ and satisfies $t\phi^t(u(t)) \in L^{\infty}(\tau, T)$ and $\phi^t(u(t)) \in L^1(\tau, T)$. Furthermore, for any $\tau < \delta < T$, $\phi^t(u(t))$ is of bounded variation on $[\delta, T] - Z$.

(ii) For any $\tau < \delta < T$, u is strongly absolutely continuous on $[\delta, T]$, and $t^{1/2}\frac{du}{dt} \in L^2(\tau, T; H)$.

In particular, if $u_0 \in D(\phi^{\tau})$, then u satisfies

(i)′ For all $t \in [\tau, T] - Z$, $u(t)$ is in $D(\phi^t)$ and $\phi^t(u(t))$ is of bounded variation on $[\tau, T] - Z$.

(ii)′ u is strongly absolutely continuous on $[\tau, T]$ and satisfies $\frac{du}{dt} \in L^2(\tau, T; H)$.

For our specific problem, we consider $H := L^2(\Omega)$ with a bounded smooth domain $\Omega \subset \mathbb{R}^n$, $n \ge 1$, $p(\cdot) \in C(\bar{\Omega}, \mathbb{R})$, $p^+ := \max_{x \in \bar{\Omega}} p(x) \ge p^- := \min_{x \in \bar{\Omega}} p(x) > 2$, where $C : [\tau, T] \times \Omega \to \mathbb{R}$ is a function in $L^{\infty}([\tau, T] \times \Omega)$ satisfying conditions (C1) and (C2).

Consider the Lebesgue space with variable exponents

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

Define $\rho(u) := \int_{\Omega} |u(x)|^{p(x)} dx$ and

$$\|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \rho\left(\frac{u}{\lambda}\right) \le 1 \right\}$$

for $u \in L^{p(\cdot)}(\Omega)$. The generalized Sobolev space is defined as

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

It is well-known that $Y_p := W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{Y_p} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Consider the operator $A(t)$ defined in Y_p such that for each $u \in Y_p$ associate the following element of its dual space ${Y_p}^*$, $A(t)u : Y_p \to \mathbb{R}$ given by

$$\langle A(t)u, v \rangle_{{Y_p}^*, Y_p} := D \int_{\Omega} |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla v(x)\, dx + \int_{\Omega} C(t, x)|u(x)|^{p(x)-2} u(x) v(x)\, dx.$$

It was shown in [21] that the operator $A(t) : Y_p \to {Y_p}^*$ is monotone, hemicontinuous and coercive. Moreover, we have the following estimates on the operator.

Lemma 2.5 ([21]). *Let $u \in Y_p := W^{1,p(\cdot)}(\Omega)$. For each $t \geq 0$ we have*

$$\langle A(t)u, u\rangle_{Y_p{}^*,Y_p} \geq \frac{\min\{1,\gamma\}}{2^{p^+}} \begin{cases} \|u\|_{Y_p}^{p^+}, & \text{if } \|u\|_{Y_p} < 1, \\ \|u\|_{Y_p}^{p^-}, & \text{if } \|u\|_{Y_p} \geq 1. \end{cases} \tag{2.1}$$

It is easy to see that the operator $A(t) : H \to H$ defined by

$$A(t)u := -D\text{div}(|\nabla u|^{p(x)-2}\nabla u) + C(t)|u|^{p(x)-2}u,$$

satisfies **Condition A** and, consequently, by applying Proposition 2.4 we have that problem (E) has a unique strong solution.

We will also consider the following IVP:

$$\begin{cases} \dfrac{du}{dt} + A(t)u \ni 0, \quad t > \tau, \\ u(\tau) = u_0, \end{cases} \tag{P_t}$$

where for each $t > \tau$, $A(t)$ is maximal monotone in a Hilbert space H.

Definition 2.6. Define $\{V(t,\tau);\ V(t,\tau) : H \longrightarrow H,\ t \geq \tau\}$ by $V(t,\tau)(u_0) = u(t, u(\tau)) = u(t, u_0)$, where $u(t, u_0)$ is the unique strong solution of problem (P_t), and call $\{V(t,\tau);\ V(t,\tau) : H \longrightarrow H, t \geq \tau\}$ the evolution process generated by $A := \{A(t)\}_{t>\tau}$ in H. We say that the evolution process is compact if $V(t,\tau)$ is a compact operator for each $t > \tau$.

Let us review the concept of an evolution process in the next

Definition 2.7. An evolution process in a metric space X is a family $\{U(t,\tau) : X \to X,\ t \geq \tau \in \mathbb{R}\}$ satisfying:

i) $U(\tau,\tau) = I$;

ii) $U(t,\tau) = U(t,s)U(s,\tau)$, $\tau \leq s \leq t$.

Varying f_n and u_{0_n} in ($P_{t,n}$) we obtain a family of problems and consequently a family of solutions. Consider the following solution sets

$$M(K) := \{u^n; u^n \text{ is the unique strong solution of } (P_{t,n}), \text{ with } f_n \in K \text{ and } u_{0_n} \in H\}.$$

Theorems in [36] establish conditions which ensure that the set $M(K)$ possesses some property of compactness.

We now review some concepts and results from the literature which will be useful in the sequel to understand the conditions on the multivalued functions F and G. We refer the reader to [3, 4, 43] for more details about multivalued analysis theory. Let X be a real Banach space and M a Lebesgue measurable subset in $\mathbb{R}^q$, $q \geq 1$.

Definition 2.8. The map $G : M \to \mathcal{P}(X)$ is called measurable if for each closed subset C in X the set $G^{-1}(C) = \{y \in M; G(y) \cap C \neq \emptyset\}$ is Lebesgue measurable.

If G is a univalued map, the above definition is equivalent to the usual definition of a measurable function.

Definition 2.9. By a *selection* of $E : M \to \mathcal{P}(X)$ we mean a function $f : M \to X$ such that $f(y) \in E(y)$ a.e. $y \in M$, and we denote by Sel E the set Sel $E := \{f,\ f : M \to X$ is a measurable selection of $E\}$.

In what follows U denotes a topological space.

Definition 2.10. A mapping $G : U \to \mathcal{P}(X)$ is called upper semicontinuous [weakly upper semicontinuous] at $u \in U$, if

(i) $G(u)$ is nonempty, bounded, closed and convex.

(ii) For each open subset [open set in the weak topology] D in X satisfying $G(u) \subset D$, there exists a neighborhood V of u such that $G(v) \subset D$, for each $v \in V$.

If G is upper semicontinuous [weakly upper semicontinuous] at each $u \in U$, then it is called upper semicontinuous [weakly upper semicontinuous] on U.

Definition 2.11. $F, G : H \times H \to \mathcal{P}(H)$ are said to be bounded if, whenever B_1, B_2 are bounded, then $F(B_1, B_2) = \bigcup_{(u,v)\in B_1\times B_2} F(u,v)$ and $G(B_1, B_2) = \bigcup_{(u,v)\in B_1\times B_2} G(u,v)$ are bounded in H.

In order to obtain global solutions we impose suitable conditions on the external forces F and G.

Definition 2.12. The pair (F, G) of mappings $F, G : H \times H \to \mathcal{P}(H)$, which maps bounded subsets of $H \times H$ into bounded subsets of H, is called positively sublinear if there exist $a > 0$, $b > 0$, $c > 0$ and $m_0 > 0$ such that for each $(u, v) \in H \times H$ with $\|u\| > m_0$ or $\|v\| > m_0$ for which either there exists $f_0 \in F(u, v)$ satisfying $\langle u, f_0\rangle > 0$ or there exists $g_0 \in G(u, v)$ with $\langle v, g_0\rangle > 0$, we have both

$$\|f\| \le a\|u\| + b\|v\| + c \quad \text{and} \quad \|g\| \le a\|u\| + b\|v\| + c$$

for each $f \in F(u, v)$ and each $g \in G(u, v)$.

3 Existence of solution

Now we will establish the existence of a global solution for the system (S). The idea is to show that an appropriately defined multivalued map has at least one fix point whose existence is equivalent to the existence of at least one solution of (S).

We can rewrite the system in an abstract form as

$$\begin{cases} u_t + A(t)u \in F(u,v), & t > \tau, \\ v_t + B(t)v \in G(u,v), & t > \tau, \\ (u(\tau), v(\tau)) = (u_\tau, v_\tau) \in H \times H, \end{cases} \tag{$\tilde{S}$}$$

where, for each $t > \tau$, $A(t)$ and $B(t)$ are univalued maximal monotone operators in a real separable Hilbert space H of subdifferential type, i.e., $A(t) = \partial\varphi^t$, $B(t) = \partial\psi^t$ with φ^t, ψ^t non-negative maps satisfying **Condition A** with $\partial\varphi^t(0) = \partial\psi^t(0) = 0$, $\forall\ t \in \mathbb{R}$ and F and G are bounded, upper semicontinuous and positively sublinear multivalued maps.

Definition 3.1. A strong solution of ($\tilde{S}$) is a pair (u, v) satisfying: $u, v \in C([\tau, T]; H)$ for which there exist $f, g \in L^1(\tau, T; H)$, $f(t) \in F(u(t), v(t))$, $g(t) \in G(u(t), v(t))$ *a.e.* in (τ, T), and such that (u, v) is a strong solution (see Definition 2.3) over (τ, T) to the system (P_1) below:

$$\begin{cases} u_t + A(t)u = f, \\ v_t + B(t)v = g, \\ u(\tau) = u_0, v(\tau) = v_0. \end{cases} \tag{P_1}$$

We obtain the global existence for our system ($\tilde{S}$) by applying the following

Theorem 3.2 ([36]). *Let $A = \{A(t)\}_{t>\tau}$ and $B = \{B(t)\}_{t>\tau}$ be families of univalued operators $A(t) = \partial\varphi^t$, $B(t) = \partial\psi^t$ with φ^t, ψ^t non negative maps satisfying* ***Condition A*** *with $\partial\varphi^t(0) = \partial\psi^t(0) = 0$. Also suppose each one A and B generates a compact evolution process, and let $F, G : H \times H \to \mathcal{P}(H)$ be upper semicontinuous and bounded multivalued maps. Then given a bounded subset $B_0 \subset H \times H$, there exists $T_0 > \tau$ such that for each $(u_0, v_0) \in B_0$ there exists at least one strong solution (u, v) of ($\tilde{S}$) defined on $[\tau, T_0]$. If, in addition, the pair (F, G) is positively sublinear, given $T > \tau$, the same conclusion is true with $T_0 = T$.*

4 Exact generalized process and pullback attractor

We will prove that the system ($\tilde{S}$) generates an exact generalized process. Let us review this concept in the following

Definition 4.1 ([37]). Let (X, ρ) be a complete metric space. A *generalized process* $\mathcal{G} = \{\mathcal{G}(\tau)\}_{\tau\in\mathbb{R}}$ on X is a family of function sets $\mathcal{G}(\tau)$ consisting of maps $\varphi : [\tau, \infty) \to X$, satisfying the properties:

[P1] For each $\tau \in \mathbb{R}$ and $z \in X$ there exists at least one $\varphi \in \mathcal{G}(\tau)$ with $\varphi(\tau) = z$;

[P2] If $\varphi \in \mathcal{G}(\tau)$ and $s \geq 0$, then $\varphi^{+s} \in \mathcal{G}(\tau + s)$, where $\varphi^{+s} := \varphi_{|[\tau+s,\infty)}$;

[P3] If $\{\varphi_j\} \subset \mathcal{G}(\tau)$ and $\varphi_j(\tau) \to z$, then there exists a subsequence $\{\varphi_\mu\}$ of $\{\varphi_j\}$ and $\varphi \in \mathcal{G}(\tau)$ with $\varphi(\tau) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq \tau$.

Definition 4.2 ([37]). A generalized process $\mathcal{G} = \{\mathcal{G}(\tau)\}_{\tau\in\mathbb{R}}$ which satisfies the concatenation property:

[P4] If $\varphi, \psi \in \mathcal{G}$ with $\varphi \in \mathcal{G}(\tau)$, $\psi \in \mathcal{G}(r)$ and $\varphi(s) = \psi(s)$ for some $s \geq r \geq \tau$, then $\theta \in \mathcal{G}(\tau)$, where

$$\theta(t) := \begin{cases} \varphi(t), & t \in [\tau, s], \\ \psi(t), & t \in (s, \infty), \end{cases} \tag{4.1}$$

is called an *exact (or strict) generalized process.*

Property [P1] follows from the existence of a solution for the system ($\tilde{S}$), which was guaranteed in the previous section.

Let $D(u(\tau), v(\tau))$ be the set of solutions of ($\tilde{S}$) with initial data (u_τ, v_τ). Moreover, let us consider $G(\tau) := \bigcup_{(u_\tau, v_\tau)\in H\times H} D(u(\tau), v(\tau))$ and $G := \{G(\tau)\}_{\tau\in\mathbb{R}}$.

Theorem 4.3 ([36]). *Under the conditions of Theorem 3.2, G is an exact generalized process.*

The authors of [36] provided a result that gives sufficient conditions on $A = \{A(t)\}_{t>\tau}$ to ensure that the evolution process $\{V(t,\tau)\}_{t\geq\tau}$ generated by A (see Definition 2.6) is compact. Suppose that the following conditions are true for A:

(i) $\mathcal{D}(A(t)) = V$ for all $t \in [\tau, T]$ with V compactly embedded into H and $\overline{V} = H$, where V is a reflexive Banach space and H a Hilbert space;

(ii) for each $t \in [\tau, T]$, $A(t) = \partial\varphi^t$, with $\varphi^t(\cdot) := \varphi(t,\cdot) : H \to \mathbb{R} \cup \{\infty\}$ a convex, proper and lower semicontinuous map;

(iii) there exist constants $\alpha, \alpha_1, \alpha_2 > 0$ such that for each $t \in [\tau, T]$, $\alpha\|w\|_V^{\alpha_1} \leq \varphi^t(w)$ if $\|w\|_V < 1$ and $\alpha\|w\|_V^{\alpha_2} \leq \varphi^t(w)$ if $\|w\|_V \geq 1$;

(iv) for each $t \in [\tau, T]$, $\varphi^t(x) \geq 0$ for all $x \in H$ and $\varphi^t(0) = 0$;

(v) for each $x \in V$, there exists $\frac{\partial\varphi}{\partial s}(s,x)$ and $\frac{\partial\varphi}{\partial s}(s,x) \leq 0$ for a.a. $s \in [\tau, T]$.

We will use the following result.

Theorem 4.4 ([36])**.** *If A satisfies hypotheses (i)–(v), then the generated process $\{V(t,\tau)\}_{t\geq\tau}$ by $A = \{A(t)\}_{t>\tau}$ is compact.*

Returning to our specific problem, i.e., if we consider $A(t) : H \to H$ given by $A(t)u = -D\,\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + C(t)|u|^{p(x)-2}u$, where $H = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ a bounded smooth domain, $p(\cdot) \in C(\bar{\Omega}, \mathbb{R})$, $p^+ := \max_{x\in\bar{\Omega}} p(x) \geq p^- := \min_{x\in\bar{\Omega}} p(x) > 2$. and $C : [\tau, T] \times \Omega \to \mathbb{R}$ is a function in $L^\infty([\tau, T] \times \Omega)$ such that $0 < \gamma \leq C(t,x)$ for almost all $(t,x) \in [\tau, T] \times \Omega$, for some positive constant γ, and $C(t,x) \geq C(s,x)$ for a.a. $x \in \Omega$ and $t \leq s$ in $[\tau, T]$. In particular, we have $D(A(t)) = V := W^{1,p(\cdot)}(\Omega) \subset\subset H$ for all $t \in [\tau, T]$, $\bar{V} = H$ and $A(t) = \partial\varphi^t$ where $\varphi^t : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\varphi^t(u) := \begin{cases} \left[\displaystyle\int_\Omega \frac{D}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{C(t,x)}{p(x)}|u|^{p(x)}dx\right], & \text{if } u \in W^{1,p(x)}(\Omega) \\ +\infty, & \text{otherwise} \end{cases} \tag{4.2}$$

is a convex, proper and lower semicontinuous map. It is easy to see that $A = \{A(t)\}_{t>\tau}$ satisfies all the abstract hypotheses (i)–(v) above. Moreover, we had already seen that **Condition A** is also satisfied.

Hence, considering $D(u(\tau), v(\tau))$ the set of the solutions of (S) with initial data (u_τ, v_τ) and defining $G(\tau) := \bigcup_{(u_\tau,v_\tau)\in H\times H} D(u(\tau), v(\tau))$ and $\mathbb{G} := \{G(\tau)\}_{\tau\in\mathbb{R}}$, we have

Theorem 4.5. *$\mathbb{G}$ is an exact generalized process.*

A multivalued process $\{U_{\mathcal{G}}(t,\tau)\}_{t\geq\tau}$ defined by a generalized process $\mathcal{G}$ is a family of multivalued operators $U_{\mathcal{G}}(t,\tau) : P(X) \to P(X)$ with $-\infty < \tau \leq t < +\infty$, such that for each $\tau \in \mathbb{R}$

$$U_{\mathcal{G}}(t,\tau)E = \{\varphi(t); \varphi \in \mathcal{G}(\tau), \text{ with } \varphi(\tau) \in E\}, \qquad t \geq \tau.$$

Theorem 4.6 ([37])**.** *Let $\mathcal{G}$ be an exact generalized process. Suppose that $\{U_{\mathcal{G}}(t,\tau)\}_{t\geq\tau}$ is a multivalued process defined by $\mathcal{G}$, then we have that $\{U_{\mathcal{G}}(t,\tau)\}_{t\geq\tau}$ is an exact multivalued process on $P(X)$, i.e.,*

1. *$U_{\mathcal{G}}(t,t) = Id_{P(X)}$,*
2. *$U_{\mathcal{G}}(t,\tau) = U_{\mathcal{G}}(t,s)U_{\mathcal{G}}(s,\tau)$ for all $-\infty < \tau \leq s \leq t < +\infty$.*

A family of sets $K = \{K(t) \subset X : t \in \mathbb{R}\}$ will be called a nonautonomous set. The family K is closed (compact, bounded) if $K(t)$ is closed (compact, bounded) for all $t \in \mathbb{R}$. The ω-limit set $\omega(t, E)$ consists of the pullback limits of all converging sequences $\{\xi_n\}_{n\in\mathbb{N}}$ where $\xi_n \in U_{\mathcal{G}}(t, s_n)E, s_n \to -\infty$. Let $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ be a family of subsets of X. We have the following concepts of invariance:

- $\mathcal{A}$ is positively invariant if $U_{\mathcal{G}}(t,\tau)\mathcal{A}(\tau) \subset \mathcal{A}(t)$ for all $-\infty < \tau \leq t < \infty$;
- $\mathcal{A}$ is negatively invariant if $\mathcal{A}(t) \subset U_{\mathcal{G}}(t,\tau)\mathcal{A}(\tau)$ for all $-\infty < \tau \leq t < \infty$;
- $\mathcal{A}$ is invariant if $U_{\mathcal{G}}(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $-\infty < \tau \leq t < \infty$.

Definition 4.7. Let $t \in \mathbb{R}$.

1. A set $\mathcal{A}(t) \subset X$ pullback attracts a set $B \in X$ at time t if
$$\operatorname{dist}(U_{\mathcal{G}}(t,s)B, \mathcal{A}(t)) \to 0 \quad \text{as } s \to -\infty.$$
2. A family $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ pullback attracts bounded sets of X if $\mathcal{A}(\tau) \subset X$ pullback attracts all bounded subsets at τ, for each $\tau \in \mathbb{R}$. In this case, we say that the nonautonomous set $\mathcal{A}$ is pullback attracting.
3. A set $\mathcal{A}(t) \subset X$ pullback absorbs bounded subsets of X at time t if, for each bounded set B in X, there exists $T = T(t, B) \leq t$ such that $U_{\mathcal{G}}(t,\tau)B \subset \mathcal{A}(t)$ for all $\tau \leq T$.
4. A family $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ pullback absorbs bounded subsets of X if for each $t \in \mathbb{R}$ $\mathcal{A}(t)$ pullback absorbs bounded sets at time t.

Following the ideas of [25] we obtain

Lemma 4.8. *Let (u_1, u_2) be a solution of problem (S). Then there exist a positive number r_0 and a constant T_0, which do not depend on the initial data, such that*
$$\|(u_1(t), u_2(t))\|_{H\times H} \leq r_0, \qquad \forall\, t \geq T_0 + \tau.$$

Considering $Y_q := W^{1,q(\cdot)}(\Omega)$, we have

Lemma 4.9. *Let (u_1, u_2) be a solution of problem (S). Then there exist positive constants r_1 and $T_1 > T_0$, which do not depend on the initial data, such that*
$$\|(u_1(t), u_2(t))\|_{Y_p\times Y_q} \leq r_1, \qquad \forall\, t \geq T_1 + \tau.$$

Let $U_{\mathbb{G}}$ be the multivalued process defined by the generalized process $\mathbb{G}$. We know from [33] that for all $t \geq s$ in $\mathbb{R}$ the map $x \mapsto U_{\mathbb{G}}(t,s)x \in P(H \times H)$ is closed, so we obtain from Theorem 18 in [10] the following result

Theorem 4.10. *If for any $t \in \mathbb{R}$ there exists a nonempty compact set $D(t)$ which pullback attracts all bounded sets of $H \times H$ at time t, then the set $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ with $\mathcal{A}(t) = \overline{\bigcup_{B\in\mathcal{B}(H\times H)} \omega_{pb}(t, B)}$, is the unique compact, negatively invariant pullback attracting set which is minimal in the class of closed pullback attracting nonautonomous sets. Moreover, the sets $\mathcal{A}(t)$ are compact.*

Here $\omega_{pb}(t, B)$ is the pullback omega limit set starting in the set B and ending at time t.

Theorem 4.11. *The multivalued evolution process $U_{\mathbb{G}}$ associated with system (S) has a compact, negatively invariant pullback attracting set $\mathfrak{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ which is minimal in the class of closed pullback attracting nonautonomous sets. Moreover, the sets $\mathcal{A}(t)$ are compact.*

Proof. By Lemma 4.9 we have that the family $D(t) := \overline{B_{Y_p\times Y_q}(0, r_1)}^{H\times H}$ of compact sets of $H \times H$ is attracting. The result thus follows from Theorem 4.10. □

5 Limit problems and convergence properties

In the remainder of the paper we restrict attention to the case that the coefficient functions $C_1(t)$ and $C_2(t)$ depend only on the time variable t and not on the spatial variable $x \in \Omega$.

Our main objective is to consider what happens when D_s increases to infinity and $p_s(\cdot) \to p > 2$, $q_s(\cdot) \to q > 2$ in $L^\infty(\Omega)$ as $s \to \infty$ in the system

$$\begin{cases} \frac{\partial u_s}{\partial t} - \operatorname{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + C_1(t)|u_s|^{p_s(x)-2}u_s \in F(u_s,v_s), \ t > \tau, \\ \frac{\partial v_s}{\partial t} - \operatorname{div}(D_s|\nabla v_s|^{q_s(x)-2}\nabla v_s) + C_2(t)|v_s|^{q_s(x)-2}v_s \in G(u_s,v_s), \ t > \tau, \\ \dfrac{\partial u_s}{\partial n}(t,x) = \dfrac{\partial v_s}{\partial n}(t,x) = 0, \ t \geq \tau, \ x \in \partial\Omega, \\ u_s(\tau,x) = u_{\tau s}(x), \ v_s(\tau,x) = v_{\tau s}(x), x \in \Omega, \end{cases} \tag{5.1}$$

where $u_{\tau s}, v_{\tau s} \in H := L^2(\Omega)$, and to prove that the limit problem is described by an ordinary differential system.

Firstly, we observe that the gradients of the solutions of problem (5.1) converge in norm to zero as $s \to \infty$, which allows us to guess the limit problem

$$\begin{cases} \dot{u} + \phi_p^t(u) \in \widetilde{F}(u,v), \\ \dot{v} + \phi_q^t(v) \in \widetilde{G}(u,v), \\ u(\tau) = u_\tau, v(\tau) = v_\tau, \end{cases} \tag{5.2}$$

where $\phi_p^t(w) := C_1(t)|w|^{p-2}w$, $\phi_q^t(w) := C_2(t)|w|^{q-2}w$, $\widetilde{F} := F_{|\mathbb{R}\times\mathbb{R}}, \widetilde{G} := G_{|\mathbb{R}\times\mathbb{R}} : \mathbb{R}\times\mathbb{R} \to P(\mathbb{R})$ if we identify $\mathbb{R}$ with the constant functions which are in H, since Ω is a bounded set.

The next theorem confirms that the system (5.2) is a good candidate for the limit problem. The proof of the next result follows the ideas of [35] and will not present the proof here since the nonautonomous terms $C_{1,2}(t)$ do not present difficulties for the proof (see also [21] for the problem with only one equation).

Theorem 5.1. *If (u_s, v_s) is a solution of (5.1), then for each $t > T_1 + \tau$, the sequences of real numbers $\{\|\nabla u_s(t)\|_H\}_{s\in\mathbb{N}}$ and $\{\|\nabla v_s(t)\|_H\}_{s\in\mathbb{N}}$ both possess subsequences $\{\|\nabla u_{s_j}(t)\|_H\}$ and $\{\|\nabla v_{s_j}(t)\|_H\}$ that converge to zero as $j \to +\infty$, where T_1 is the positive constant in Lemma 4.9.*

In order to prove the existence of a global solution for the limit problem we consider the following abstract result of Barbu's book [6] for a Banach space X : Let $\tau \in \mathbb{R}$ and $T > \tau$ and consider a family of nonlinear operators $\mathcal{H}(t) : X \to X^*$, $t \in [\tau, T]$ satisfying:

(i) $\mathcal{H}(t)$ is monotone and hemicontinuous from X to X^* for almost every $t \in]\tau, T)$.

(ii) Function $\mathcal{H}(\cdot)u(\cdot) : [\tau, T] \to X^*$ is measurable for every $u \in L^p(\tau, T; X)$.

(iii) There is a constant C such that

$$\|\mathcal{H}(t)u\|_{X^*} \leq C(\|u\|_X^{p-1} + 1) \quad \text{for } u \in X \text{ and } t \in]\tau, T).$$

(iv) There are constants α, ω ($\omega > 0$) such that

$$\langle \mathcal{H}(t)u, u\rangle \geq \omega\|u\|_X^p + \alpha \quad \text{for } u \in X \text{ and } t \in]\tau, T).$$

Proposition 5.2 ([6, Theorem 4.2]). *Consider a Gelfand triple given by (X, H, X^*) and suppose that (i)–(iv) hold. If $u_\tau \in H$ and $f \in L^q(\tau, T; X^*)$ $(\frac{1}{p} + \frac{1}{q} = 1)$, then there exists a unique function $u(t)$ which is X^*-valued absolutely continuous on $[\tau, T]$ and satisfies*

$$u \in L^p(\tau, T; X) \cap C([\tau, T]; H), \qquad \frac{du}{dt} \in L^q(\tau, T, X^*),$$
$$\frac{du}{dt}(t) + \mathcal{H}(t)u(t) = f(t), \quad \text{a.e. on } (\tau, T), \quad u(\tau) = u_\tau.$$

Lemma 5.3. *The problem (5.2) has a global solution.*

Proof. Considering $\mathcal{H}(t) : \mathbb{R} \to \mathbb{R}$, defined by $\mathcal{H}(t)u := C(t)|u|^{p-2}u$, it is trivial to check (i)–(iv) above for $\mathcal{H}(t)$ with $X = H = X^* = \mathbb{R}$. Thus, for a given $f \in L^2(\tau, T; \mathbb{R})$, we have from Proposition 5.2 that there exists a unique function $u \in C([\tau, T]; \mathbb{R})$ which is a strong solution to the problem

$$\frac{du}{dt}(t) + \mathcal{H}(t)u(t) = f(t), \qquad u(\tau) = u_\tau \in \mathbb{R}.$$

Hence, with the same argument as in the proof of Theorem 41 in [36] we conclude that the limit problem (5.2) has a global strong solution. □

Remark 5.4. In the proof of the previous theorem we only need that $C(\cdot)$ is measurable and $\gamma \leq C(t)$. The constant γ is taken uniform in τ and T in order to yield global solutions.

The next result guarantees that (5.2) is in fact the limit problem for (5.1), as $s \to \infty$. The proof is analogous to what was done in [35] for the autonomous case, so will not be give here since the nonautonomous terms $C_{1,2}(t)$ do not present any difficulties.

Theorem 5.5. *Let (u_s, v_s) be a solution of the problem (5.1). Suppose that $(u_s(\tau), v_s(\tau)) = (u_{\tau s}, v_{\tau s})$ $\to (u_\tau, v_\tau) \in \mathbb{R} \times \mathbb{R}$ in the topology of $H \times H$ as $s \to +\infty$. Then there exists a solution (u, v) of the problem (5.2) satisfying $(u(\tau), v(\tau)) = (u_\tau, v_\tau)$ and a subsequence $\{(u_{s_j}, v_{s_j})\}_j$ of $\{(u_s, v_s)\}_s$ such that, for each $T > \tau$, $u_{s_j} \to u$, $v_{s_j} \to v$ in $C([\tau, T]; H)$ as $j \to +\infty$.*

Remark 5.6. Theorem 5.5 remains valid without the hypothesis $(u_\tau, v_\tau) \in \mathbb{R} \times \mathbb{R}$, whenever $(u_{\tau s}, v_{\tau s}) \in \mathcal{A}_s(\tau)$, $\forall\, s \in \mathbb{N}$, because in this case we prove, analogously to Lemma 6.2 in [21], that u_τ and v_τ are independent of x.

5.1 Upper semicontinuity of the family of pullback attractors

We start this section proving the existence of the pullback attractor for the limit problem.

Theorem 5.7. *The limit problem (5.2) defines a generalized process $\mathbb{G}^\infty$ which has a pullback attractor $\mathcal{U}_\infty = \{\mathcal{A}_\infty(t);\ t \in \mathbb{R} \times \mathbb{R}\}$.*

Proof. That limit problem (5.2) defines a generalized process $\mathbb{G}^\infty$ follows in the same way as before for the system (S).

Let us focus on the existence of the pullback attractor. Multiplying the equation $\dot{u} + C_1(t)|u|^{p-2}u = f(t)$ by u and using the assumption that (F, G) is positively sublinear and Young's Inequality to estimate $f(t).u(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 \leq -\frac{\gamma}{2}|u(t)|^p + c, \qquad t \geq \tau$$

where $c > 0$ is a constant. Therefore, the map $y(t) := |u(t)|^2$ satisfies the inequality

$$\frac{d}{dt}y(t) \le -\gamma(y(t))^{p/2} + 2c, \qquad t \ge \tau.$$

So, by Lemma 5.1 in [39],

$$|u(t)|^2 \le \left(\frac{2c}{\gamma}\right)^{2/p} + \left(\gamma\left(\frac{p}{2}-1\right)(t-\tau)\right)^{-\frac{2}{p-2}}, \qquad \forall\, t \ge \tau. \tag{5.3}$$

Let $\xi_1 > 0$ such that $\left(\gamma(\frac{p}{2}-1)\xi_1\right)^{-\frac{2}{p-2}} \le 1$, then

$$|u(t)| \le \left[\left(\frac{2c}{\gamma}\right)^{2/p} + 1\right]^{1/2} =: \kappa_1, \qquad \forall\, t \ge \xi_1 + \tau.$$

Analogously, we can prove that

$$|v(t)| \le \left[\left(\frac{2c}{\gamma}\right)^{2/q} + 1\right]^{1/2} =: \kappa_2, \qquad \forall\, t \ge \xi_2 + \tau.$$

Thus, considering $\kappa := \max\{\kappa_1, \kappa_2\}$, we have that the family $K(t) := B_{\mathbb{R}\times\mathbb{R}}[0,\kappa]$ of compact sets of $\mathbb{R}\times\mathbb{R}$ pullback attracts bounded sets of $\mathbb{R}\times\mathbb{R}$ at time t. Consequently, we have by Theorem 4.10 that the evolution process $\{S_\infty(t,s)\}_{t\ge s}$ defined by $\mathbb{G}^\infty$ has a pullback attractor $\mathcal{U}_\infty = \{\mathcal{A}_\infty(t);\ t \in \mathbb{R}\}$. □

Theorem 5.8. *The family of pullback attractors $\{\mathcal{U}_s;\ s \in \mathbb{N}\}$ associated with system (5.1) is upper semicontinuous on s at infinity, in the topology of H, i.e., for each $\tau \in \mathbb{R}$,*

$$\lim_{s\to+\infty} \operatorname{dist}(\mathcal{A}_s(\tau), \mathcal{A}_\infty(\tau)) = 0.$$

Proof. The proof follows the same ideas used in the autonomous version considered in [35], but instead of constructing a bounded complete orbit for a generalized process here we have to construct a complete bounded trajectory for a generalized process using Theorem 5.5 and working in an analogous way as in the proof of Theorem 6.1 in [34]. □

Remark 5.9. Note that if $p_s(\cdot) \equiv p$ and $q_s(\cdot) \equiv q$ the family of attractors is also lower semicontinuous since each solution of (5.2) is also a solution of (5.1) when we consider the constants C_1 and C_2 depend only on time in (5.1). For the general case of a variable exponent, lower semicontinuity is an open problem.

Remark 5.10. The assumption on the nonincreasing nature of $C_i(t)$ implies that the pointwise limit C_i^* as $t \to \infty$ exists and satisfies $0 < \gamma \le C_i^*$. Then the limit problem with C_i^* is autonomous and has an autonomous attractor $\mathcal{A}_\infty$ as a particular case of the results in this paper. This means that the original problem is asymptotic autonomous. It would be interesting to compare the asymptotic behaviour as $t \to \infty$ of its pullback attractor with this autonomous attractor. Applying Theorem 5.3 in [25] we obtain $\lim_{t\to+\infty} \operatorname{dist}(\mathcal{A}_\infty(t), \mathcal{A}_\infty) = 0$.

5.2 Forward attraction and omega limit sets

Pullback attractors describe the behaviour of a system from the past and, in general, have little to say about the future behaviour of the system. There is a corresponding concept of forward attractor involving the usual forward attraction instead of pullback attraction, but such forward attractors rarely exist and, when they do, need not be unique. See Kloeden & Yang [26], where an alternative characterization of forward attraction is developed using omega limit sets.

By (5.3) the closed and bounded (hence compact) absorbing set $B_{\mathbb{R}\times\mathbb{R}}[0,\kappa]$ is forward absorbing for the generalized process $\mathbb{G}^\infty$ on $\mathbb{R}^2$ generated by the limit problem (5.2). Moreover, the set $B := \cup_{0\le t\le T_\kappa}\mathbb{G}^\infty(t, B_{\mathbb{R}\times\mathbb{R}}[0,\kappa])$, where T_κ is the time for the set $B_{\mathbb{R}\times\mathbb{R}}[0,\kappa]$ to absorb itself under $\mathbb{G}^\infty$, is also positive invariant under $\mathbb{G}^\infty$. In addition, its absorbing property here is uniform in the sense that for any bounded subset D of $\mathbb{R}^2$ and every τ there exists a $T_D \ge 0$ such that

$$\mathbb{G}^\infty(t,\tau,x_0) \subset B \qquad \forall t \ge \tau + T_D,\ x_0 \in D,$$

since the estimate (5.3) depends just on the elapsed time and not the actual times.

ω-limit sets were defined and investigated in [26, Chapter 12] for single valued processes, but analogous definitions hold for a generalized process $\mathbb{G}^\infty$. Specifically, the ω-limit set is defined by

$$\omega_{B,\tau} := \bigcap_{t\ge\tau}\overline{\bigcup_{s\ge t}\mathbb{G}^\infty(s,\tau,B)}.$$

It is a nonempty compact set of B for each τ. Note that

$$\lim_{t\to\infty}\mathrm{dist}_{\mathbb{R}^2}\left(\mathbb{G}^\infty(t,\tau,B),\omega_{B,\tau}\right) = 0 \tag{5.4}$$

for each τ and that $\omega_{B,\tau} \subset \omega_{B,\tau'} \subset B$ for $\tau \le \tau'$. Hence, the set

$$\omega_B := \overline{\bigcup_{\tau\in\mathbb{R}}\omega_{B,\tau}} \subset B$$

is nonempty and compact. It contains all of the future limit points of the generalized process $\mathbb{G}^\infty$ starting in the set B at some time $\tau \ge T^*$. In particular, it contains the omega limit points of the pullback attractor, i.e.,

$$\bigcap_{t\ge\tau}\overline{\bigcup_{s\ge t}\mathcal{A}_\infty(s)} = \bigcap_{t\ge\tau}\overline{\bigcup_{s\ge t}\mathbb{G}^\infty(s,\tau,\mathcal{A}_\infty(\tau))} \subset \omega_{B,\tau} \subset \omega_B$$

for each $\tau \in \mathbb{R}$.

The set ω_B characterises the forward asymptotic behaviour of the nonautonomous system $\mathbb{G}^\infty$. It was called the *forward attracting set* of the nonautonomous system in [26] and is closely related to the Haraux–Vishik uniform attractor, but it may be smaller and does not require the generating system to be defined for all time or the attraction to be uniform in the initial time.

The forward attracting set ω_B need not be invariant for the generalized process $\mathbb{G}^\infty$, but in view of the above uniform absorbing property it is *asymptotically positive invariant* [26, Chapter 12], i.e., if for every $\varepsilon > 0$ here exists a $T(\varepsilon)$ such that

$$\mathbb{G}^\infty(t,\tau,\omega_B) \subset B_\varepsilon(\omega_B), \qquad t \ge \tau,$$

for each $\tau \ge T(\varepsilon)$.

References

[1] S. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions. Existence, uniqueness, localization, blow-up*, Atlantis Studies in Differential Equations, Vol. 4, Atlantis Press, Paris, 2015. https://doi.org/10.2991/978-94-6239-112-3; MR3328376; Zbl 1410.35001

[2] S. Antonsev, S. Shmarev, A model porous medium equation with variable exponent of nonlinearity: Existence, uniqueness and localization properties of solutions, *Nonlinear Anal.* **60**(2005), 515–545. https://doi.org/10.1016/j.na.2004.09.026; MR2103951

[3] J.-P. Aubin, A. Cellina, *Differential inclusions: Set-valued maps and viability theory,* Springer-Verlag, Berlin, 1984. https://doi.org/10.1007/978-3-642-69512-4; MR0755330; Zbl 0538.34007

[4] J.-P. Aubin, H. Frankowska, *Set-valued analysis*, Birkhäuser, Berlin, 1990. MR1048347; Zbl 0713.49021

[5] M. Avci, Ni-Serrin type equations arising from capillarity phenomena with non-standard growth, *Bound. Value Probl.* **2013**, 2013:55, 13 pp. https://doi.org/10.1186/1687-2770-2013-55; MR3081709; Zbl 1291.35102

[6] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International, 1976. MR0390843; Zbl 0328.47035

[7] A. Bressan, Differential inclusions and the control of forest fires, *J. Differential Equations* **243**(2007), 179–207. https://doi.org/10.1016/j.jde.2007.03.009; MR2371785; Zbl 1138.34002

[8] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Company, Amsterdam, 1973. MR0348562; Zbl 0252.47055

[9] M. I. Budyko, The effects of solar radiation variations on the climate of the Earth, *Tellus* **21**(1969), 611–619. https://doi.org/10.3402/tellusa.v21i5.10109

[10] T. Caraballo, J. A. Langa, V. S. Melnik, J. Valero, Pullback attractors for nonautonomous and stochastic multivalued dynamical systems, *Set-valued analysis* **11**(2003), 153–201. https://doi.org/10.1023/A:1022902802385; MR1966698; Zbl 1018.37048

[11] A. N. Carvalho, J. A. Langa, J. C. Robinson, *Attractors of infinite dimensional nonautonomous dynamical systems,* Springer, New York, 2013. https://doi.org/10.1007/978-1-4614-4581-4; MR2976449; Zbl 1263.37002

[12] M. Avci, B. Cekic, A. V. Kalinin, R.A. Mashiyev, $L^{p(x)}(\Omega)$-estimates of vector fields and some applications to magnetostatics problems, *J. Math. Anal. Appl.* **389**(2012), No. 2, 838–851. https://doi.org/10.1016/j.jmaa.2011.12.029; MR2879262; Zbl 1234.35192

[13] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Math.* **66**(2006), No. 4, 1383–1406. https://doi.org/10.1137/050624522; MR2246061; Zbl 1102.49010

[14] J. I. Díaz, I. I. Vrabie, Existence for reaction diffusion systems. A compactness method approach, *J. Math. Anal. Appl.* **188**(1994), 521–540. https://doi.org/10.1006/jmaa.1994.1443; MR1305466; Zbl 0815.35132

[15] J. I. Díaz, J. Hernández, L. Tello, Some results about multiplicity and bifurcation of stationary solutions of a reaction diffusion climatological model, *Rev. R. Acad. Cien. Serie A. Mat.* **96**(2002), 357–366. MR1985741; Zbl 1140.35414

[16] J. I. Díaz, J. Hernández, L. Tello, On the multiplicity of equilibrium solutions to a nonlinear diffusion equation on a manifold arising in climatology, *J. Math. Anal. Appl.* **216**(1997), 593–613. https://doi.org/10.1006/jmaa.1997.5691; MR1489600; Zbl 0892.35065

[17] J. I. Díaz, G. Hetzer, L. Tello, An energy balance climate model with hysteresis, *Nonlinear Anal.* **64**(2006), 2053–2074. https://doi.org/10.1016/j.na.2005.07.038; MR2211199; Zbl 1098.35090

[18] J. I. Díaz, E. Schiavi, On a degenerate parabolic/hyperbolic system in glaciology giving rise to a free boundary, *Nonlinear Anal.* **38**(1999), 649–673. https://doi.org/10.1016/S0362-546X(99)00101-7; MR1709993; Zbl 0935.35179

[19] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Springer-Verlag, Berlin, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8; MR2790542; Zbl 1222.46002

[20] E. Feireisl, J. Norbury, Some existence and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities, *Proc. Roy. Soc. Edinburgh Sect. A* **119**(1991), 1–17. https://doi.org/10.1017/S0308210500028262; MR1130591; Zbl 0784.35117

[21] A. C. Fernandes, M. C. Gonçalves, J. Simsen, Non-autonomous reaction-diffusion equations with variable exponents and large diffusion, *Discrete Contin. Dyn. Syst. Ser. B* **24**(2019), No. 4, 1485–1510. https://doi.org/10.3934/dcdsb.2018217; MR3927466; Zbl 1411.35151

[22] Z. Guo, Q. Liu, J. Sun, B. Wu, Reaction-diffusion systems with $p(x)$-growth for image denoising, *Nonlinear Anal. Real World Appl.* **12**(2011), 2904–2918. https://doi.org/10.1016/j.nonrwa.2011.04.015; MR2813233; Zbl 1219.35340

[23] O. V. Kapustyan, V. S. Melnik, J. Valero, V. V. Yasinski, *Global attractors of multi-valued dynamical systems and evolution equations without uniqueness*, National Academy of Sciences of Ukraine, Naukova Dumka, 2008.

[24] P. E. Kloeden, J. Simsen, Pullback attractors for non-autonomous evolution equation with spatially variable exponents, *Commun. Pure Appl. Anal.* **13**(2014), No. 6, 2543–2557. https://doi.org/10.3934/cpaa.2014.13.2543; MR3248403; Zbl 1304.35120

[25] P. E. Kloeden, J. Simsen, P. Wittbold, Asymptotic behavior of coupled inclusions with variable exponents, *Commun. Pure Appl. Anal.* **19**(2020), No. 2, 1001–1016. https://doi.org/10.3934/cpaa.2020046; MR4043776; Zbl 1430.35282

[26] P. E. Kloeden, M. Yang, *An introduction to nonautonomous dynamical systems and their attractors*, World Scientific Publishing, Singapore, 2021. https://doi.org/10.1142/12053; Zbl 07275178

[27] S. Kondo, M. Mimura, A reaction-diffusion system and its shadow system describing harmful algal blooms, *Tamkang J. Math.* **47**(2016), No. 1, 71–92. https://doi.org/10.5556/j.tkjm.47.2016.1916; MR3474537; Zbl 1343.35236

[28] V. S. Melnik, J. Valero, On attractors of multi-valued semi-flows and differential inclusions, *Set-Valued Anal.* **6**(1998), 83–111. MR1631081; Zbl 0915.58063

[29] V. S. Melnik, J. Valero, On global attractors of multivalued semiprocesses and nonautonomous evolution inclusions, *Set-Valued Anal.* **8**(2000), 375–403. MR1802241; Zbl 1063.35040

[30] D. E. Norman, Chemically reacting flows and continuous flow stirred tank reactors: the large diffusivity limit, *J. Dynam. Differential Equations* **12**(2000), No. 2, 309–355. https://doi.org/10.1142/12053; MR1790658; Zbl 0973.35155

[31] K. Rajagopal, M. Růžička, Mathematical modelling of electrorheological fluids, *Contin. Mech. Thermodyn.* **13**(2001), 59–78. Zbl 0971.76100

[32] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, Vol. 1748, Springer-Verlag, Berlin, 2000. https://doi.org/10.1007/BFb0104029; MR1810360; Zbl 0968.76531

[33] J. Simsen, E. Capelato, Some Properties for Exact Generalized Processes, in: *Continuous and distributed systems II*, Studies in Systems, Decision and Control, Vol. 30, Springer International Publishing, 2015, pp. 209–219. https://doi.org/10.1007/978-3-319-19075-4_12; MR3381632; Zbl 1335.37004

[34] J. Simsen, M. S. Simsen, J. Valero, Convergence of nonautonomous multivalued problems with large diffusion to ordinary differential inclusions, *Commun. Pure Appl. Anal.* **19**(2020), No. 4, 2347–2368. https://doi.org/10.3934/cpaa.2020102; MR4153512; Zbl 07195140

[35] J. Simsen, M. S. Simsen, P. Wittbold, Reaction-diffusion coupled inclusions with variable exponents and large diffusion, submitted preprint. https://www.uni-due.de/mathematik/ruhrpad/preprints (20-02).

[36] J. Simsen, P. Wittbold, Compactness results with applications for nonautonomous coupled inclusions, *J. Math. Anal. Appl.* **479**(2019), No. 1, 426–449. https://doi.org/10.1016/j.jmaa.2019.06.033; MR3987041; Zbl 07096845

[37] J. Simsen, J. Valero, Characterization of pullback attractors for multivalued nonautonomous dynamical systems, in: *Advances in dynamical systems and control*, Studies in Systems, Decision and Control, Vol. 69, Springer, Cham, 2016, pp. 179–195. https://doi.org/10.1007/978-3-319-40673-2_8; MR3616353; Zbl 1362.37037

[38] J. Simsen, M. J. D. Nascimento, M. S. Simsen, Existence and upper semicontinuity of pullback attractors for non-autonomous p-Laplacian parabolic problems, *J. Math. Anal. Appl.* **413**(2014), 685–699. https://doi.org/10.1016/j.jmaa.2013.12.019; MR3159798; Zbl 06420290

[39] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1988. https://doi.org/10.1007/978-1-4612-0645-3; MR0953967; Zbl 0662.35001

[40] D. Terman, A free boundary problem arising from a bistable reaction-diffusion equation, *SIAM J. Math. Anal.* **14**(1983), 1107–1129. https://doi.org/10.1137/0514086; MR0718812; Zbl 0534.35085

[41] D. Terman, A free boundary arising from a model for nerve conduction, *J. Differential Equations* **58**(1985), 345–363. https://doi.org/10.1016/0022-0396(85)90004-X; MR0797315; Zbl 0652.35055

[42] A. A. Tolstonogov, Solutions of evolution inclusions I, *Sibirsk. Mat. Zh.* **33**(1992), 161–174, 221; translation in *Siberian Math. J.* **33**(1992), No. 3, 500–511 (1993). https://doi.org/10.1007/BF00970899; MR1178468; Zbl 0787.34052

[43] I. I. Vrabie, *Compactness methods for nonlinear evolutions*, Second Edition, Pitman Monographs and Surveys in Pure and Applied Mathematics, New York, 1995. MR1375237; Zbl 0842.47040

[44] S. Yotsutani, Evolution equations associated with the subdifferentials, *J. Math. Soc. Japan* **31**(1979), 623–646. https://doi.org/10.2969/jmsj/03140623; MR0544681; Zbl 0405.35043