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Hopf bifurcation in a reaction-diffusive-advection two-species competition model with one delay

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Abstract. In this paper, we investigate a reaction-diffusive-advection two-species competition model with one delay and Dirichlet boundary conditions. The existence and multiplicity of spatially non-homogeneous steady-state solutions are obtained. The stability of spatially nonhomogeneous steady-state solutions and the existence of Hopf bifurcation with the changes of the time delay are obtained by analyzing the distribution of eigenvalues of the infinitesimal generator associated with the linearized system. By the normal form theory and the center manifold reduction, the stability and bifurcation direction of Hopf bifurcating periodic orbits are derived. Finally, numerical simulations are given to illustrate the theoretical results.

Keywords: reaction-diffusive, advection, delay, Hopf bifurcation, spatial heterogeneity.

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1 Introduction

In this paper, we consider a two-species competition model in a reaction-diffusive-advection with one delay

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \nabla \cdot [d_1 \nabla u(x,t) - a_1 u(x,t) \nabla m] + u(x,t)[m(x) - b_1 u(x,t-r) - c_1 v(x,t-r)], \\ \frac{\partial v(x,t)}{\partial t} = \nabla \cdot [d_2 \nabla v(x,t) - a_2 v(x,t) \nabla m] + v(x,t)[m(x) - b_2 u(x,t-r) - c_2 v(x,t-r)], \end{cases} \tag{1.1}$$

where $u(x,t), v(x,t)$ represents the population density at location $x \in \Omega$ and time t, time delay $r > 0$ represents the maturation time, and Ω is a bounded domain in $\mathbb{R}^k$ $(1 \le k \le 3)$ in (1.1) with a smooth boundary $\partial\Omega$. $a_i, b_i, c_i, d_i > 0$ $(i = 1, 2)$.

In (1.1), we assume that both species have the same per-capita growth rates at place $x \in \Omega$, denoted by $m(x)$. This scenario can occur if the two species are competing for the same resources. To reflect the heterogeneity of environment, we assume that $m(x)$ is a nonconstant function. In some sense, $m(x)$ can reflect the quality and quantity of resources available at the location x, where the favorable region $\{x \in \Omega : m(x) > 0\}$ acts as a source and the unfavorable part $\{x \in \Omega : m(x) < 0\}$ is a sink region, see [26]. When $m(x) \equiv 1$, see [15, 18].

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Under our assumptions in (1.1), the dispersal of the two competitors can be described in terms of their fluxes

$$J_u = -d_1 \nabla u + a_1 u \nabla m, \quad J_v = -d_2 \nabla v + a_2 v \nabla m,$$

respectively, where $d_1 \nabla u$ and $d_2 \nabla v$ account for random diffusion, and $a_1 u \nabla m$ and $a_2 v \nabla m$ represent movement upward along the environmental gradient. The two non-negative constants a_1 and a_2 measure the tendency of the two populations to move up along the gradient of $m(x)$, and d_1 and d_2 represent the random diffusion rates of two species, respectively. See [1,2,4–8,10,11,13,17,20,22–29].

When $b_1 = b_2 = c_1 = c_2 = 1,\ r = 0$ in (1.1), Chen, Hambrock and Lou [6] investigated the following model

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \nabla \cdot [d_1 \nabla u(x,t) - a_1 u(x,t) \nabla m] + u(x,t)[m(x) - u(x,t) - v(x,t)], \\ \frac{\partial v(x,t)}{\partial t} = \nabla \cdot [d_2 \nabla v(x,t) - a_2 v(x,t) \nabla m] + v(x,t)[m(x) - u(x,t) - v(x,t)]. \end{cases} \tag{1.2}$$

They showed that at least two scenarios can occur: if only one species has a strong tendency to move upward the environmental gradients, the two species can coexist since one species mainly pursues resources at places of locally most favorable environments while the other relies on resources from other parts of the habitat; if both species have such strong biased movements, it can lead to overcrowding of the whole population at places of locally most favorable environments, which causes the extinction of the species with stronger biased movement. These results provided a new mechanism for the coexistence of competing species, and they also implied that selection is against excessive advection along environmental gradients, and an intermediate biased movement rate may evolve.

When $v = 0$ in (1.1), Chen, Lou and Wei [8] investigated the following model,

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \nabla \cdot [d \nabla u - a_1 u \nabla m] + u(x,t)[m(x) - u(x,t-r)], \\ u(x,t) = 0. \end{cases} \tag{1.3}$$

They investigated a reaction-diffusion-advection model with time delay effect. The stability and instability of the spatially nonhomogeneous positive steady state were investigated when the given parameter of the model is near the principle eigenvalue of an elliptic operator. Their results implied that time delay can make the spatially nonhomogeneous positive steady state unstable for a reaction-diffusion-advection model, and the model can exhibit oscillatory pattern through Hopf bifurcation. The effect of advection on Hopf bifurcation values was also considered, and their results suggested that Hopf bifurcation is more likely to occur when the advection rate increases. See [3,9,12,14–16,18,19,21,30–34].

When $d_1 = d_2 = d,\ a_2 = a_1$ in (1.1), we study the following model with homogeneous Dirichlet boundary and initial value conditions

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \nabla \cdot [d \nabla u - a_1 u \nabla m] + u(x,t)[m(x) - b_1 u(x,t-r) - c_1 v(x,t-r)], \\ \frac{\partial v(x,t)}{\partial t} = \nabla \cdot [d \nabla v - a_1 v \nabla m] + v(x,t)[m(x) - b_2 u(x,t-r) - c_2 v(x,t-r)], \\ x \in \Omega, \quad t > 0, \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ u(x,t) = \varphi_1(x,t) \geq 0, \quad v(x,t) = \varphi_2(x,t) \geq 0, \quad (x,t) \in \overline{\Omega} \times [-r,0], \end{cases} \tag{1.4}$$

with the initial value functions

$$\varphi_i(x,\cdot) \in C([-r,0], \mathbb{R}_0^+) \ \ (x \in \overline{\Omega}), \quad \varphi_i(\cdot,t) \in H_0^1(\overline{\Omega}) \ \ (t \in [-r,0]), \quad i = 1,2.$$

In this paper, we mainly investigate whether time delay r can induce Hopf bifurcation for reaction-diffusion-advection model (1.4).

As in [2,8], Let $\widetilde{u} = e^{(-a_1/d)m(x)}u, \widetilde{v} = e^{(-a_1/d)m(x)}v, \widetilde{t} = td$, dropping the tilde sign, and denoting $\lambda = 1/d, a = a_1/d, \tau = dr$, system (1.4) can be transformed as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla u] + \lambda u[m(x) - b_1 e^{am(x)}u(x,t-\tau) - c_1 e^{am(x)}v(x,t-\tau)], \\ \frac{\partial v}{\partial t} = e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla v] + \lambda v[m(x) - b_2 e^{am(x)}u(x,t-\tau) - c_2 e^{am(x)}v(x,t-\tau)], \\ x\in\Omega, \quad t>0, \\ u(x,t) = v(x,t) = 0, \quad x\in\partial\Omega, \quad t\geq 0, \\ u(x,t) = \varphi_1(x,t)\geq 0, \quad v(x,t) = \varphi_2(x,t)\geq 0, \quad (x,t)\in\overline{\Omega}\times[-\tau,0]. \end{cases} \tag{1.5}$$

Throughout the paper, unless otherwise specified, $m(x)$ satisfies the following assumption

(H) $m\in C^2(\overline{\Omega})$, and $\max_{x\in\overline{\Omega}} m(x) > 0$.

The following eigenvalue problem

$$\begin{cases} -e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla u] = -\Delta u - a\nabla m\cdot\nabla u = \lambda m(x)u, \quad x\in\Omega, \\ u(x) = 0, \quad x\in\partial\Omega, \end{cases} \tag{1.6}$$

is crucial to derive our main results. It follows from [2,8,26] that, under assumption (H), (1.6) has a unique positive principal eigenvalue λ_* admitting a strictly positive eigenfunction $\varphi\in C_0^{1+\delta}(\overline{\Omega})$ for some $\delta\in(0,1)$ and $\int_\Omega \varphi^2 dx = 1$.

The rest of the paper is organized as follows. In Section 2, we study the existence of positive steady state solutions of (1.5). In Section 3, we focus on the eigenvalue problem of the linearized system of the steady-state solution of (1.5). In Section 4, we study the stability and Hopf bifurcation of the spatially nonhomogeneous positive steady state of (1.5). In Section 5, we derive an explicit formula, which can be used to determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic orbits. In Section 6, we give some numerical simulations are illustrated to support our analytical results.

Throughout the paper, we also denote the spaces $X = H^2(\Omega)\cap H_0^1(\Omega)$, $Y = L^2(\Omega)$. Moreover, we denote the complexification of a linear space Z to be $Z_{\mathbb{C}} = Z\oplus iZ = \{x_1 + ix_2 \mid x_1, x_2\in Z\}$, the domain of a linear operator L by $\mathcal{D}(L)$, the kernel of L by $\mathcal{N}(L)$, and the range of L by $\mathcal{R}(L)$. For Hilbert space $Y_{\mathbb{C}}$, we use the standard inner product $\langle u,v\rangle = \int_\Omega \overline{u}(x)^T v(x)dx, u,v\in Y_{\mathbb{C}}^2$.

2 Existence of positive steady state solutions

Denote

$$L := \nabla\cdot[e^{am(x)}\nabla] + \lambda_* e^{am(x)}m(x),$$

where λ_* is a unique positive principal eigenvalue of problem (1.6) admitting a strictly positive eigenfunction $\varphi\in C_0^{1+\delta}(\overline{\Omega})$ for some $\delta\in(0,1)$ and $\int_\Omega \varphi^2 dx = 1$.

Now, we have the following decompositions:

$$X = \mathcal{N}(L)\oplus X_1, \quad Y = \mathcal{N}(L)\oplus Y_1, \quad \mathcal{N}(L) = \operatorname{span}\{\varphi\},$$

$$X_1 = \left\{y\in X : \int_\Omega \varphi(x)y(x)dx = 0\right\}, \quad Y_1 = \mathcal{R}(L) = \left\{y\in Y : \int_\Omega \varphi(x)y(x)dx = 0\right\}.$$

Clearly, the operator $L : X \to Y$ is Fredholm with index zero. $L|_{X_1} : X_1 \to Y_1$ is invertible and has a bounded inverse.

In this section, we consider the existence of positive spatially nonhomogeneous steady states solutions of system (1.5), which satisfy

$$\begin{cases} \nabla \cdot [e^{am(x)}\nabla u] + \lambda e^{am(x)}u[m(x) - b_1 e^{am(x)}u(x,t) - c_1 e^{am(x)}v(x,t)] = 0, \\ \nabla \cdot [e^{am(x)}\nabla v] + \lambda e^{am(x)}v[m(x) - b_2 e^{am(x)}u(x,t) - c_2 e^{am(x)}v(x,t)] = 0, \end{cases} \tag{2.1}$$

Suppose that the solution of (2.1) has the following expressions:

$$\begin{cases} u_\lambda = \alpha(\lambda - \lambda_*)[\varphi + (\lambda - \lambda_*)\xi(x)], \\ v_\lambda = \beta(\lambda - \lambda_*)[\varphi + (\lambda - \lambda_*)\eta(x)], \end{cases} \tag{2.2}$$

where $\alpha, \beta \in \mathbb{R}, \xi, \eta \in X_1$. Substitute (2.2) into (2.1) we have

$$\begin{cases} L\xi + m(x)e^{am(x)}[\varphi + (\lambda - \lambda_*)\xi] - \lambda\alpha b_1 e^{2am(x)}[\varphi + (\lambda - \lambda_*)\xi]^2 \\ \qquad\qquad - \lambda\beta c_1 e^{2am(x)}[\varphi + (\lambda - \lambda_*)\xi][\varphi + (\lambda - \lambda_*)\eta] = 0, \\ L\eta + m(x)e^{am(x)}[\varphi + (\lambda - \lambda_*)\eta] - \lambda\alpha c_2 e^{2am(x)}[\varphi + (\lambda - \lambda_*)\eta]^2 \\ \qquad\qquad - \lambda\beta b_2 e^{2am(x)}[\varphi + (\lambda - \lambda_*)\xi][\varphi + (\lambda - \lambda_*)\eta] = 0. \end{cases} \tag{2.3}$$

When $\lambda = \lambda_*$, (2.3) becomes following equations

$$\begin{cases} L\xi + m(x)e^{am(x)}\varphi - \lambda_*\alpha b_1 e^{2am(x)}\varphi^2 - \lambda_*\beta c_1 e^{2am(x)}\varphi^2 = 0, \\ L\eta + m(x)e^{am(x)}\varphi - \lambda_*\beta b_2 e^{2am(x)}\varphi^2 - \lambda_*\alpha c_2 e^{2am(x)}\varphi^2 = 0. \end{cases} \tag{2.4}$$

Multiplying both sides of each equation in (2.4) by φ and integrating on Ω, we have

$$\alpha_{\lambda_*} = \frac{c_2 - c_1}{b_1 c_2 - b_2 c_1} d_1, \quad \beta_{\lambda_*} = \frac{b_1 - b_2}{b_1 c_2 - b_2 c_1} d_1,$$

where $d_1 = \frac{\int_\Omega m(x)e^{am(x)}\varphi^2 dx}{\lambda_* \int_\Omega e^{2am(x)}\varphi^3 dx} > 0$, see [8]. And $\xi_{\lambda_*}, \eta_{\lambda_*} \in X_1$ is the unique solution of the following equations

$$\begin{cases} L\xi + m(x)e^{am(x)}\varphi - \lambda_*\alpha_{\lambda_*} b_1 e^{2am(x)}\varphi^2 - \lambda_*\beta_{\lambda_*} c_1 e^{2am(x)}\varphi^2 = 0, \\ L\eta + m(x)e^{am(x)}\varphi - \lambda_*\beta_{\lambda_*} b_2 e^{2am(x)}\varphi^2 - \lambda_*\alpha_{\lambda_*} c_2 e^{2am(x)}\varphi^2 = 0. \end{cases} \tag{2.5}$$

To guarantee positive steady states solutions of system (2.1), we need following conditions:

(H1) $(\lambda - \lambda_*)\frac{c_2 - c_1}{b_1 c_2 - b_2 c_1} > 0, \quad (\lambda - \lambda_*)\frac{b_1 - b_2}{b_1 c_2 - b_2 c_1} > 0.$

Theorem 2.1. *Assume that (H1) holds. Then there exist a constant $\delta > 0$ and a continuously differentiable mapping which defined by $\lambda \to (\xi_\lambda, \eta_\lambda, \alpha_\lambda, \beta_\lambda)$, from $(\lambda_* - \delta, \lambda_* + \delta)$ to $X_1^2 \times (\mathbb{R}^+)^2$ such that system (1.5) has a positive spatially nonhomogeneous steady-state solution:*

$$\begin{cases} u_\lambda = \alpha_\lambda(\lambda - \lambda_*)[\varphi + (\lambda - \lambda_*)\xi_\lambda(x)], \\ v_\lambda = \beta_\lambda(\lambda - \lambda_*)[\varphi + (\lambda - \lambda_*)\eta_\lambda(x)]. \end{cases} \tag{2.6}$$

Proof. Let $F = (F_1, F_2, F_3, F_4)$ be defined as the following

$$\begin{cases} F_1(\xi,\eta,\alpha,\beta,\lambda) = L\xi + m(x)e^{am(x)}[\varphi+(\lambda-\lambda_*)\xi] - \lambda\alpha b_1 e^{2am(x)}[\varphi+(\lambda-\lambda_*)\xi]^2 \\ \qquad\qquad - \lambda\beta c_1 e^{2am(x)}[\varphi+(\lambda-\lambda_*)\xi][\varphi+(\lambda-\lambda_*)\eta] = 0, \\ F_2(\xi,\eta,\alpha,\beta,\lambda) = L\eta + m(x)e^{am(x)}[\varphi+(\lambda-\lambda_*)\eta] - \lambda\alpha c_2 e^{2am(x)}[\varphi+(\lambda-\lambda_*)\eta]^2 \\ \qquad\qquad - \lambda\beta b_2 e^{2am(x)}[\varphi+(\lambda-\lambda_*)\xi][\varphi+(\lambda-\lambda_*)\eta] = 0, \\ F_3(\xi,\eta,\alpha,\beta,\lambda) = \langle\varphi,\xi\rangle = 0, \\ F_4(\xi,\eta,\alpha,\beta,\lambda) = \langle\varphi,\eta\rangle = 0. \end{cases}$$

It is easy to obtain that from (2.5)

$$F_i(\xi_{\lambda_*},\eta_{\lambda_*},\alpha_{\lambda_*},\beta_{\lambda_*},\lambda_*) = 0, \qquad (i = 1,2,3,4).$$

The Fréchet derivative of F at $(\xi_{\lambda_*},\eta_{\lambda_*},\alpha_{\lambda_*},\beta_{\lambda_*},\lambda_*)$ is

$$\frac{\partial F}{\partial(\xi,\eta,\alpha,\beta)}\Big|_{(\xi_{\lambda_*},\eta_{\lambda_*},\alpha_{\lambda_*},\beta_{\lambda_*},\lambda_*)} \begin{pmatrix} \widehat{\xi} \\ \widehat{\eta} \\ \widehat{\alpha} \\ \widehat{\beta} \end{pmatrix} = \begin{pmatrix} L\widehat{\xi} - \lambda_*(\widehat{\alpha}b_1 + \widehat{\beta}c_1)e^{am(x)}\varphi^2 \\ L\widehat{\eta} - \lambda_*(\widehat{\alpha}c_2 + \widehat{\beta}b_2)e^{am(x)}\varphi^2 \\ \langle\varphi,\widehat{\xi}\rangle \\ \langle\varphi,\widehat{\eta}\rangle \end{pmatrix}.$$

It is clear that the derivative operator $\frac{\partial F}{\partial(\xi,\eta,\alpha,\beta)}\big|_{(\xi_{\lambda_*},\eta_{\lambda_*},\alpha_{\lambda_*},\beta_{\lambda_*},\lambda_*)}$ is bijective. By using the implicit function theorem we know that there exist a constant $\delta > 0$ and a continuously differentiable mapping which defined by $\lambda \to (\xi_\lambda, \eta_\lambda, \alpha_\lambda, \beta_\lambda)$ from $(\lambda_* - \delta, \lambda_* + \delta)$ to $X_1^2 \times (\mathbb{R}^+)^2$ such that system (1.5) has a positive spatially nonhomogeneous steady-state solution (2.6). □

3 Eigenvalue problems of the linearized system

For the convenience of discussion, we always suppose that $\Lambda = (\lambda_* - \delta, \lambda_*) \cup (\lambda_*, \lambda_* + \delta)$.

Let $(u_\lambda, v_\lambda)^T$ is a spatially nonhomogeneous steady-state solution of (1.5) which is determined by (2.6). Let

$$\tilde{u} = u - u_\lambda, \quad \tilde{v} = v - v_\lambda,$$

dropping the tilde sign, system (1.5) can be transformed as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla u] + \lambda u(x,t)[m(x) - b_1 e^{am(x)}u_\lambda - c_1 e^{am(x)}v_\lambda] \\ \qquad - \lambda e^{am(x)}u_\lambda[b_1 u(x,t-\tau) + c_1 v(x,t-\tau)], \\ \frac{\partial v}{\partial t} = e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla v] + \lambda v(x,t)[m(x) - b_2 e^{am(x)}u_\lambda - c_2 e^{am(x)}v_\lambda] \\ \qquad - \lambda e^{am(x)}v_\lambda[b_2 u(x,t-\tau) + c_2 v(x,t-\tau)]. \end{cases} \tag{3.1}$$

Denote A_λ, B_λ :

$$A_\lambda = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B_\lambda\psi = \begin{pmatrix} \lambda e^{am(x)}u_\lambda[b_1\psi_1(-\tau) + c_1\psi_2(-\tau)] \\ \lambda e^{am(x)}v_\lambda[b_2\psi_1(-\tau) + c_2\psi_2(-\tau)] \end{pmatrix}$$

where

$$A_1 = e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla] + \lambda[m(x) - b_1 e^{am(x)}u_\lambda - c_1 e^{am(x)}v_\lambda],$$
$$A_2 = e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla] + \lambda[m(x) - b_2 e^{am(x)}u_\lambda - c_2 e^{am(x)}v_\lambda],$$

and $\psi = (\psi_1, \psi_2)^T \in X^2_{\mathbb{C}}$.

It follows from [14,33] that the semigroup induced by the solutions of the linearized system (3.1) has the infinitesimal generator $T_{\tau,\lambda}$ satisfying

$$T_{\tau,\lambda}\psi = \dot{\psi}, \tag{3.2}$$

where

$$\mathcal{D}(T_{\tau,\lambda}) = \{\psi \in C_{\mathbb{C}} \cap C^1_{\mathbb{C}} \mid \psi(0) \in X_{\mathbb{C}},\ \dot{\psi}(0) = A_\lambda \psi(0) - B_\lambda \psi(-\tau)\},$$

where

$$C_{\mathbb{C}} = C([-\tau, 0], Y^2_{\mathbb{C}}), \qquad C^1_{\mathbb{C}} = C^1([-\tau, 0], Y^2_{\mathbb{C}}).$$

Moreover, $\mu \in \mathbb{C}$ an eigenvalue of $T_{\tau,\lambda}$ if and only if there exists $\psi = (\psi_1, \psi_2)^T \in X^2_{\mathbb{C}} \setminus \{(0,0)^T\}$ such that

$$\Delta(\lambda, \mu, \tau)\psi = A_\lambda \psi - B_\lambda \psi e^{-\mu\tau} - \mu\psi = 0. \tag{3.3}$$

Lemma 3.1. *0 is not an eigenvalue of $T_{\tau,\lambda}$.*

Proof. If 0 is an eigenvalue of $T_{\tau,\lambda}$, that is, there exists some $\psi = (\psi_1, \psi_2)^T \in X^2_{\mathbb{C}} \setminus \{(0,0)^T\}$ such that

$$\Delta(\lambda, 0, \tau)\psi = 0. \tag{3.4}$$

Note that $\Delta(\lambda_*, 0, \tau) = \left(\begin{smallmatrix} L & 0 \\ 0 & L \end{smallmatrix}\right)$ and $\mathcal{N}(L) = \operatorname{span}\{\varphi\}$. We let that ψ takes the form

$$\begin{cases} \psi_1 = p_1\varphi + (\lambda - \lambda_*)q_1(x), \\ \psi_2 = p_2\varphi + (\lambda - \lambda_*)q_2(x), \end{cases} \tag{3.5}$$

where $p_1, p_2 \in \mathbb{R}$, $q_1(x), q_2(x) \in X_1$. Then substituting (3.5) into (3.4) and let $\lambda = \lambda_*$, by calculation, we have

$$\begin{cases} Lq_1 + [m(x)e^{am(x)}\varphi - \lambda_* e^{2am(x)}(b_1\alpha_{\lambda_*} + c_1\beta_{\lambda_*})\varphi^2]p_1 - \lambda_* e^{2am(x)}\alpha_{\lambda_*}\varphi^2(b_1p_1 + c_1p_2) = 0, \\ Lq_2 + [m(x)e^{am(x)}\varphi - \lambda_* e^{2am(x)}(b_2\alpha_{\lambda_*} + c_2\beta_{\lambda_*})\varphi^2]p_2 - \lambda_* e^{2am(x)}\beta_{\lambda_*}\varphi^2(b_2p_1 + c_2p_2) = 0. \end{cases} \tag{3.6}$$

By (2.5), (3.6) becomes

$$\begin{cases} L(q_1 - \xi_{\lambda_*}p_1) - \lambda_* e^{2am(x)}\alpha_{\lambda_*}\varphi^2(b_1p_1 + c_1p_2) = 0, \\ L(q_2 - \eta_{\lambda_*}p_2) - \lambda_* e^{2am(x)}\beta_{\lambda_*}\varphi^2(b_2p_1 + c_2p_2) = 0. \end{cases} \tag{3.7}$$

Multiplying both sides of each equation in (3.7) by φ and integrating on Ω, we have

$$\begin{cases} b_1p_1 + c_1p_2 = 0, \\ b_2p_1 + c_2p_2 = 0. \end{cases} \tag{3.8}$$

By the condition (H1), we have $b_1c_2 - b_2c_1 \neq 0$. So we get $p_1 = p_2 = 0$ from (3.8). By (3.6), we get $q_1 = q_2 = 0$. Then $\psi_1 = 0, \psi_2 = 0$. The Lemma 3.1 is now proved. $\square$

We will show that the eigenvalues of $T_{\tau,\lambda}$ could pass through the imaginary axis when time delay τ increases. It is obvious that $T_{\tau,\lambda}$ has an imaginary eigenvalue $\mu = i\omega$ ($\omega \neq 0$) for some $\tau \geq 0$ if and only if

$$m(\lambda, \omega, \theta)\psi = \Delta(\lambda, \omega, \theta)\psi = A_\lambda\psi - B_\lambda\psi e^{-i\theta} - i\omega\psi = 0 \tag{3.9}$$

is solvable for some $\omega > 0, \theta \in [0, 2\pi), \tau = \frac{\theta + 2n\pi}{\omega}$, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\psi = (\psi_1, \psi_2)^T \in X^2_{\mathbb{C}} \setminus \{(0,0)^T\}$.

Lemma 3.2. *If* $(\omega, \theta, \psi) \in (0, \infty) \times [0, 2\pi) \times (X_{\mathbb{C}}^2 \setminus \{(0,0)^T\})$ *solves* (3.9), *then* $\frac{\omega}{\lambda - \lambda_*}$ *is bounded for* $\lambda \in \Lambda$.

Proof. Assume that $(\omega, \theta, \psi) \in (0, \infty) \times [0, 2\pi) \times (X_{\mathbb{C}}^2 \setminus \{(0,0)^T\})$ satisfy the following equation

$$\langle A_\lambda \psi - B_\lambda \psi e^{-i\theta} - i\omega\psi, \psi \rangle = 0. \tag{3.10}$$

Separating the real and imaginary parts of system (3.10), we obtain

$$\omega \langle \psi, \psi \rangle = \sin\theta \langle B_\lambda \psi, \psi \rangle.$$

$$\begin{aligned} \frac{|\omega|}{|\lambda - \lambda_*|} &= \lambda e^{am(x)} |\sin\theta| \frac{\left| \left\langle \begin{pmatrix} \alpha_\lambda [\varphi + (\lambda - \lambda_*)\xi_\lambda(x)](b_1\psi_1 + c_1\psi_2) \\ \beta_\lambda [\varphi + (\lambda - \lambda_*)\eta_\lambda(x)](b_2\psi_1 + c_2\psi_2) \end{pmatrix}, \psi \right\rangle \right|}{\langle \psi, \psi \rangle} \\ &\leq (\lambda_* + \delta) e^{a \max_{x \in \Omega} m(x)} \max\{M, N\} \max\{|b_1|, |c_1|, |b_2|, |c_2|\}. \end{aligned}$$

where $M = \max_{\lambda \in \Lambda}\{|\alpha_\lambda|[\|\varphi\|_\infty + (\lambda + \lambda_*)\|\xi_\lambda(x)\|_\infty]\}$,
$N = \max_{\lambda \in \Lambda}\{|\beta_\lambda|[\|\varphi\|_\infty + (\lambda + \lambda_*)\|\eta_\lambda(x)\|_\infty]\}$. The boundedness of $\frac{\omega}{\lambda - \lambda_*}$ follows from the continuity of $\lambda \mapsto (\alpha_\lambda, \beta_\lambda, \|\xi_\lambda(x)\|_\infty, \|\eta_\lambda(x)\|_\infty)$. The Lemma 3.2 is now proved. $\square$

Note that $X = \mathcal{N}(L) \oplus X_1$. If (ω, θ, ψ) satisfies (3.9), let $\psi = (\psi_1, \psi_2)^T \in X_{\mathbb{C}}^2 \setminus \{(0,0)^T\}$ can be represented as

$$\begin{cases} \psi_1 = p_1\varphi + (\lambda - \lambda_*)q_1(x), \\ \psi_2 = p_2\varphi + (\lambda - \lambda_*)q_2(x), \end{cases} \tag{3.11}$$

where $p_1, p_2 \in \mathbb{R}$, $q_1(x), q_2(x) \in X_1$. Let

$$G(q_1, q_2, p_1, p_2, h, \theta, \lambda)\psi = \frac{m(\lambda, (\lambda - \lambda_*)h, \theta)}{\lambda - \lambda_*}\psi = 0, \tag{3.12}$$

where $m(\lambda, \omega, \theta)$ is defined as in (3.9).

Obviously, we have

$$G(q_1, q_2, p_1, p_2, h, \theta, \lambda_*)\psi = 0,$$

that is

$$\begin{cases} L(q_1 - \xi_{\lambda_*} p_1) - \lambda_* e^{2am(x)} \alpha_{\lambda_*} \varphi^2 (b_1 p_1 + c_1 p_2) e^{-i\theta} - ih\varphi e^{am(x)} p_1 = 0, \\ L(q_2 - \eta_{\lambda_*} p_2) - \lambda_* e^{2am(x)} \beta_{\lambda_*} \varphi^2 (b_2 p_1 + c_2 p_2) e^{-i\theta} - ih\varphi e^{am(x)} p_2 = 0. \end{cases} \tag{3.13}$$

Multiplying both sides of each equation in (3.13) by φ and integrating on Ω, we have

$$-\lambda_* d_2 e^{-i\theta} M p = ihp, \tag{3.14}$$

where $p = (p_1, p_2)^T, d_2 = \frac{\int_\Omega e^{2am(x)}\varphi^3 dx}{\int_\Omega e^{am(x)}\varphi^2 dx} > 0$, $M = \begin{pmatrix} \alpha_{\lambda_*} b_1 & \alpha_{\lambda_*} c_1 \\ \beta_{\lambda_*} b_2 & \beta_{\lambda_*} c_2 \end{pmatrix}$.

Separating the real and imaginary parts of (3.14), we get

$$\begin{cases} \lambda_* d_2 \sin\theta M p = hp, \\ \lambda_* d_2 \cos\theta M p = 0. \end{cases} \tag{3.15}$$

It is easy to obtain the following lemma.

Lemma 3.3.

(1) *When* $\theta = \frac{\pi}{2}$ *in* (3.15), $\lambda_* d_2 M$ *has two real eigenvalues* $h_1 = \lambda_* d_1 d_2$, $h_2 = \lambda_* d_1 d_2 \frac{(c_2-c_1)(b_1-b_2)}{b_1c_2-b_2c_1}$, *and* $(c_2-c_1, b_1-b_2)^T$ *and* $(-c_1, b_2)^T$ *are two eigenvectors associated with eigenvalues* h_1 *and* h_2*, respectively.*

(2) *When* $\theta = \frac{3\pi}{2}$ *in* (3.15), $-\lambda_* d_2 M$ *has two real eigenvalues* $h_1 = -\lambda_* d_1 d_2$, $h_2 = -\lambda_* d_1 d_2 \frac{(c_2-c_1)(b_1-b_2)}{b_1c_2-b_2c_1}$, *and* $(c_2-c_1, b_1-b_2)^T$ *and* $(-c_1, b_2)^T$ *are two eigenvectors associated with eigenvalues* h_1 *and* h_2*, respectively.*

For each $j = 1, 2$, set

$$h^j_{\lambda*} = \begin{cases} |h_j|, & \text{if } \lambda > \lambda_*, \\ -|h_j|, & \text{if } \lambda < \lambda_*, \end{cases} \tag{3.16}$$

which satisfies $\omega^j_{\lambda*} = (\lambda - \lambda_*) h^j_{\lambda*} > 0$, and their corresponding eigenvectors

$$\begin{cases} (p^1_{1\lambda*}, p^1_{2\lambda*})^T = (c_2 - c_1, b_1 - b_2)^T, & \text{if } h^1_{\lambda*} = |h_1|, \\ (p^2_{1\lambda*}, p^2_{2\lambda*})^T = (-c_1, b_2)^T, & \text{if } h^2_{\lambda*} = |h_2|, \end{cases} \tag{3.17}$$

$$\begin{cases} (p^1_{1\lambda*}, p^1_{2\lambda*})^T = (c_2 - c_1, b_1 - b_2)^T, & \text{if } h^1_{\lambda*} = -|h_1|, \\ (p^2_{1\lambda*}, p^2_{2\lambda*})^T = (-c_1, b_2)^T, & \text{if } h^2_{\lambda*} = -|h_2|. \end{cases} \tag{3.18}$$

And set

$$\theta^j_{\lambda*} = \begin{cases} \frac{\pi}{2}, & \text{if } \lambda > \lambda_*, \\ \frac{3\pi}{2}, & \text{if } \lambda > \lambda_*, \end{cases} \tag{3.19}$$

which satisfies $-e^{-i\theta^j_{\lambda*}} h^j_{\lambda*} = i h^j_{\lambda*}$.

Thus, $q^j_{1\lambda*}, q^j_{2\lambda*} \in X_1$ is the unique solution of the following equations

$$\begin{cases} L(q^j_1 - \xi_{\lambda_*} p^j_{1\lambda*}) - \lambda_* e^{2am(x)} \alpha_{\lambda_*} \phi^2 (b_1 p^j_{1\lambda*} + c_1 p^j_{2\lambda*}) e^{-i\theta^j_{\lambda_*}} - i h^j_{\lambda*} \phi e^{am(x)} p^j_{1\lambda*} = 0, \\ L(q^j_2 - \eta_{\lambda_*} p^j_{2\lambda*}) - \lambda_* e^{2am(x)} \beta_{\lambda_*} \phi^2 (b_2 p^j_{1\lambda*} + c_2 p^j_{2\lambda*}) e^{-i\theta^j_{\lambda_*}} - i h^j_{\lambda*} \phi e^{am(x)} p^j_{2\lambda*} = 0, \end{cases} \tag{3.20}$$

Remark 3.4.

(1) When $v = 0, b_1 = 1$ in (1.4), h_1 in Lemma 3.3 (1) is the same as h_{λ_*} in (2.20) in [8].

(2) When $a_1 = 0$ in (1.4), h_1, h_2 in Lemma 3.3 (1) are the same as that in Lemma 3.4 (i) in [15].

Then we get the following lemma.

Lemma 3.5. *Assume that (H1) holds. For* $j = 1, 2$*, the following equation*

$$\begin{cases} G(q^j_1, q^j_2, p^j_1, p^j_2, \theta^j, h^j, \lambda_*) = 0, \\ q^j_1,\ q^j_2 \in X_1,\ p^j_1,\ p^j_2,\ h^j \in \mathbb{R},\ \theta^j \in [0, 2\pi] \end{cases} \tag{3.21}$$

has a unique solution $(q^j_{1\lambda_*}, q^j_{2\lambda_*}, p^j_{1\lambda_*}, p^j_{2\lambda_*}, \theta^j_{\lambda_*}, h^j_{\lambda_*})$*, see* (3.16)–(3.20).

Theorem 3.6. *Assume that (H1) holds. Then for* $j = 1, 2$*, there exist a constant* $\delta > 0$ *and a continuously differentiable mapping which defined by* $\lambda \to (q^j_{1\lambda}, q^j_{2\lambda}, p^j_{1\lambda}, p^j_{2\lambda}, \theta^j_\lambda, h^j_\lambda)$ *from* Λ *to* $X_1^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}$ *such that* $G(q^j_{1\lambda}, q^j_{2\lambda}, p^j_{1\lambda}, p^j_{2\lambda}, \theta^j_\lambda, h^j_\lambda, \lambda) = 0$.

Proof. Let $G = (g_1, g_2, g_3, g_4, g_5, g_6)$ be defined as the following:

$$\begin{cases} g_1 = m(\lambda, (\lambda-\lambda_*)h^j_\lambda, \theta^j_\lambda)q^j_{1\lambda} + m(x)e^{am(x)}\varphi p^j_{1\lambda} \\ \qquad - \lambda e^{am(x)}\varphi[\alpha_\lambda b_1(\varphi + (\lambda-\lambda_*)\xi) + \beta_\lambda c_1(\varphi + (\lambda-\lambda_*)\xi)]p^j_{1\lambda} \\ \qquad + \lambda\varphi\alpha_\lambda e^{am(x)}(\varphi + (\lambda-\lambda_*)\xi)(b_1 p^j_{1\lambda} + c_1 p^j_{2\lambda})e^{-i\theta^j_\lambda} - ih^j_\lambda \varphi p^j_{1\lambda} = 0, \\ g_2 = m(\lambda, (\lambda-\lambda_*)h^j_\lambda, \theta^j_\lambda)q^j_{2\lambda} + m(x)e^{am(x)}\varphi p^j_{2\lambda} \\ \qquad - \lambda e^{am(x)}\varphi[\alpha_\lambda b_2(\varphi + (\lambda-\lambda_*)\xi) + \beta_\lambda c_2(\varphi + (\lambda-\lambda_*)\xi)]p^j_{2\lambda} \\ \qquad + \lambda\varphi\alpha_\lambda e^{am(x)}(\varphi + (\lambda-\lambda_*)\xi)(b_2 p^j_{1\lambda} + c_2 p^j_{2\lambda})e^{-i\theta^j_\lambda} - ih^j_\lambda \varphi p^j_{2\lambda} = 0, \\ g_3 = \mathrm{Re}\langle \varphi, q^j_{1\lambda}\rangle = 0, \quad g_4 = \mathrm{Im}\langle \varphi, q^j_{1\lambda}\rangle = 0, \\ g_5 = \mathrm{Re}\langle \varphi, q^j_{2\lambda}\rangle = 0, \quad g_6 = \mathrm{Im}\langle \varphi, q^j_{2\lambda}\rangle = 0. \end{cases}$$

The Fréchet derivative of G at $(q^j_{1\lambda_*}, q^j_{2\lambda_*}, p^j_{1\lambda_*}, p^j_{2\lambda_*}, \theta^j_{\lambda_*}, h^j_{\lambda_*}, \lambda_*)$ is

$$\frac{\partial G(q^j_{1\lambda_*}, q^j_{2\lambda_*}, p^j_{1\lambda_*}, p^j_{2\lambda_*}, \theta^j_{\lambda_*}, h^j_{\lambda_*}, \lambda_*)}{\partial(q^j_{1\lambda}, q^j_{2\lambda}, p^j_{1\lambda}, p^j_{2\lambda}, \theta^j_\lambda, h^j_\lambda)} \begin{pmatrix} \hat{q}^j_{1\lambda} \\ \hat{q}^j_{2\lambda} \\ \hat{p}^j_{1\lambda} \\ \hat{p}^j_{2\lambda} \\ \hat{\theta}^j_\lambda \\ \hat{h}^j_\lambda \end{pmatrix} = \begin{pmatrix} e^{-am(x)}L\hat{q}^j_{1\lambda} + \widetilde{g}_1\hat{p}^j_{1\lambda} + \widetilde{g}_2\hat{p}^j_{2\lambda} + \widetilde{g}_3\hat{\theta}^j_\lambda + \widetilde{g}_4\hat{h}^j_\lambda \\ e^{-am(x)}L\hat{q}^j_{2\lambda} + \widetilde{g}_5\hat{p}^j_{1\lambda} + \widetilde{g}_6\hat{p}^j_{2\lambda} + \widetilde{g}_7\hat{\theta}^j_\lambda + \widetilde{g}_8\hat{h}^j_\lambda \\ \mathrm{Re}\langle \varphi, \hat{q}^j_{1\lambda}\rangle \\ \mathrm{Im}\langle \varphi, \hat{q}^j_{1\lambda}\rangle \\ \mathrm{Re}\langle \varphi, \hat{q}^j_{2\lambda}\rangle \\ \mathrm{Im}\langle \varphi, \hat{q}^j_{2\lambda}\rangle \end{pmatrix},$$

where

$$\begin{cases} \widetilde{g}_1 = m(x)e^{am(x)}\varphi - \lambda_* e^{am(x)}(\alpha_{\lambda_*}b_1 + \beta_{\lambda_*}c_1)\varphi^2 - \lambda_*\alpha_{\lambda_*}e^{am(x)}b_1 e^{-i\theta^j_{\lambda_*}} - ih^j_{\lambda_*}\varphi, \\ \widetilde{g}_2 = -\lambda_*\alpha_{\lambda_*}e^{am(x)}c_1 e^{-i\theta^j_{\lambda_*}},\ \widetilde{g}_3 = -i\theta^j_{\lambda_*}\lambda_*\varphi^2\alpha_{\lambda_*}(b_1 p^j_{1\lambda_*} + b_2 p^j_{2\lambda_*})e^{-i\theta^j_{\lambda_*}},\ \widetilde{g}_4 = -i\varphi p^j_{1\lambda_*}, \\ \widetilde{g}_5 = m(x)e^{am(x)}\varphi - \lambda_* e^{am(x)}(\alpha_{\lambda_*}b_2 + \beta_{\lambda_*}c_2)\varphi^2 - \lambda_*\alpha_{\lambda_*}e^{am(x)}b_2 e^{-i\theta^j_{\lambda_*}} - ih^j_{\lambda_*}\varphi, \\ \widetilde{g}_6 = -\lambda_*\alpha_{\lambda_*}e^{am(x)}c_2 e^{-i\theta^j_{\lambda_*}},\ \widetilde{g}_7 = -i\theta^j_{\lambda_*}\lambda_*\varphi^2\alpha_{\lambda_*}(b_1 p_{1\lambda_*} + b_2 p_{2\lambda_*})e^{-i\theta^j_{\lambda_*}},\ \widetilde{g}_8 = -i\varphi p^j_{2\lambda_*}. \end{cases}$$

It is clear that the derivative operator

$$\frac{\partial G(q^j_{1\lambda_*}, q^j_{2\lambda_*}, p^j_{1\lambda_*}, p^j_{2\lambda_*}, \theta^j_{\lambda_*}, h^j_{\lambda_*}, \lambda_*)}{\partial(q^j_{1\lambda}, q^j_{2\lambda}, p^j_{1\lambda}, p^j_{2\lambda}, \theta^j_\lambda, h^j_\lambda)}$$

is bijective. By using the implicit function theorem we know that there exist a constant $\delta > 0$ and a continuously differentiable mapping which defined by $\lambda \to (q^j_{1\lambda}, q^j_{2\lambda}, p^j_{1\lambda}, p^j_{2\lambda}, \theta^j_\lambda, h^j_\lambda)$, from from Λ to $X_1^2 \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}$ such that $G(q^j_{1\lambda}, q^j_{2\lambda}, p^j_{1\lambda}, p^j_{2\lambda}, \theta^j_\lambda, h^j_\lambda, \lambda) = 0$. The proof of Theorem 3.6 is complete. □

From Theorem 3.6, we derive the following result.

Theorem 3.7. *Assume that (H1) holds. For $j = 1, 2, \lambda \in \Lambda$ and $n \in \mathbb{N}_0$, let*

$$\tau^j_n = \frac{\theta^j_{1\lambda} + 2n\pi}{\omega^j_\lambda}, \qquad \omega^j_\lambda = (\lambda - \lambda_*)h^j_\lambda,$$

and $\psi^j_\lambda = (\psi^j_{1\lambda}, \psi^j_{2\lambda})^T$,

$$\begin{cases} \psi^j_{1\lambda} = p^j_{1\lambda}\varphi + (\lambda - \lambda_*)q^j_{1\lambda}(x), \\ \psi^j_{2\lambda} = p^j_{2\lambda}\varphi + (\lambda - \lambda_*)q^j_{2\lambda}(x), \end{cases} \tag{3.22}$$

where $q^j_{1\lambda}$, $q^j_{2\lambda}$, $p^j_{1\lambda}$, $p^j_{2\lambda}$, θ^j_λ, h^j_λ *are defined as in Theorem 3.6. Then*

(1) $T_{\tau^j_n,\lambda}$ *has a pair of purely imaginary eigenvalues* $\pm i\omega^j_\lambda$;

(2) $T_{\tau^j_n,\lambda} e^{i\omega^j_\lambda}\psi^j = i\omega^j_\lambda e^{i\omega^j_\lambda}\psi^j_\lambda$, $T_{\tau^j_n,\lambda} e^{-i\omega^j_\lambda}\overline{\psi}^j = -i\omega^j_\lambda e^{-i\omega^j_\lambda}\overline{\psi}^j_\lambda$.

Now, we give some estimates to prove the simplicity of $i\omega^j_\lambda$. The proof of the following Lemmas 3.8–3.10 is similar to [14]. For the sake of the integrity of the article, we are going to prove them again.

Lemma 3.8. *Assume that (H1) holds. For* $j = 1, 2, \lambda \in \Lambda$ *and* $n \in \mathbb{N}_0$, *we have*

$$S^j_n = \langle \overline{\psi}^j_\lambda, \psi^j_\lambda - \tau^j_n e^{-i\theta^j_\lambda} B_\lambda \psi^j_\lambda \rangle \neq 0, \tag{3.23}$$

where ψ^j_λ, τ^j_n *and* θ^j_λ *are defined as in Theorem 3.7.*

Proof. It is easy to obtain that

$$\text{Re }\{S^j_n\} = \langle \overline{\psi}_\lambda, (\psi_\lambda - \tau^j_n e^{-i\theta^j_\lambda} B_\lambda \psi_\lambda) \rangle \to [(p^j_{1\lambda_*})^2 + (p^j_{2\lambda_*})^2] \neq 0, \quad \text{as } \lambda \to \lambda_*,$$

where using $\int_\Omega \varphi^2 dx = 1$ in Section 1. □

Lemma 3.9. *Assume that (H1) holds. For* $j = 1, 2, \lambda \in \Lambda$ *and* $n \in \mathbb{N}_0$, $i\omega^j_\lambda$ *is a simple eigenvalue of* $T_{\tau^j_n,\lambda}$.

Proof. It follows from Theorem 3.7 that $\mathcal{N}[T_{\tau^j_n,\lambda} - i\omega^j_\lambda] = \text{span}[\psi^j_\lambda e^{i\omega^j_\lambda(\cdot)}]$. If $\widetilde{\psi} \in \mathcal{N}[T_{\tau^j_n,\lambda} - i\omega^j_\lambda]^2$, that is

$$(T_{\tau^j_n,\lambda} - i\omega^j_\lambda)^2 \widetilde{\psi} = 0,$$

then

$$(T_{\tau^j_n,\lambda} - i\omega^j_\lambda)\widetilde{\psi} \in \mathcal{N}[T_{\tau^j_n,\lambda} - i\omega^j_\lambda] = \text{span}[\psi^j_\lambda e^{i\omega^j_\lambda(\cdot)}].$$

We assume that a constant ρ satisfies

$$(T_{\tau^j_n,\lambda} - i\omega^j_\lambda)\widetilde{\psi} = \rho\psi^j_\lambda e^{i\omega^j_\lambda(\cdot)},$$

which leads to

$$\begin{cases} \widetilde{\psi}'(s) = i\omega^j_\lambda \widetilde{\psi}(s) + \rho\psi^j_\lambda e^{i\omega^j_\lambda s}, \quad s \in [-\tau^j_n, 0), \\ \widetilde{\psi}'(0) = A_\lambda \widetilde{\psi}(0) - B_\lambda \widetilde{\psi}(-\tau^j_n). \end{cases} \tag{3.24}$$

From the first equation of (3.24), we have

$$\begin{cases} \widetilde{\psi}(s) = \widetilde{\psi}(0)e^{i\omega^j_\lambda s} + \rho s\psi^j_\lambda e^{i\omega^j_\lambda s}, \quad s \in [-\tau^j_n, 0), \\ \widetilde{\psi}'(0) = i\omega^j_\lambda \widetilde{\psi}(0) + \rho\psi^j_\lambda. \end{cases} \tag{3.25}$$

Eq. (3.24) and Eq. (3.25) imply that

$$\begin{cases} A_\lambda \widetilde{\psi}(0) - B_\lambda \widetilde{\psi}(-\tau_n^j) = i\omega_\lambda^j \widetilde{\psi}(0) + \rho \psi_\lambda^j, \\ \widetilde{\psi}(-\tau_n^j) = \widetilde{\psi}(0) e^{-i\omega_\lambda^j \tau_n^j} - \tau_n^j \rho \psi_\lambda^j e^{-i\omega_\lambda^j \tau_n^j}. \end{cases} \tag{3.26}$$

Then we have

$$\begin{aligned} \Delta(\lambda, i\omega_\lambda^j, \tau_n^j)\widetilde{\psi}(0) &= (A_\lambda - i\omega_\lambda^j)\widetilde{\psi}(0) - B_\lambda \widetilde{\psi}(0) e^{-i\theta_\lambda^j} \\ &= \rho(\psi_\lambda^j - \tau_n^j e^{-i\theta_\lambda^j} B_\lambda \psi_\lambda^j). \end{aligned}$$

Since $\Delta(\lambda, i\omega_\lambda^j, \tau_n^j)\psi_\lambda^j = 0$, so

$$\Delta(\lambda, -i\omega_\lambda^j, \tau_n^j)\overline{\psi}_\lambda^j = 0. \tag{3.27}$$

Then

$$0 = \langle \Delta(\lambda, -i\omega_\lambda^j, \tau_n^j)\overline{\psi}_\lambda^j, \widetilde{\psi}(0)\rangle = \langle \overline{\psi}_\lambda^j, \Delta(\lambda, i\omega_\lambda^j, \tau_n^j)\widetilde{\psi}(0)\rangle = \rho\langle \overline{\psi}_\lambda^j, (\psi_\lambda^j - \tau_n^j e^{-i\theta_\lambda^j} B_\lambda \psi_\lambda^j)\rangle.$$

As a consequence of Lemma 3.8, we have $\rho = 0$ and $(T_{\tau_n^j,\lambda} - i\omega_\lambda^j)\widetilde{\psi} = 0$, that is $\widetilde{\psi} \in \mathcal{N}[T_{\tau_n^j,\lambda} - i\omega_\lambda^j]$. By induction, we obtain

$$\mathcal{N}([T_{\tau_n^j,\lambda} - i\omega_\lambda^j]^s) = \mathcal{N}[T_{\tau_n^j,\lambda} - i\omega_\lambda^j]$$

for all $s \in \{1, 2, 3, \dots\}$. Hence, $i\omega_\lambda^j$ is a simple eigenvalue of $T_{\tau_n^j,\lambda}$. □

Note that $\mu = i\omega_\lambda^j$ is a simple eigenvalue of $T_{\tau_n^j,\lambda}$. It follows from the implicit function theorem that there are a neighborhood On $O_n \times D_n \times H_n \subset \mathbb{R} \times \mathbb{C} \times X_\mathbb{C}$ of $(\tau_n^j, i\omega_\lambda^j, \psi_\lambda^j)$ and a continuously differential function $(\mu(\tau), \psi(\tau)) : O_n \to D_n \times H_n$ such that for each $\tau \in O_n$, the only eigenvalue of $T_{\tau,\lambda}$ in D_n is $\mu(\tau)$, and the following equality holds

$$\Delta(\lambda, \mu(\tau), \tau)\psi(\tau) = A_\lambda \psi(\tau) - e^{-\mu(\tau)\tau} B_\lambda \psi(\tau) - \mu(\tau)\psi(\tau) = 0. \tag{3.28}$$

Moreover, $\mu(\tau_n^j) = i\omega_\lambda^j$ and $\psi(\tau_n^j) = \psi_\lambda^j$. Then we have the following transversality condition, see [21].

Lemma 3.10. *Assume that (H1) holds. For $j = 1, 2$, $\lambda \in \Lambda$ and $n \in \mathbb{N}_0$, assume that $\mu(\tau)$ is the eigenvalue of $T_{\tau,\lambda}$, then*

$$\left.\frac{d\,\mathrm{Re}\{\mu(\tau)\}}{d\tau}\right|_{\tau=\tau_n^j} > 0.$$

Proof. Differentiating Eq. (3.28) with respect to τ at $\tau = \tau_n^j$, we have

$$\Delta(\lambda, i\omega_\lambda^j, \tau_n^j)\frac{d\psi_\lambda^j(\tau_n^j)}{d\tau} + [\tau_n^j e^{-i\theta_\lambda^j} B_\lambda \psi_\lambda^j - \psi_\lambda^j]\frac{d\mu(\tau_n^j)}{d\tau} + i\omega_\lambda^j e^{-i\theta_\lambda^j} B_\lambda \psi_\lambda^j = 0. \tag{3.29}$$

By (3.27), we get

$$\left\langle \overline{\psi}_\lambda^j, \Delta(\lambda, i\omega_\lambda^j, \tau_n^j)\frac{d\psi_\lambda^j(\tau_n^j)}{d\tau}\right\rangle = \left\langle \Delta(\lambda, -i\omega_\lambda^j, \tau_n^j)\overline{\psi}_\lambda^j, \frac{d\psi_\lambda^j(\tau_n^j)}{d\tau}\right\rangle = 0. \tag{3.30}$$

Calculating the inner product with ψ_λ^j in Eq. (3.29) and using Eq. (3.30), we have

$$S_n^j \frac{d\mu(\tau_n^j)}{d\tau} = \langle \overline{\psi}_\lambda^j, i\omega_\lambda^j e^{-i\theta_\lambda^j} B_\lambda \psi_\lambda^j \rangle,$$

where S_n^j is defined as in Lemma 3.8. Then we have

$$\frac{d\mu(\tau_n^j)}{d\tau} = \frac{I_1 + I_2}{(S_n^j)^2},$$

where

$$I_1 = \langle \overline{\psi}_\lambda^j, \psi_\lambda^j \rangle \langle \overline{\psi}_\lambda^j, i\omega_\lambda^j e^{-i\theta_\lambda^j} B_\lambda \psi_\lambda^j \rangle, \qquad I_2 = i\omega_\lambda^j \tau_n^j |\langle \overline{\psi}_\lambda^j, B_\lambda \psi_\lambda^j \rangle|^2.$$

Hence, it is clear that

$$\left.\frac{dRe\{\mu(\tau)\}}{d\tau}\right|_{\tau=\tau_n^j} = \mathrm{Re}\frac{I_1}{|S_n^j|^2}.$$

In fact,

$$\langle \overline{\psi}_\lambda^j, \psi_\lambda^j \rangle \to [(p_{1\lambda_*}^j)^2 + (p_{2\lambda_*}^j)^2], \quad \text{as } \lambda \to \lambda_*,$$

and

$$\frac{1}{(\lambda-\lambda_*)^2} \langle \overline{\psi}_\lambda^j, i\omega_\lambda^j e^{-i\theta_\lambda^j} B_\lambda \psi_\lambda^j \rangle \to (h_{\lambda_*}^j)^2, \quad \text{as } \lambda \to \lambda_*.$$

Therefore, for δ enough small, we have $\frac{d\,\mathrm{Re}\{\mu(\tau)\}}{d\tau}|_{\tau=\tau_n^j} > 0$.

From above analysis, we obtain that a pair of purely imaginary eigenvalues will occur as τ passes $\tau = \tau_n^j$. The proof of Lemma 3.10 is complete. □

From Lemmas 3.8–3.10, we have the result on the distribution of eigenvalues of $T_{\tau,\lambda}$.

Theorem 3.11. *Assume that (H1) holds. For $j = 1, 2$, $\lambda \in \Lambda$ and $n \in \mathbb{N}_0$, the infinitesimal generator $T_{\tau,\lambda}$ has exactly $2(n+1)$ eigenvalues with positive real parts when $\tau \in (\tau_n^j, \tau_{n+1}^j)$.*

4 Stability analysis

In this section, we study the stability of the steady state solutions (u_λ, v_λ) of (1.5) by regarding the delay τ as a parameter. We first investigate the stability of when $\tau = 0$, and then discuss the stability and bifurcation when $\tau \neq 0$. We need the following condition (H2).

(H2) $b_1c_2 - b_2c_1 > 0$.

Theorem 4.1. *Assume (H1)–(H2) hold. For $\lambda \in (\lambda_*, \lambda_* + \delta)$ (respectively, $\lambda \in (\lambda_* - \delta, \lambda_*)$, then all eigenvalues of $T_{0,\lambda}$ have negative (respectively, positive) real parts, and hence the steady state solution $(u_\lambda, v_\lambda)^T$ of (1.5) with $\tau = 0$ is locally asymptotically stable (respectively, unstable).*

Proof. When $\tau = 0$, the eigenvalue problem (3.3) reduces to

$$\Delta(\lambda, \mu, 0)\psi = A_\lambda \psi - B_\lambda \psi = \mu\psi. \tag{4.1}$$

with $\psi = (\psi_1, \psi_2)^T \in X_{\mathbb{C}}^2 \setminus \{(0,0)^T\}$. We suppose that

$$\begin{cases} \psi_1 = p_1\varphi + (\lambda - \lambda_*)q_1(x), \\ \psi_2 = p_2\varphi + (\lambda - \lambda_*)q_2(x), \end{cases} \tag{4.2}$$

where $p_1, p_2 \in \mathbb{R}$, $q_1(x), q_2(x) \in X_1$. Then substituting (4.2) into (4.1) and let $\lambda \to \lambda_*$, by calculation, we have

$$\begin{cases} Lq_1 + [m(x)e^{am(x)}\varphi - \lambda_* e^{2am(x)}(b_1\alpha_{\lambda_*} + c_1\beta_{\lambda_*})\varphi^2]p_1 \\ \qquad\qquad - \lambda_* e^{2am(x)}\alpha_{\lambda_*}\varphi^2(b_1p_1 + c_1p_2) = \tilde{\mu}e^{am(x)}\varphi p_1, \\ Lq_2 + [m(x)e^{am(x)}\varphi - \lambda_* e^{2am(x)}(b_2\alpha_{\lambda_*} + c_2\beta_{\lambda_*})\varphi^2]p_2 \\ \qquad\qquad - \lambda_* e^{2am(x)}\beta_{\lambda_*}\varphi^2(b_2p_1 + c_2p_2) = \tilde{\mu}e^{am(x)}\varphi p_2, \end{cases} \tag{4.3}$$

where $\tilde{\mu} = \lim_{\lambda\to\lambda_*} \frac{\mu}{\lambda-\lambda_*}$. By (2.5), (4.3) becomes

$$\begin{cases} L(q_1 - \xi_{\lambda_*}p_1) - \lambda_* e^{2am(x)}\alpha_{\lambda_*}\varphi^2(b_1p_1 + c_1p_2) = \tilde{\mu}\varphi p_1, \\ L(q_2 - \eta_{\lambda_*}p_2) - \lambda_* e^{2am(x)}\beta_{\lambda_*}\varphi^2(b_2p_1 + c_2p_2) = \tilde{\mu}\varphi p_2. \end{cases} \tag{4.4}$$

Multiplying both sides of each equation in (4.4) by φ and integrating on Ω, we have

$$\begin{cases} \lambda_* d_2\alpha_{\lambda_*}(b_1p_1 + c_1p_2) + \tilde{\mu}p_1 = 0, \\ \lambda_* d_2\beta_{\lambda_*}(b_2p_1 + c_2p_2) + \tilde{\mu}p_2 = 0. \end{cases} \tag{4.5}$$

Thus, we get that the eigenvalue equation of $\tilde{\mu}$

$$\tilde{\mu}^2 + \lambda_* d_2(\alpha_{\lambda_*}b_1 + \beta_{\lambda_*}c_2)\tilde{\mu} + \lambda_*^2 d_2^2\alpha_{\lambda_*}\beta_{\lambda_*}(b_1c_2 - b_2c_1) = 0. \tag{4.6}$$

By (H2), we have that the eigenvalue of (4.6) $\tilde{\mu}_1, \tilde{\mu}_2 < 0$. Then the conclusion of Theorem 4.1 is obtained. □

Theorem 4.2. *Assume (H1)–(H2) hold. For $j = 1, 2$, $\lambda \in \Lambda, n \in \mathbb{N}_0$, then*

(1) *the steady state solution $(u_\lambda, v_\lambda)^T$ of (1.5) is locally asymptotically stable when $\tau \in [0, \tau_0)$, where $\tau_0 = \min\{\tau_0^1, \tau_0^2\}$;*

(2) *the system (1.5) undergoes a Hopf bifurcation at the steady state solution $(u_\lambda, v_\lambda)^T$ when $\tau = \tau_n^j$, i.e., system (1.5) has a branch of periodic solutions bifurcating from the steady state solution $(u_\lambda, v_\lambda)^T$ near $\tau = \tau_n^j$.*

5 Direction of Hopf bifurcation

From the analysis of section 4, we obtained conditions for Hopf bifurcation to occur when $\tau = \tau_n^j$ $(j = 1, 2, n \in \mathbb{N}_0)$. In this section,we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from the equilibrium $(u_\lambda, v_\lambda)^T$ at $\tau = \tau_n^j$ $(j = 1, 2, n \in \mathbb{N}_0)$, by using techniques from normal form and center manifold theory [9,12,14,19,33].

Let $(u_\lambda, v_\lambda)^T$ is a spatially nonhomogeneous steady-state solution of (1.5). Let

$$\tilde{u}(t) = u(\cdot, \tau t) - u_\lambda, \tilde{v}(t) = v(\cdot, \tau t) - v_\lambda.$$

For the simple, let $U(t) = (u(t), v(t))^T = (\tilde{u}(t), \tilde{v}(t))^T$, then system (1.5) can be written as follows:

$$\frac{dU(t)}{dt} = \tau L_0(U_t) - \tau L_1(U_t) + f(U_t, \tau), \tag{5.1}$$

where $U_t \in \mathcal{C} = C^1([-1,0], Y^2)$, and

$$L_0(U(t)) = \begin{pmatrix} e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla u] + \lambda u(t)[m(x) - b_1 e^{am(x)}u_\lambda - c_1 e^{am(x)}v_\lambda] \\ e^{-am(x)}\nabla\cdot[e^{am(x)}\nabla v] + \lambda v(t)[m(x) - b_2 e^{am(x)}u_\lambda - c_2 e^{am(x)}v_\lambda] \end{pmatrix}, \tag{5.2}$$

$$L_1(U_t) = \begin{pmatrix} \lambda e^{am(x)}u_\lambda[b_1 u(t-1) + c_1 v(t-1)] \\ \lambda e^{am(x)}v_\lambda[b_2 u(t-1) + c_2 v(t-1)] \end{pmatrix}, \tag{5.3}$$

$$f(U_t,\tau) = \begin{pmatrix} -\tau\lambda e^{am(x)}[b_1 u(t)u(t-1) + c_1 u(t)v(t-1)] \\ -\tau\lambda e^{am(x)}[b_2 v(t)u(t-1) + c_2 v(t)v(t-1)] \end{pmatrix}. \tag{5.4}$$

Let $\tau = \tau_n^j + \varepsilon$, then (5.1) can be rewritten as

$$\frac{dU(t)}{dt} = \tau_n^j L_0(U(t)) - \tau_n^j L_1(U_t) + F(U_t,\varepsilon), \tag{5.5}$$

where

$$F(U_t,\varepsilon) = \varepsilon L_0(U(t)) - \varepsilon L_1(U_t) + f(U_t, \tau_n^j + \varepsilon). \tag{5.6}$$

From the previous discussion, it is clear that when $\varepsilon = 0$ (i.e., $\tau = \tau_n^j$) system (5.5) undergoes Hopf bifurcation at the equilibrium (0,0).

It follows from [14,33] that

$$T_{\tau_n^j}\psi = \dot{\psi}, \tag{5.7}$$

and the domain

$$\mathcal{D}(T_{\tau_n^j}) = \{\psi \in \mathcal{C}_{\mathbb{C}} \cap \mathcal{C}^1_{\mathbb{C}} : \psi(0) \in X_{\mathbb{C}},\ \dot{\psi}(0) = \tau_n^j L_0\psi(0) - \tau_n^j L_1\psi(-1)\},$$

where

$$\mathcal{C}_{\mathbb{C}} = C([-1,0], Y^2_{\mathbb{C}}), \qquad \mathcal{C}^1_{\mathbb{C}} = C^1([-1,0], Y^2_{\mathbb{C}}).$$

We can compute the formal adjoint operator $T^*_{\tau_n^j}$ of $T_{\tau_n^j}$ with respect to the formal duality,

$$T^*_{\tau_n^j}\phi = -\dot{\phi}, \tag{5.8}$$

and the domain

$$\mathcal{D}(T^*_{\tau_n^j}) = \{\phi \in \mathcal{C}^*_{\mathbb{C}} \cap (\mathcal{C}^*_{\mathbb{C}})^1 : \phi(0) \in X_{\mathbb{C}},\ -\dot{\phi}(0) = \tau_n^j L_0\phi(0) - \tau_n^j L_1\phi(1)\},$$

where

$$\mathcal{C}^*_{\mathbb{C}} = C([0,1], Y^2_{\mathbb{C}}), \qquad (\mathcal{C}^*_{\mathbb{C}})^1 = C^1([0,1], Y^2_{\mathbb{C}}).$$

Following [30], we introduce the formal duality $\langle\langle\cdot,\cdot\rangle\rangle$ in $\mathcal{C}_{\mathbb{C}} \times \mathcal{C}^*_{\mathbb{C}}$ by

$$\langle\langle\phi,\psi\rangle\rangle = \langle\phi(0),\psi(0)\rangle_1 - \tau_n^j \int_{-1}^0 \langle\phi(s+1), L_1\psi(s)\rangle_1 ds, \tag{5.9}$$

for $\psi \in \mathcal{C}_{\mathbb{C}}$ and $\phi \in \mathcal{C}^*_{\mathbb{C}}$, where $\langle\psi,\phi\rangle_1 = \int_\Omega e^{am(x)}\bar{\psi}^T\phi dx$, see [8].

Lemma 5.1. *$T_{\tau_n^j}$ and $T^*_{\tau_n^j}$ are adjoint operators, that is*

$$\langle\langle\phi, T_{\tau_n^j}\psi\rangle\rangle = \langle\langle T^*_{\tau_n^j}\phi,\psi\rangle\rangle,$$

for $\psi \in \mathcal{C}_{\mathbb{C}}$, $\phi \in \mathcal{C}^*_{\mathbb{C}}$.

Proof. It follows from (5.9) and the definition of $T_{\tau_n^j}$, $T^*_{\tau_n^j}$ that,

$$\begin{aligned}
\langle\langle \phi, T_{\tau_n^j}\psi\rangle\rangle &= \langle \phi(0), T_{\tau_n^j}\psi(0)\rangle_1 - \tau_n^j \int_{-1}^0 \langle \phi(s+1), L_1\dot{\psi}(s)\rangle_1 ds \\
&= \langle \phi(0), \tau_n^j L_0\psi(0) - \tau_n^j L_1\psi(-1)\rangle_1 - \tau_n^j [\langle \phi(s+1), L_1\psi(s)\rangle_1]_{-1}^0 \\
&\quad + \tau_n^j \int_{-1}^0 \langle \dot{\phi}(s+1), L_1\psi(s)\rangle_1 ds \\
&= \langle T^*_{\tau_n^j}\phi(0), \psi(0)\rangle_1 + \tau_n^j \int_{-1}^0 \langle -\dot{\phi}(s+1), L_1\psi(s)\rangle_1 ds \\
&= \langle\langle T^*_{\tau_n^j}\phi, \psi\rangle\rangle.
\end{aligned}$$

The proof of Lemma 5.1 is complete. □

From Theorem 3.7 and Lemma 5.1, we have that $\pm i\omega_\lambda^j\tau_n^j$ are the eigenvalues of $T_{\tau_n^j}$, and they are the eigenvalues of $T^*_{\tau_n^j}$. The vectors $p(\theta) = e^{i\omega_\lambda^j\tau_n^j\theta}\psi_\lambda^j$ $(\theta \in [-1,0])$ and $q(s) = e^{i\omega_\lambda^j\tau_n^j s}\overline{\psi}_\lambda^j$ $(s \in [0,1])$ satisfy

$$T_{\tau_n^j} p = i\omega_\lambda^j\tau_n^j p, \quad \text{and} \quad T^*_{\tau_n^j} q = i\omega_\lambda^j\tau_n^j q,$$

respectively. Let

$$\Phi = (p(\theta), \bar{p}(\theta))^T, \qquad \Psi = (\frac{q(s)}{\bar{R}_n^j}, \frac{\bar{q}(s)}{R_n^j})^T,$$

$$R_n^j = \langle \overline{\psi}_\lambda^j, \psi_\lambda^j - \tau_n^j e^{-i\theta_\lambda^j} B_\lambda \psi_\lambda^j\rangle_1.$$

and $\mathrm{Re}\{R_n^j\} = \langle \overline{\psi}_\lambda, (\psi_\lambda - \tau_n^j e^{-i\theta_\lambda^j} B_\lambda\psi_\lambda)\rangle_1 \to [(p_{1\lambda_*}^j)^2 + (p_{2\lambda_*}^j)^2] \int_\Omega e^{am(x)}\varphi^2 dx \neq 0$, as $\lambda \to \lambda_*$. One can easily check that $\langle\langle \Psi, \Phi\rangle\rangle = I$, where I is the identity matrix in $\mathbb{R}^{2\times 2}$. Moreover, can be decomposed as $\mathcal{C}_\mathbb{C} = P \oplus Q$, where

$$P = \mathrm{span}\{p(\theta), \bar{p}(\theta)\}, \qquad P^* = \mathrm{span}\{q(s), \bar{q}(s)\},$$

$$Q = \{\widetilde{\psi} \in \mathcal{C}_\mathbb{C} : \langle\langle \widetilde{\psi}, \psi\rangle\rangle = 0, \quad \text{for all } \widetilde{\psi} \in P^*\}.$$

By (5.7), system (5.5) can be transformed into the following

$$\frac{dU_t}{dt} = T_{\tau_n^j} U_t + X_0 F(U_t, \varepsilon), \tag{5.10}$$

where

$$X_0(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ 1, & \theta = 0. \end{cases} \tag{5.11}$$

Let U_t be the solution of system (5.10) with $\varepsilon = 0$ and set

$$z(t) = \frac{1}{R_n^j}\langle\langle q(s), U_t\rangle\rangle, \tag{5.12}$$

then

$$\bar{z}(t) = \frac{1}{\bar{R}_n^j}\langle\langle \bar{q}(s), U_t\rangle\rangle. \tag{5.13}$$

Let

$$W(z,\bar{z},\theta)=W_{20}^j(\theta)\frac{z^2}{2}+W_{11}^j(\theta)z\bar{z}+W_{02}^j(\theta)\frac{\bar{z}^2}{2}+\cdots \tag{5.14}$$

be the center manifold with the range in Q, and the flow of Eq. (5.10) on center manifold can be written as

$$U_t=W(z,\bar{z},\theta)+p(\theta)z(t)+\bar{p}(\theta)\bar{z}(t). \tag{5.15}$$

From (5.12) and (5.10), we have that

$$\begin{aligned}\dot{z}(t)&=\frac{1}{R_n^j}\frac{d}{dt}\langle\langle q(s),U_t\rangle\rangle\\&=\frac{1}{R_n^j}\langle\langle q,T_{\tau_n^j}U_t\rangle+\frac{1}{R_n^j}\langle\langle q(s),X_0F(U_t,0)\rangle\rangle\\&=\frac{1}{R_n^j}\langle\langle T^*_{\tau_n^j}q,U_t\rangle+\frac{1}{R_n^j}\langle q(0),F(U_t,0)\rangle_1\\&=iw_\lambda^j\tau_n^jz(t)+\frac{1}{R_n^j}\langle q(0),F(W(z,\bar{z},\theta)+p(\theta)z(t)+\bar{p}(\theta)\bar{z}(t),0)\rangle_1\\&=iw_\lambda^j\tau_n^jz(t)+g(z,\bar{z}),\end{aligned} \tag{5.16}$$

where

$$g(z,\bar{z})=\frac{1}{R_n^j}\langle q(0),F(W(z,\bar{z},\theta)+p(\theta)z(t)+\bar{p}(\theta)\bar{z}(t),0)\rangle_1.$$

From (5.15), we get

$$U_t(0)=\psi_\lambda^jz+\overline{\psi}_\lambda^j\bar{z}+W_{20}^j(0)\frac{z^2}{2}+W_{11}^j(0)z\bar{z}+W_{02}^j(0)\frac{\bar{z}^2}{2}+\cdots,$$
$$U_t(-1)=\psi_\lambda^jze^{-iw_\lambda^j\tau_n^j}+\overline{\psi}_\lambda^j\bar{z}e^{e^{iw_\lambda^j\tau_n^j}}+W_{20}^j(-1)\frac{z^2}{2}+W_{11}^j(-1)z\bar{z}+W_{02}^j(-1)\frac{\bar{z}^2}{2}+\cdots.$$

From the above three equalities, we get

$$g(z,\bar{z})=-\frac{\lambda\tau_n^j}{R_n^j}\int_\Omega e^{2am(x)}(\psi_\lambda^j)^T(U(0)\times CU(-1))dx=g_{20}^j\frac{z^2}{2}+g_{11}^jz\bar{z}+g_{02}^j\frac{\bar{z}^2}{2}+g_{21}^j\frac{z^2\bar{z}}{2}+\cdots,$$

where

$$C=\begin{pmatrix}b_1&c_1\\b_2&c_2\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}a\\b\end{pmatrix}\times\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}ac\\bd\end{pmatrix}.$$

Thus we get

$$\begin{aligned}g_{20}^j&=-\tfrac{2\lambda\tau_n^j}{R_n^j}e^{-iw_\lambda^j\tau_n^j}\int_\Omega e^{2am(x)}(\psi_\lambda^j)^T(\psi_\lambda^j\times C\psi_\lambda^j)dx,\\g_{11}^j&=-\tfrac{\lambda\tau_n^j}{R_n^j}\left[e^{iw_\lambda^j\tau_n^j}\int_\Omega e^{2am(x)}(\psi_\lambda^j)^T(\psi_\lambda^j\times C\overline{\psi}_\lambda^j)dx+e^{-iw_\lambda^j\tau_n^j}\int_\Omega e^{2am(x)}(\psi_\lambda^j)^T(\overline{\psi}_\lambda^j\times C\psi_\lambda^j)dx\right]\\g_{02}^j&=-\tfrac{2\lambda\tau_n^j}{R_n^j}e^{iw_\lambda^j\tau_n^j}\int_\Omega e^{2am(x)}(\psi_\lambda^j)^T(\overline{\psi}_\lambda^j\times C\overline{\psi}_\lambda^j)dx,\\g_{21}^j&=-\tfrac{2\lambda\tau_n^j}{R_n^j}\int_\Omega e^{2am(x)}[e^{iw_\lambda^j\tau_n^j}(\psi_\lambda^j)^T(W_{20}^j(0)\times C\overline{\psi}_\lambda^j)+(\psi_\lambda^j)^T(\overline{\psi}_\lambda^j\times CW_{20}^j(-1))]dx\\&\quad-\tfrac{2\lambda\tau_n^j}{R_n^j}\int_\Omega e^{2am(x)}[e^{-iw_\lambda^j\tau_n^j}(\psi_\lambda^j)^T(W_{11}^j(0)\times C\psi_\lambda^j)+(\psi_\lambda^j)^T(\psi_\lambda^j\times CW_{11}^j(-1))]dx.\end{aligned}$$

Similarly, by (5.16), we have

$$\dot{\bar{z}}(t) = -iw_\lambda^j \tau_n^j \bar{z}(t) + \bar{g}(z,\bar{z}) = -iw_\lambda^j \tau_n^j \bar{z}(t) + \bar{g}_{20}^j \frac{z^2}{2} + \bar{g}_{11}^j z\bar{z} + \bar{g}_{02}^j \frac{\bar{z}^2}{2} + \bar{g}_{21}^j \frac{z^2\bar{z}}{2} + \cdots \tag{5.17}$$

From (5.15), we have

$$\begin{aligned}
\dot{W}_t(z,\bar{z},\theta) &= \frac{dU_t}{dt} - p(\theta)\dot{z}(t) - \bar{p}(\theta)\dot{\bar{z}}(t) \\
&= T_{\tau_n^j} U_t + X_0 F(U_t,0) - p(\theta)\dot{z}(t) - \bar{p}(\theta)\dot{\bar{z}}(t) \\
&= T_{\tau_n^j} W + T_{\tau_n^j}(p(\theta)z(t) + \bar{p}(\theta)\bar{z}(t)) + X_0 F(U_t,0) - p(\theta)\dot{z}(t) - \bar{p}(\theta)\dot{\bar{z}}(t) \\
&= T_{\tau_n^j} W + X_0 F(U_t,0) - p(\theta)g(z,\bar{z}) - \bar{p}(\theta)\bar{g}(z,\bar{z}) \\
&= T_{\tau_n^j} W + H(z,\bar{z},\theta),
\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}
H(z,\bar{z},\theta) &= X_0 F(U_t,0) - p(\theta)g(z,\bar{z}) - \bar{p}(\theta)\bar{g}(z,\bar{z}) \\
&= H_{20}^j(\theta)\frac{z^2}{2} + H_{11}^j(\theta)z\bar{z} + H_{02}^j(\theta)\frac{\bar{z}^2}{2} + \cdots
\end{aligned} \tag{5.19}$$

By using chain rule,

$$\dot{W}_t = \frac{\partial W(z,\bar{z},\theta)}{\partial z}\dot{z} + \frac{\partial W(z,\bar{z},\theta)}{\partial \bar{z}}\dot{\bar{z}}. \tag{5.20}$$

It is from (5.18)–(5.20) and (5.15) that

$$\begin{cases} (2iw_\lambda^j \tau_n^j - T_{\tau_n^j})W_{20}^j(\theta) = H_{20}^j(\theta), \\ -T_{\tau_n^j} W_{11}^j(\theta) = H_{11}^j(\theta). \end{cases} \tag{5.21}$$

From (5.19), we get for $\theta \in [-1,0)$,

$$\begin{cases} H_{20}^j(\theta) = -g_{20}^j p(\theta) - \bar{g}_{20}^j \bar{p}(\theta), \\ H_{11}^j(\theta) = -g_{11}^j p(\theta) - \bar{g}_{11}^j \bar{p}(\theta), \end{cases} \tag{5.22}$$

and for $\theta = 0$,

$$H_{20}^j(0) = -g_{20}^j p(0) - \bar{g}_{20}^j \bar{p}(0) - 2\lambda \tau_n^j e^{-iw_\lambda^j \tau_n^j} e^{am(x)} (\psi_\lambda^j \times C\psi_\lambda^j),$$

$$H_{11}^j(0) = -g_{11}^j p(0) - \bar{g}_{11}^j \bar{p}(0) - \lambda \tau_n^j e^{2am(x)} [e^{iw_\lambda^j \tau_n^j} (\psi_\lambda^j \times C\overline{\psi}_\lambda^j) + e^{-iw_\lambda^j \tau_n^j} (\overline{\psi}_\lambda^j \times C\psi_\lambda^j)].$$

It follows (5.21)-(5.22) and the definition of $T_{\tau_n^j}$ that

$$(W_{20}^j)'(\theta) = 2iw_\lambda^j \tau_n^j W_{20}^j(\theta) + g_{20}^j p(\theta) + \bar{g}_{20}^j \bar{p}(\theta).$$

Hence,

$$W_{20}^j(\theta) = \frac{ig_{20}^j}{w_\lambda^j \tau_n^j} p^j(0) e^{iw_\lambda^j \tau_n^j \theta} + \frac{i\bar{g}_{20}^j}{3w_\lambda^j \tau_n^j} \bar{p}(0) e^{-iw_\lambda^j \tau_n^j \theta} + C_{1\lambda}^j e^{2iw_\lambda^j \tau_n^j \theta}, \tag{5.23}$$

where $C_{1\lambda}^j \in \mathbb{R}^2$ is a constant vector. From (5.22), we have that

$$T_{\tau_n^j} W_{20}^j(0) = 2iw_\lambda^j \tau_n^j W_{20}^j(0) - H_{20}^j(0). \tag{5.24}$$

From (5.23)–(5.25) and the definition of $T_{\tau_n^j}$ in (5.7), we get that

$$(2iw_\lambda^j\tau_n^j - T_{\tau_n^j})C_1 e^{2iw_\lambda^j\tau_n^j\theta}|_{\theta=0} = -2\lambda\tau_n^j e^{-iw_\lambda^j\tau_n^j} e^{am(x)}(\psi_\lambda^j \times C\psi_\lambda^j),$$

or equivalently,

$$\triangle(\lambda, 2iw_\lambda^j, \tau_n^j)C_{1\lambda}^j = 2\lambda e^{-iw_\lambda^j\tau_n^j} e^{am(x)}(\psi_\lambda^j \times C\psi_\lambda^j). \tag{5.25}$$

Note that $2iw_\lambda^j$ is not the eigenvalue of $T_{\tau_n^j}$ for $\lambda \in \Lambda$ and hence

$$C_{1\lambda}^j = 2\lambda e^{-iw_\lambda^j\tau_n^j}\triangle(\lambda, 2iw_\lambda^j, \tau_n^j)^{-1}(e^{am(x)}(\psi_\lambda^j \times C\psi_\lambda^j)). \tag{5.26}$$

Similarly, from (5.21)-(5.22) and the definition of $T_{\tau_n^j}$, we get that

$$(W_{11}^j)'(\theta) = g_{11}p(\theta) + \bar{g}_{11}\bar{p}(\theta).$$

Hence,

$$W_{11}^j(\theta) = \frac{ig_{11}}{w_\lambda^j\tau_n^j}p(0)e^{iw_\lambda^j\tau_n^j\theta} + \frac{i\bar{g}_{11}}{3w_\lambda^j\tau_n^j}\bar{p}(0)e^{-iw_\lambda^j\tau_n^j\theta} + C_{2\lambda}^j, \tag{5.27}$$

where

$$C_{2\lambda}^j = \lambda(e^{-iw_\lambda^j\tau_n^j} + e^{iw_\lambda^j\tau_n^j})\triangle(\lambda, w_\lambda^j, \tau_n^j)^{-1}(e^{am(x)}(\psi_\lambda^j \times C\psi_\lambda^j)). \tag{5.28}$$

Lemma 5.2. *For $j = 1, 2$, $\lambda \in \Lambda$ and $n \in N_0$, $C_{1\lambda}^j$ and $C_{2\lambda}^j$ are defined in (5.26) and (5.27), then*

$$C_{1\lambda}^j = \frac{1}{\lambda - \lambda_*}(c_\lambda^j U_\lambda + \eta_\lambda^j), \qquad C_{2\lambda}^j = \frac{1}{\lambda - \lambda_*}\zeta_\lambda^j, \tag{5.29}$$

where $U_\lambda = (u_\lambda, v_\lambda)^T$,

$$\langle U_\lambda, \eta_\lambda^j\rangle = 0, \quad \lim_{\lambda\to\lambda_*}\|\eta_\lambda^j\|_{Y_{\mathbb{C}}^2} = 0, \quad \lim_{\lambda\to\lambda_*}\|\zeta_\lambda^j\|_{Y_{\mathbb{C}}^2} = 0. \tag{5.30}$$

Moreover,

$$\lim_{\lambda\to\lambda_*}(\lambda - \lambda_*)c_\lambda^j = \begin{cases} \frac{2i}{(2i-1)}\frac{(b_1c_2-b_2c_1)^2}{d_1^2}, & j = 1, \\ \frac{2i\alpha_{\lambda_*}\beta_{\lambda_*}(c_1^2+b_2^2)(b_1c_2-b_2c_1)}{d_1^2(\alpha_{\lambda_*}^2+\beta_{\lambda_*}^2)[2i\alpha_{\lambda_*}\beta_{\lambda_*}(b_1c_2-b_2c_1)-1]}, & j = 2. \end{cases} \tag{5.31}$$

Proof. Since $e^{am(x)}A_\lambda U_\lambda = 0$. Substituting (5.29) into the equation $e^{am(x)}\times$(5.25), we obtain

$$\begin{aligned} &e^{am(x)}A_\lambda\eta_\lambda^j - e^{am(x)}B_\lambda\eta_\lambda^j e^{-2iw_\lambda^j\tau_n^j} - 2iw_\lambda^j e^{am(x)}\eta_\lambda^j - c_\lambda^j B_\lambda U_\lambda e^{am(x)}e^{-2iw_\lambda^j\tau_n^j} - 2iw_\lambda^j c_\lambda^j e^{am(x)}U_\lambda \\ &\qquad = 2\lambda(\lambda - \lambda_*)e^{-iw_\lambda^j\tau_n^j}e^{2am(x)}(\psi_\lambda^j \times C\psi_\lambda^j). \end{aligned} \tag{5.32}$$

Calculating the inner product of (5.32) with U_λ, we get

$$\begin{aligned} &\langle U_\lambda, e^{am(x)}B_\lambda\eta_\lambda^j\rangle e^{-2iw_\lambda^j\tau_n^j} + c_\lambda^j\langle U_\lambda, e^{am(x)}B_\lambda U_\lambda\rangle e^{-2iw_\lambda^j\tau_n^j} + 2iw_\lambda^j c_\lambda^j\langle U_\lambda, e^{am(x)}U_\lambda\rangle \\ &\qquad = -2\lambda(\lambda - \lambda_*)e^{-iw_\lambda^j\tau_n^j}\langle U_\lambda, e^{2am(x)}(\psi_\lambda^j \times C\psi_\lambda^j)\rangle. \end{aligned} \tag{5.33}$$

Then

$$(\lambda - \lambda_*)c_\lambda^j = -\frac{2\lambda(\lambda - \lambda_*)^2 e^{-iw_\lambda^j\tau_n^j}\langle U_\lambda, e^{2am(x)}(\psi_\lambda^j \times C\psi_\lambda^j)\rangle + (\lambda - \lambda_*)\langle U_\lambda, e^{am(x)}B_\lambda\eta_\lambda\rangle e^{-2iw_\lambda^j\tau_n^j}}{\langle U_\lambda, e^{am(x)}B_\lambda U_\lambda\rangle e^{-2iw_\lambda^j\tau_n^j} + 2iw_\lambda^j\langle U_\lambda, e^{am(x)}U_\lambda\rangle}.$$

Since

$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)^{-3}\langle U_\lambda, e^{am(x)}B_\lambda U_\lambda\rangle = \lambda_* d_1(\alpha^2_{\lambda_*}+\beta^2_{\lambda_*})\int_\Omega e^{2am(x)}\varphi^3 dx,$$

$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)^{-3}w^j_\lambda\langle U_\lambda, e^{am(x)}U_\lambda\rangle = \begin{cases}\lambda_* d_1(\alpha^2_{\lambda_*}+\beta^2_{\lambda_*})\int_\Omega e^{2am(x)}\varphi^3 dx, & j=1,\\ \lambda_* d_1(\alpha^2_{\lambda_*}+\beta^2_{\lambda_*})\frac{(c_2-c_1)(b_1-b_2)}{b_1c_2-b_2c_1}\int_\Omega e^{2am(x)}\varphi^3 dx, & j=2,\end{cases}$$

$$\begin{aligned}&\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)^{-1}\lambda\langle U_\lambda, e^{2am(x)}(\psi^j_\lambda\times C\psi^j_\lambda)\rangle\\ &\quad=\begin{cases}\frac{1}{d_1}\lambda_*(\alpha^2_{\lambda_*}+\beta^2_{\lambda_*})(b_1c_2-b_2c_1)^2\int_\Omega e^{2am(x)}\varphi^3 dx, & j=1,\\ \frac{1}{d_1}\lambda_*\alpha_{\lambda_*}\beta_{\lambda_*}(c_1^2+b_2^2)(b_1c_2-b_2c_1)\int_\Omega e^{2am(x)}\varphi^3 dx, & j=2.\end{cases}\end{aligned}$$

Hence, there exist $\delta_1<\delta, M_0, M_1>0$ such that for any $\lambda\in(\lambda-\delta_1,\lambda+\delta_1)$,

$$(\lambda-\lambda_*)c^j_\lambda \le M_0\|\eta^j_\lambda\|_{Y^2_{\mathbb{C}}}+M_1. \tag{5.34}$$

Calculating the inner product of (5.32) with η_λ, we obtain

$$\begin{aligned}&\langle\eta^j_\lambda, e^{am(x)}A_\lambda\eta^j_\lambda\rangle-\langle\eta^j_\lambda, e^{am(x)}B_\lambda\eta^j_\lambda\rangle e^{-2iw^j_\lambda\tau^j_n}\\ &\qquad-2iw^j_\lambda\langle\eta^j_\lambda, e^{am(x)}\eta^j_\lambda\rangle - c^j_\lambda\langle\eta^j_\lambda, e^{am(x)}B_\lambda U_\lambda\rangle e^{-2iw^j_\lambda\tau^j_n}\\ &\quad=2\lambda(\lambda-\lambda_*)e^{-iw^j_\lambda\tau^j_n}\langle\eta^j_\lambda, e^{2am(x)}(\psi^j_\lambda\times C\psi^j_\lambda)\rangle.\end{aligned}\tag{5.35}$$

From (5.35), it follows that there exist constants $\delta_2<\delta_1, M_2, M_3>0$ such that for any $\lambda\in(\lambda-\delta_2,\lambda+\delta_2)$,

$$\lambda_2(\lambda)\|\eta^j_\lambda\|^2_{Y^2_{\mathbb{C}}}\le(\lambda-\lambda_*)M_2\|\eta^j_\lambda\|^2_{Y_{\mathbb{C}}}+(\lambda-\lambda_*)M_3\|\eta^j_\lambda\|_{Y_{\mathbb{C}}}. \tag{5.36}$$

Similar to the proof of Lemma 2.3 of [3], we have $|\langle e^{am(x)}A_\lambda U_\lambda, U_\lambda\rangle|\ge|\lambda_2(\lambda)|\|U_\lambda\|^2_{Y^2_{\mathbb{C}}}$ and $\lambda_2(\lambda)$ is the second eigenvalue of $e^{am(x)}A_\lambda$. Then we have $\lim_{\lambda\to\lambda_*}\|\zeta^j_\lambda\|_{Y^2_{\mathbb{C}}}=0$. From all, we can obtain (5.31). This completes the proof of Lemma 5.2. □

Remark 5.3.

(1) When $v=0, b_1=1$ in (1.4), (5.31) in Lemma 5.2 is as same as that in Lemma 3.2 in [8].

(2) When $a_1=0$ in (1.4), (5.31) in Lemma 5.2 is as same as that [18, p. 106].

Therefore, one can easily check

$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)g^j_{11}=0,\qquad \lim_{\lambda\to\lambda_*}\mathrm{Re}[(\lambda-\lambda_*)^2g^j_{21}]<0.$$

It is well-known that the real part of the following quantity determines the direction and stability of bifurcating periodic orbits (see [14,19,33]):

$$c^j_1(0)=\frac{i}{2w^j_\lambda\tau^j_n}\left(g^j_{20}g^j_{11}-2|g^j_{11}|^2-\frac{1}{3}|g^j_{02}|^2\right)+\frac{1}{2}g^j_{21}.$$

It follows from (5.31) that $\lim_{\lambda\to\lambda_*}\mathrm{Re}[(\lambda-\lambda_*)^2c^j_1(0)]<0$. Hence we have the following result.

Theorem 5.4. *Assume (H1)–(H2) hold. Then for $j=1,2$, $\lambda\in\Lambda$ and $n\in\mathbb{N}_0$, system (1.5) has a branch of bifurcating periodic solutions emerging from the steady state solution $(u_\lambda, v_\lambda)^T$ for τ near τ^j_n. More precisely, the direction of the Hopf bifurcation at τ^j_n is forward and the bifurcating periodic solution from τ^j_n have the same stability as the steady state solution $(u_\lambda, v_\lambda)^T$.*

6 Simulations

In this section, some numerical simulations for model (1.4) are given to illustrate the results of Theorem 5.4.

In (1.4), choose

$$\Omega = (0, \pi), \quad m(x) = \sin x,$$

$$d = 0.20, \quad b_1 = 0.04, \quad b_2 = 0.02, \quad c_1 = 0.03, \quad c_2 = 0.04.$$

and the initial value conditions:

$$u(x,t) = v(x,t) = \sin x, \quad \text{for } t \in [-r, 0].$$

Example 6.1. Model (1.4) without advection, that is $a_1 = 0$.

(1) When $r = 1$, solutions of model (1.4) without advection tend to a positive steady state. See Fig. 6.1.

(2) When $r = 5$, solutions of model (1.4) without advection tend to periodically oscillatory orbit, that is, model (1.4) undergoes a Hopf bifurcation. See Fig. 6.2.

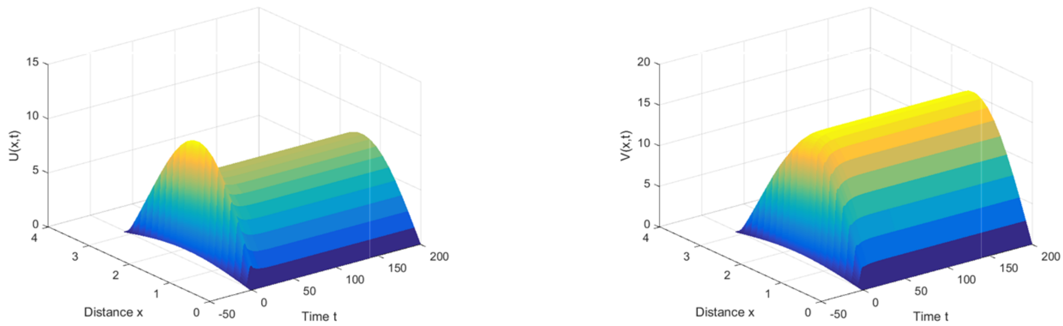

Figure 6.1: Solutions of model (1.4) without advection ($a_1 = 0$) tend to a positive steady state when $r = 1$.

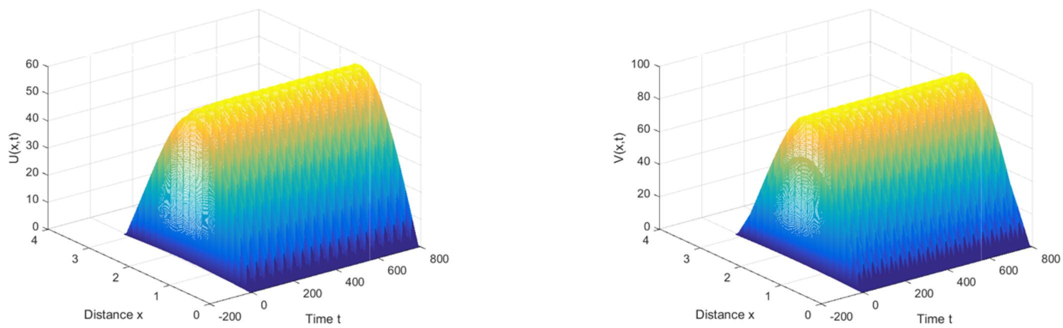

Figure 6.2: Model (1.4) without advection ($a_1 = 0$) undergoes a Hopf bifurcation when $r = 4$.

Example 6.2. Model (1.4) with advection, that is $a_1 = 0.5$.

When $r = 1$, solutions of model (1.4) with advection tend to a positive steady state. See Fig. 6.3.

When $r = 5$, solutions of model (1.4) with advection tend to periodically oscillatory orbit, that is, model (1.4) undergoes a Hopf bifurcation. See Fig. 6.4.

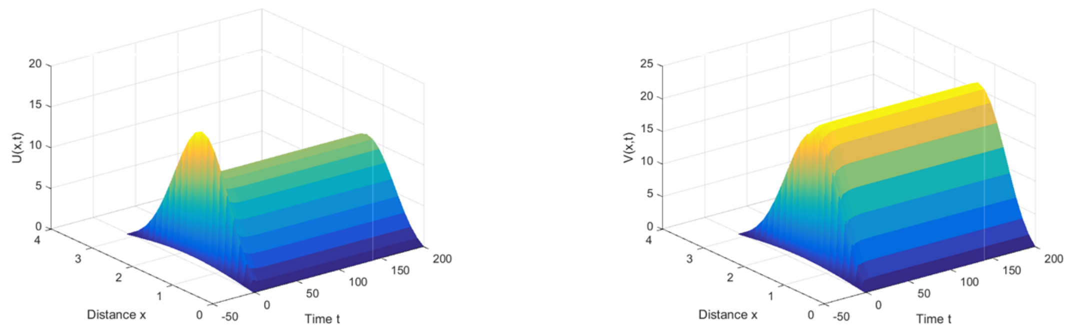

Figure 6.3: Solutions of model (1.4) with advection $(a_1 = 0.5)$ tend to a positive steady state when $r = 1$.

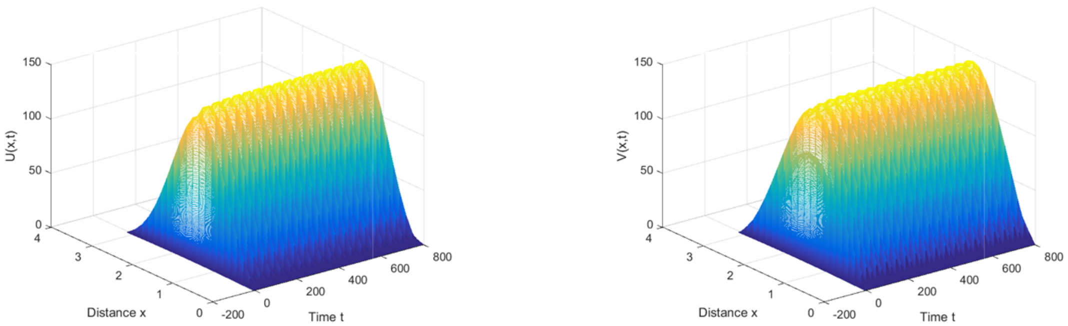

Figure 6.4: Model (1.4) with advection $(a_1 = 0.5)$ undergoes a Hopf bifurcation when $r = 4$.

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