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Reduction of order in the oscillation theory of half-linear differential equations

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Abstract. Oscillation of solutions of even order half-linear differential equations of the form

$$D(\alpha_n,\dots,\alpha_1)x + q(t)|x|^\beta \operatorname{sgn} x = 0, \qquad t \ge a > 0, \tag{1.1}$$

where α_i, $1 \le i \le n$, and β are positive constants, q is a continuous function from $[a,\infty)$ to $(0,\infty)$ and the differential operator $D(\alpha_n,\dots,\alpha_1)$ is defined by

$$D(\alpha_1)x = \frac{d}{dt}\left(|x|^{\alpha_1} \operatorname{sgn} x\right)$$

and

$$D(\alpha_i,\dots,\alpha_1)x = \frac{d}{dt}\left(|D(\alpha_{i-1},\dots,\alpha_1)x|^{\alpha_i} \operatorname{sgn} D(\alpha_{i-1},\dots,\alpha_1)x\right), \qquad i = 2,\dots,n,$$

is proved in the case where $\alpha_1 \cdots \alpha_n = \beta$ through reduction to the problem of oscillation of solutions of some lower order differential equations associated with (1.1).

Keywords: half-linear differential equation, oscillation test.

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1 Introduction

Consider differential equations of the form

$$D(\alpha_n,\dots,\alpha_1)x + q(t)|x|^\beta \operatorname{sgn} x = 0, \qquad t \ge a > 0, \tag{1.1}$$

where $n \ge 2$ is an even integer, $\alpha_1, \alpha_2, \dots, \alpha_n$ and β are positive constants, $q : [a,\infty) \to (0,\infty)$, $a > 0$, is a continuous function and the differential operator $D(\alpha_n,\dots,\alpha_1)x$ is defined recursively by

$$D(\alpha_1)x = \frac{d}{dt}\left(|x|^{\alpha_1} \operatorname{sgn} x\right)$$

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and

$$D(\alpha_i,\dots,\alpha_1)x = \frac{d}{dt}\left(|D(\alpha_{i-1},\dots,\alpha_1)x|^{\alpha_i}\operatorname{sgn} D(\alpha_{i-1},\dots,\alpha_1)x\right), \qquad i=2,\dots,n.$$

It is convenient to denote by $C(\alpha_j,\dots,\alpha_1)[t_0,\infty), 1 \le j \le n$, the set of continuous functions $x : [t_0,\infty) \to \mathbb{R}$ such that $D(\alpha_i,\dots,\alpha_1)x,\ i = 1,\dots,j$, exist and are continuous on $[t_0,\infty)$.

A function $x(t)$ from $C(\alpha_n,\dots,\alpha_1)[t_0,\infty)$ is called a solution of equation (1.1) on $[t_0,\infty)$ if it satisfies (1.1) at each $t \in [t_0,\infty)$. We restrict our consideration to the so called proper solutions of (1.1), i.e., solutions which are not trivial in any neighborhood of infinity. Such a solution is called *oscillatory* if it has an unbounded set of zeros, and it is called *nonoscillatory* otherwise.

It is known that for any nonoscillatory solution $x(t)$ of (1.1) there exist a $t_0 \ge a$ and an odd integer l, $1 \le l \le n-1$, such that for $t \ge t_0$

$$x(t)D(\alpha_j,\dots,\alpha_1)x(t) > 0 \quad \text{for } j = 1,\dots,l, \tag{1.2}$$

and

$$(-1)^{n+j}x(t)D(\alpha_j,\dots,\alpha_1)x(t) < 0 \quad \text{for } j = l+1,\dots,n, \tag{1.3}$$

(see Naito [19]). Functions belonging to $C(\alpha_n,\dots,\alpha_1)[t_0,\infty)$ and satisfying (1.2) and (1.3) for $t \ge t_0$, will be called *nonoscillatory functions of Kiguradze's degree l*. We denote by $\mathcal{N}_l$ the set of all nonoscillatory solutions of equation (1.1) which are of degree l. The elements of $\mathcal{N}_1$ (resp. $\mathcal{N}_{n-1}$) will be called nonoscillatory solutions of the *minimal* (resp. *maximal*) Kiguradze's degree.

Existence and asymptotic behavior of positive solutions of nonlinear differential equations of the form (1.1) in the case where the exponents satisfied either $\beta < \alpha_1\cdots\alpha_n$ or $\beta > \alpha_1\cdots\alpha_n$ were studied by Naito in [18,19] (for some particular cases see also [7,8,11–13,16,17,20–22,24,25]), but the important special case in which $\beta = \alpha_1\cdots\alpha_n$ seems to remain untouched until now. As far as we know, the paper by Došlý et al. [4] devoted to the study of nonoscillation of solutions of higher order half-linear differential equations of the form

$$\sum_{k=0}^{n}(-1)^k\left(r_k(t)|x^{(k)}|^{\alpha}\operatorname{sgn} x^{(k)}\right)^{(k)} = 0,$$

where r_k, $0 \le k \le n$, are continuous functions with $r_n(t) > 0$ in the interval under consideration, is the only work on the subject.

Recently, the present author in [6] gave an oscillation criterion which (when specialized to equation (1.1)) says that all solutions of (1.1) are oscillatory if there exists an $\varepsilon \in (0,1]$ such that

$$\int_a^\infty t^{\alpha_2\cdots\alpha_n+\alpha_3\cdots\alpha_n+\cdots+(1-\varepsilon)\alpha_n}q(t)dt = \infty. \tag{1.4}$$

The result is sharp in the sense that if $\varepsilon = 0$ in (1.1)), then equation (1.1) may have nonoscillatory solutions. On the other hand, the above criterion does not apply to such an important special case of (1.1) as the nonlinear Euler-type differential equation

$$D(\alpha_n,\dots,\alpha_1)x + \frac{\gamma}{t^{\alpha_2\cdots\alpha_n+\alpha_3\cdots\alpha_n+\cdots+\alpha_n+1}}|x|^{\alpha_1\cdots\alpha_n}\operatorname{sgn} x = 0, \qquad t \ge a > 0, \tag{1.5}$$

where $\gamma > 0$ is a constant.

Thus, our main purpose here is to obtain criteria which would be more sensitive to oscillatory behaviour of solutions of equations of the form (1.1) and would apply also to higher order half-linear equations of the Euler type. Our approach is based on reduction of the problem of oscillation of equation (1.1) to the problem of oscillation of solutions of some lower order equations and inequalities. In the linear case this approach was used successfully by various authors in [1,2,5,9,10,14,15,23].

2 Preliminaries

We begin with some preparatory results which will be needed in the sequel.

Lemma 2.1. *Let $\alpha > 0$ and $y \in C(\alpha)[t_0, \infty)$ be such that either*

$$y(t)D(\alpha)y(t) > 0 \quad \text{for } t \geq t_0, \tag{2.1}$$

or

$$y(t)D(\alpha)y(t) < 0 \quad \text{for } t \geq t_0 \tag{2.2}$$

and

$$\int_{t_0}^{\infty} |D(\alpha)y(t)|dt < \infty. \tag{2.3}$$

Then $y \in C^1[t_0, \infty)$, i.e., the usual derivative $y'(t)$ exists and is continuous on $[t_0, \infty)$.

Proof. We will assume that $y(t) > 0$ on $[t_0, \infty)$. (The proof in the case $y(t) < 0$ for $t \geq t_0$ is similar and is omitted.)

If y satisfies (2.1), then we can integrate $D(\alpha)y(t)$ from t_0 to t and raise the result to the power $1/\alpha$ to get

$$y(t) = \left[y(t_0)^{\alpha} + \int_{t_0}^{t} D(\alpha)y(s)ds \right]^{\frac{1}{\alpha}}, \qquad t \geq t_0. \tag{2.4}$$

Similarly, if y satisfies (2.2) and (2.3), then $D(\alpha)y(t) < 0$ for $t \geq t_0$ implies that $y(\infty)^{\alpha} = \lim_{t\to\infty} y(t)^{\alpha}$ exists as a nonnegative finite number and after integration of $D(\alpha)y(t)$ from $t(\geq t_0)$ to ∞ we arrive at

$$y(t) = \left[y(\infty)^{\alpha} - \int_{t}^{\infty} D(\alpha)y(s)ds \right]^{\frac{1}{\alpha}}, \qquad t \geq t_0. \tag{2.5}$$

From (2.4) (resp. (2.5)) it is clear that in both cases the function $y(t)$ is continuously differentiable on $[t_0, \infty)$. □

Remark 2.2. Repeated application of Lemma 2.1 shows that if y is a nonoscillatory solution of equation (1.1) on an interval $[t_0, \infty)$, then y and $D(\alpha_i, \dots, \alpha_1)y$, $i = 1, \dots, n-1$, are continuously differentiable functions, that is,

$$\frac{d}{dt}y(t) \quad \text{and} \quad \frac{d}{dt}\left[D(\alpha_i, \dots, \alpha_1)y(t)\right], \qquad i = 1, \dots, n-1,$$

exist and are continuous on $[t_0, \infty)$.

To formulate and prove our next lemma, we define the numbers $r_i(k), 1 \leq i \leq n-1$ and $k = 0, 1, \dots, i$, by

$$r_i(0) = 1 \quad \text{and} \quad r_i(k) = \frac{1}{\alpha_{i-k+1}} r_i(k-1) + 1 \quad \text{for } k = 1, \dots, i. \tag{2.6}$$

We also set

$$r_i := r_i(i) = 1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_1\alpha_2} + \cdots + \frac{1}{\alpha_1\alpha_2\cdots\alpha_i}.$$

Lemma 2.3. *If $y \in C(\alpha_l, \dots, \alpha_1)[t_0, \infty)$ satisfies $D(\alpha_i, \dots, \alpha_1)y(t) > 0$, $i = 0, \dots, l$ and $D(\alpha_{l+1}, \dots, \alpha_1)y(t) < 0$ for $t \geq t_0$, then*

$$(t-t_0)D(\alpha_{l-k}, \dots, \alpha_1)y(t) \leq r_l(k)\big[D(\alpha_{l-k-1}, \dots, \alpha_1)y(t)\big]^{\alpha_{l-k}}, \qquad k = 0, 1, \dots, l-1, \tag{2.7$_k$}$$

for $t \geq t_0$.

Proof. Since $D(\alpha_l, \dots, \alpha_1)y(t)$ is decreasing for $t \geq t_0$, integrating on $[t_0, t]$ we obtain

$$\begin{aligned}(t-t_0)D(\alpha_l, \dots, \alpha_1)y(t) &\leq \int_{t_0}^t D(\alpha_l, \dots, \alpha_1)y(s)ds = \int_{t_0}^t \left([D(\alpha_{l-1}, \dots, \alpha_1)y(s)]^{\alpha_l}\right)' ds \\ &= \big[D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]^{\alpha_l} - \big[D(\alpha_{l-1}, \dots, \alpha_1)y(t_0)\big]^{\alpha_l} \\ &\leq \big[D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]^{\alpha_l},\end{aligned} \tag{2.8}$$

which gives inequality (2.7$_k$) for $k = 0$. Next, since by the remark after Lemma 2.1, $D(\alpha_{l-1}, \dots, \alpha_1)y(t)$ is continuously differentiable function, we can express (2.8) explicitly as

$$\alpha_l(t-t_0)\big[D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]^{\alpha_l - 1}\big(D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big)' \leq \big[D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]^{\alpha_l},$$

or, equivalently,

$$\big[(t-t_0)D(\alpha_{l-1}, \dots, \alpha_1)y(t)\big]' \leq \frac{1+\alpha_l}{\alpha_l} D(\alpha_{l-1}, \dots, \alpha_1)y(t), \tag{2.9}$$

for $t \geq t_0$. Integrating (2.9) from t_0 to t we obtain

$$(t-t_0)D(\alpha_{l-1}, \dots, \alpha_1)y(t) \leq \frac{1+\alpha_l}{\alpha_l}\big[D(\alpha_{l-2}, \dots, \alpha_1)y(t)\big]^{\alpha_{l-1}}, \qquad t \geq t_0, \tag{2.10}$$

which is (2.7$_k$) for $k = 1$.

Repeated application of the above procedure yields (2.7$_k$) also for $k = 2, \dots, l-1$ where $D(\alpha_j, \dots, \alpha_1)y(t)$ for $j = 0$ should be interpreted as $y(t)$. □

The following comparison lemma will play an important role in our later discussions. For the proof see Naito [19].

Lemma 2.4. *Let $l \in \{1, 3, \dots, n-1\}$ be a fixed odd number and let the differential inequality*

$$D(\alpha_n, \dots, \alpha_1)y + q(t)|y|^{\alpha_1 \cdots \alpha_n} \operatorname{sgn} y \leq 0, \qquad t \geq a > 0, \tag{2.11}$$

where $q : [a, \infty) \to (0, \infty)$ is a continuous function, have a positive solution $y(t)$ of degree l for $t \geq t_0$. Then there exists a positive solution $x(t)$ of equation (1.1) which has the same degree l.

3 Reduction to the existence of solutions of minimal degree

Define numbers R_i, $1 \leq i \leq n-1$, by

$$R_1 = 1 \quad \text{and} \quad R_i = \left(\frac{1}{r_i(i-1)}\right)^{\frac{1}{\alpha_1}} \left(\frac{1}{r_i(i-2)}\right)^{\frac{1}{\alpha_1 \alpha_2}} \cdots \left(\frac{1}{r_i(1)}\right)^{\frac{1}{\alpha_1 \cdots \alpha_{i-1}}}, \qquad i = 2, \dots, n-1,$$

where $r_i(k)$, $k = 0, 1, \dots, i$, are given by (2.6).

Theorem 3.1. *Eq.* (1.1) *has a nonoscillatory solution of the Kiguradze's degree* $l, 1 \le l \le n-1$, *if and only if the differential equation*

$$D(\alpha_n, \dots, \alpha_l)z + R_l^{\beta}(t-t_0)^{(r_{l-1}-1)\beta} q(t)|z|^{\alpha_l \cdots \alpha_n} \operatorname{sgn} z = 0, \qquad t \ge t_0, \tag{3.1_l}$$

has a nonoscillatory solution of the Kiguradze's degree 1.

Proof. (Necessity.) Suppose that (1.1) has a nonoscillatory solution $x(t)$ whose Kiguradze's degree is l, $1 \le l \le n-1$. We may assume that $x(t)$ is positive and satisfies (1.2) and (1.3) on $[t_0, \infty)$. If we chain the inequalities (2.7_k), $k = 1, \dots, l-1$, together, we obtain

$$x(t) \ge R_l (t-t_0)^{r_{l-1}-1} \big[D(\alpha_{l-1}, \dots, \alpha_1)x(t)\big]^{\frac{1}{\alpha_1 \cdots \alpha_{l-1}}}, \qquad t \ge t_0. \tag{3.2}$$

Substituting this inequality into (1.1), we obtain that $x(t)$ satisfies the inequality

$$D(\alpha_n, \dots, \alpha_1)x(t) + R_l^{\alpha_1 \cdots \alpha_n}(t-t_0)^{(r_{l-1}-1)\alpha_1 \cdots \alpha_n} q(t) \big[D(\alpha_{l-1}, \dots, \alpha_1)x(t)\big]^{\alpha_l \cdots \alpha_n} \le 0.$$

Put $y(t) = D(\alpha_{l-1}, \dots, \alpha_1)x(t)$. Then the function $y(t)$ satisfies

$$D(\alpha_n, \dots, \alpha_l)y(t) + R_l^{\alpha_1 \cdots \alpha_n}(t-t_0)^{(r_{l-1}-1)\alpha_1 \cdots \alpha_n} q(t)|y(t)|^{\alpha_l \cdots \alpha_n} \operatorname{sgn} y(t) \le 0, \qquad t \ge t_0, \tag{3.3}$$

and its Kiguradze's degree is 1. By Lemma 2.4, the corresponding differential equation (3.1_l) has a positive solution $z(t)$ of the same degree 1.

(Sufficiency.) Let (3.1_l) have a nonoscillatory solution $z(t)$ of degree 1. We may assume that $z(t) > 0$ for $t \ge t_0$. Then the function

$$w(t) = (R_l/R_{l-1}) \left(\int_{t_0}^{t} \left(\int_{t_0}^{s_1} \dots \left(\int_{t_0}^{s_{l-2}} z(s_{l-1}) ds_{l-1} \right)^{\frac{1}{\alpha_{l-1}}} \dots \, ds_2 \right)^{\frac{1}{\alpha_2}} ds_1 \right)^{\frac{1}{\alpha_1}} \tag{3.4}$$

satisfies

$$D(\alpha_{l-1}, \dots, \alpha_1)w(t) = (R_l/R_{l-1})^{\alpha_1 \cdots \alpha_{l-1}} z(t)$$

and since $z(t)$ has degree 1, the function $w(t)$ satisfies

$$D(\alpha_k, \dots, \alpha_1)w(t) > 0 \quad \text{for } k = 1, \dots, l,$$

and

$$(-1)^{n+k} D(\alpha_k, \dots, \alpha_1)w(t) < 0 \quad \text{for } k = l+1, \dots, n.$$

Hence, $w(t)$ is a function having degree l for $t \ge t_0$. Since $z(t)$ is increasing, from (3.4) we obtain

$$\begin{aligned} w(t) &\le (R_l/R_{l-1}) z(t)^{1/(\alpha_1 \cdots \alpha_{l-1})} \left(\int_{t_0}^{t} \left(\int_{t_0}^{s_1} \dots \left(\int_{t_0}^{s_{l-2}} ds_{l-1} \right)^{\frac{1}{\alpha_{l-1}}} \dots ds_2 \right)^{\frac{1}{\alpha_2}} ds_1 \right)^{\frac{1}{\alpha_1}} \\ &= R_l (t-t_0)^{r_{l-1}-1} z(t)^{1/(\alpha_1 \cdots \alpha_{l-1})}. \end{aligned}$$

Now, as a consequence of the relation

$$r_l(k) = r_{l-1}(k-1) + \frac{1}{\alpha_{l-k+1} \cdots \alpha_l}, \qquad k = 1, \dots, l,$$

we get $r_l(k) \geq r_{l-1}(k-1), k = 1, \dots, l$, which implies

$$\big(R_l / R_{l-1}\big)^{\alpha_1 \cdots \alpha_{l-1}} \leq 1.$$

Thus,

$$D(\alpha_n, \dots, \alpha_1) w(t) = \big(R_l / R_{l-1}\big)^{\alpha_1 \cdots \alpha_{l-1}} D(\alpha_n, \dots, \alpha_l) z(t) \leq D(\alpha_n, \dots, \alpha_l) z(t)$$

and so for $t \geq t_0$,

$$D(\alpha_n, \dots, \alpha_1) w(t) + q(t) w(t)^{\alpha_1 \cdots \alpha_n} \leq D(\alpha_n, \dots, \alpha_l) z(t) + R_l^{\alpha_1 \cdots \alpha_n} (t - t_0)^{(r_{l-1} - 1)\alpha_1 \cdots \alpha_n} q(t) z(t)^{\alpha_l \cdots \alpha_n}$$

showing that $w(t)$ is a solution of (2.11) for $t \geq t_0$ since $z(t)$ is a solution of (3.1_l). Finally, by Lemma 2.4, there exists a positive solution $x(t)$ of (1.1) of degree l. This completes the proof of the theorem. □

Remark 3.2. If $l = n - 1$, then (3.1_l) reduces to the second-order equation

$$D(\alpha_n, \alpha_{n-1}) z + R_{n-1}^{\beta} (t - t_0)^{(r_{n-2} - 1)\beta} q(t) |z|^{\alpha_{n-1}\alpha_n} \operatorname{sgn} z = 0. \tag{3.1_{n-1}}$$

From Theorem 3.1 it follows that if (3.1_{n-1}) is nonoscillatory, then equation (1.1) is nonoscillatory, too. (More precisely, it has a nonoscillatory solution of the maximal degree $l = n - 1$.)

However, if $l < n - 1$, then equations (3.1_l) are of orders greater than 2 and it may not be an easy matter to determine whether or not (3.1_l) has a nonoscillatory solutions of degree 1.

Thus, we proceed further and associate with (1.1) a set of half-linear differential equations all of which are of the second order.

For this purpose we assume that the integrals

$$I_1(q) = \int_a^\infty q(t) dt,$$

$$I_2(q) = \int_a^\infty \left(\int_t^\infty q(s) ds \right)^{\frac{1}{\alpha_n}} dt,$$

$$\vdots$$

$$I_{n-l-1}(q) = \int_a^\infty \left(\int_{s_{l+3}}^\infty \cdots \left(\int_{s_{n-1}}^\infty q(s) ds \right)^{\frac{1}{\alpha_n}} \cdots ds_{l+4} \right)^{\frac{1}{\alpha_{l+3}}} ds_{l+3}, \qquad 1 \leq l \leq n-2,$$

converge and define continuous functions $\rho_0(t), \dots, \rho_{n-l-1}(t)$ by

$$\rho_0(t) = q(t), \qquad \rho_k(t) = \left[\int_t^\infty \rho_{k-1}(s) ds \right]^{\frac{1}{\alpha_{n-k+1}}}, \qquad k = 1, \dots, n-l-1. \tag{3.5}$$

The following theorem is the main result of this paper.

Theorem 3.3. *Suppose that* (1.1) *has a nonoscillatory solution $x(t)$ which is of degree l, $1 \leq l \leq n-1$, for $t \geq t_0$. Then, the second order half-linear differential equation*

$$D(\alpha_{l+1}, \alpha_l) z + R_l^{\alpha_1 \cdots \alpha_{l+1}} (t - t_0)^{(r_{l-1} - 1)\alpha_1 \cdots \alpha_{l+1}} \rho_{n-l-1}(t) |z|^{\alpha_l \alpha_{l+1}} \operatorname{sgn} z = 0, \qquad t \geq t_0, \tag{3.6_l}$$

has a nonoscillatory solution of degree 1.

Proof. Suppose that equation (1.1) has an eventually positive solution $x(t)$ which is of degree l, $1 \le l \le n-1$, for $t \ge t_0$. (If $x(t)$ is a solution which is eventually negative, the proof is similar and is omitted.)

By Theorem 3.1, there exists a positive solution $z(t)$ of the lower order differential equation (3.1_l) which is of degree 1, i.e., it satisfies for $t \ge t_0$

$$D(\alpha_l)z(t) > 0 \quad \text{and} \quad (-1)^{n+j}D(\alpha_j,\dots,\alpha_l)z(t) < 0 \quad \text{for } j = l+1,\dots,n. \tag{3.7}$$

Integrating (3.1_l) from t to ∞ and using (3.7), we get

$$D(\alpha_{n-1},\dots,\alpha_l)z(t) \ge R_l^{\alpha_1\cdots\alpha_{n-1}}\left(\int_t^\infty (s-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_n} q(s)z(s)^{\alpha_l\cdots\alpha_n}ds\right)^{1/\alpha_n}, \qquad t \ge t_0.$$

Continuing in this fashion and using the fact that $z(t)$ and $(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_n}$ are increasing functions for $t \ge t_0$, we obtain

$$\begin{aligned}&-\left[D((\alpha_{l+1},\alpha_l)z(t)\right]^{\alpha_{l+2}}\\ &\quad\ge R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}z(t)^{\alpha_l\alpha_{l+1}\alpha_{l+2}}\left(\int_t^\infty\left(\dots\left(\int_{s_{n-1}}^\infty q(s)ds\right)^{\frac{1}{\alpha_n}}\dots\right)^{\frac{1}{\alpha_{l+3}}}ds_{l+2}\right),\end{aligned}$$

or, equivalently,

$$D(\alpha_{l+1},\alpha_l)z(t) + R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}\rho_{n-l-1}(t)z(t)^{\alpha_l\alpha_{l+1}} \le 0, \qquad t \ge t_0, \tag{3.8}$$

where $\rho_{n-l-1}(t)$ is defined by (3.5). Thus, by Lemma 2.4, the differential equation (3.6_l) has a positive solution of degree 1 as claimed. The proof of the theorem is complete. □

As an immediate consequence of Theorem 3.3 we get the following oscillation result.

Corollary 3.4. *If all of the second order half-linear differential equations* (3.6_l), $l = 1,3,\dots,n-1$, *are oscillatory, then all solutions of the n-th order differential equation* (1.1) *are oscillatory.*

Example 3.5. Consider the Euler-type nonlinear differential equation

$$D(\alpha_n,\dots,\alpha_1)x + \gamma t^{-(\alpha_2\cdots\alpha_n+\alpha_3\cdots\alpha_n+\cdots+\alpha_n+1)}|x|^{\alpha_1\cdots\alpha_n}\operatorname{sgn} x = 0, \qquad t \ge 1, \tag{3.9}$$

where n is an even integer and $\alpha_1,\dots,\alpha_n$ and γ are positive constants.

To simplify notation and formulation of our results for equation (3.9), we define the numbers q_i and Q_i, $i = 1,\dots,n$, by

$$q_1 = 0, \qquad q_i = \alpha_i(q_{i-1}+1) \quad \text{for } i = 2,\dots,n, \tag{3.10}$$

and

$$Q_1 = 1, \qquad Q_i = \left(\frac{1}{q_i}\right)^{\frac{1}{\alpha_i}}\left(\frac{1}{q_{i+1}}\right)^{\frac{1}{\alpha_i\alpha_{i+1}}}\dots\left(\frac{1}{q_{n-1}}\right)^{\frac{1}{\alpha_i\cdots\alpha_{n-1}}}\left(\frac{1}{q_n}\right)^{\frac{1}{\alpha_i\cdots\alpha_n}}, \qquad i = 2,\dots,n. \tag{3.11}$$

It is a matter of easy computation to verify that if $q(t) = \gamma t^{-q_n-1}$, $\gamma > 0$, then the functions ρ_{n-l-1} defined by (3.5) become

$$\rho_{n-l-1}(t) = \gamma^{1/(\alpha_{l+2}\cdots\alpha_n)}Q_{l+2}t^{-q_{l+1}+1}, \qquad l = 1,\dots,n-3, \tag{3.12}$$

and the second order half-linear differential equations (3.6_l) associated with (3.9) reduce respectively to

$$\left(|z'|^{\alpha_{l+1}}\operatorname{sgn} z'\right)' + \gamma^{1/(\alpha_1\cdots\alpha_n)} R_l^{\alpha_1\cdots\alpha_{l+1}} Q_{l+2} t^{-q_{l+1}-1} |z|^{\alpha_{l+1}} \operatorname{sgn} z = 0, \qquad t \geq 1, \tag{3.13_l}$$

if $1 \leq l \leq n-3$, and

$$\left(|z'|^{\alpha_n}\operatorname{sgn} z'\right)' + \gamma R_{n-1}^{\alpha_1\cdots\alpha_n} t^{-q_n-1} |z|^{\alpha_n} \operatorname{sgn} z = 0, \qquad t \geq 1, \tag{3.14}$$

if $l = n-1$.

If we apply the well-known result which says that all solutions of the generalized second order Euler differential equation

$$\left(|z'|^{\alpha}\operatorname{sgn} z\right)' + \lambda t^{-\alpha-1} |z|^{\alpha} \operatorname{sgn} z = 0, \qquad t \geq 1, \tag{3.15}$$

are oscillatory if and only if

$$\lambda > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{3.16}$$

(see, for example, [3]), then we get that for oscillation of all solutions of equation (3.7) it is sufficient that

$$\gamma^{1/(\alpha_{l+2}\cdots\alpha_n)} R_l^{\alpha_1\cdots\alpha_{l+1}} Q_{l+2} > \left(\frac{\alpha_{l+1}}{\alpha_{l+1}+1}\right)^{\alpha_{l+1}+1}, \qquad l = 1, 3, \ldots, n-3, \tag{3.17_l}$$

and

$$\gamma R_{n-1}^{\alpha_1\cdots\alpha_n} > \left(\frac{\alpha_n}{\alpha_n+1}\right)^{\alpha_n+1}. \tag{3.18}$$

Example 3.6. Consider the fourth order half-linear differential equation

$$D(\alpha_4, \alpha_3, \alpha_2, \alpha_1)x + q(t)|x|^{\alpha_1\alpha_2\alpha_3\alpha_4} \operatorname{sgn} x = 0, \qquad t \geq a > 0, \tag{3.19}$$

where $\alpha_i, 1 \leq i \leq 4$, are positive constants and $q : [a, \infty) \to (0, \infty)$ is continuous function. Second order equations associated with (3.19) are

$$\left(|z'|^{\alpha_2}\operatorname{sgn} z'\right)' + \left(\int_t^\infty \left(\int_s^\infty q(\tau)d\tau\right)^{1/\alpha_4} ds\right)^{1/\alpha_3} |z|^{\alpha_2} \operatorname{sgn} z = 0, \qquad t \geq t_0, \tag{3.20}$$

and

$$\left(|z'|^{\alpha_4}\operatorname{sgn} z'\right)' + \left(\frac{\alpha_2\alpha_3}{1+\alpha_3+\alpha_2\alpha_3}\right)^{\alpha_2\alpha_3\alpha_4} \left(\frac{\alpha_3}{1+\alpha_3}\right)^{\alpha_3\alpha_4} (t-t_0)^{(1+\alpha_2)\alpha_3\alpha_4} q(t)|z|^{\alpha_4} \operatorname{sgn} z = 0, \quad t \geq t_0. \tag{3.21}$$

From Corollary 3.4 we know that oscillation of both equations (3.20) and (3.21) implies oscillation of all solutions of equation (3.19).

This occurs, for example, if for some $\varepsilon \in (0, 1]$

$$\int_a^\infty t^{1-\varepsilon} \left(\int_t^\infty \left(\int_s^\infty q(\tau)d\tau\right)^{1/\alpha_4} ds\right)^{1/\alpha_3} dt = \infty \tag{3.22}$$

and

$$\int_a^\infty t^{(1+\alpha_2)\alpha_3\alpha_4+1-\varepsilon} q(t)dt = \infty, \tag{3.23}$$

(see [6]).

4 Reduction to the existence of solutions of maximal degree

In the last section we indicate an alternative way how to obtain the set of second-order equations (3.6_l) associated with the even order half-linear differential equation (1.1). Here, the problem of the existence of nonoscillatory solutions of an arbitrary degree l of equation(1.1) is converted into the problem of the existence of solutions of the maximal Kiguradze's degree of certain lower order half-linear differential equation.

Theorem 4.1. *If the n-th order equation (1.1) has a nonoscillatory solution of degree l, then the $(l+1)$-order differential equation*

$$D(\alpha_{l+1},\dots,\alpha_1)z(t)+\rho_{n-l-1}(t)|z(t)|^{\alpha_1\cdots\alpha_{l+1}}\operatorname{sgn} z(t)=0, \qquad t\ge t_0, \tag{4.1_l}$$

has a nonoscillatory solution of the same degree l.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) which is of Kiguradze's degree l. We may suppose that $x(t)$ is eventually positive and satisfies (1.2) and (1.3) on $[t_0,\infty), t_0\ge a$.

If $l=n-1$, then the proof is trivial because (4.1_{n-1}) is the same as (1.1).

Let $1\le l<n-1$. Integrating (1.1) from $t(\ge t_0)$ to ∞, we get

$$D(\alpha_{n-1},\dots,\alpha_1)x(t)\ge\left(\int_t^\infty q(s)x(s)^{\alpha_1\cdots\alpha_n}ds\right)^{1/\alpha_n}, \qquad t\ge t_0.$$

Continuing in this way, we finally arrive at

$$\begin{aligned}&-D(\alpha_{l+1},\dots,\alpha_1)x(t)\\ &\qquad\ge\left(\int_t^\infty\left(\int_{s_{l+2}}^\infty\cdots\left(\int_{s_{n-1}}q(s)x(s)^{\alpha_1\cdots\alpha_n}ds\right)^{1/\alpha_n}\dots ds_{l+3}\right)^{1/\alpha_{l+3}}ds_{l+2}\right)^{1/\alpha_{l+2}}\end{aligned} \tag{4.2}$$

for $t\ge t_0$. Since $x(t)$ is increasing for $t\ge t_0$, from (4.2) it follows that

$$D(\alpha_{l+1},\dots,\alpha_1)x(t)+\rho_{n-l-1}(t)x(t)^{\alpha_1\cdots\alpha_n}\le 0, \qquad t\ge t_0.$$

Application of Lemma 2.4 shows that (4.1_l) has a positive solution $z(t)$ which satisfies (1.2) and (1.3) with n replaced by $l+1$. The proof of the theorem is complete. □

If we estimate $x(t)$ from below as in the proof of Theorem 3.1 and substitute it into (4.1_l), we obtain

$$D(\alpha_{l+1},\dots,\alpha_1)x(t)+R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}\rho_{n-l-1}(t)\left[D(\alpha_{l-1},\dots,\alpha_1)x(t)\right]^{\alpha_l\alpha_{l+1}}\le 0 \tag{4.3}$$

for $t\ge t_0$. Let $y(t)$ be given by

$$y(t)=\left[D(\alpha_{l-1},\dots,\alpha_1)x(t)\right]^{\alpha_l}.$$

Then $y(t)$ satisfies the second order differential inequality

$$\left(|y'(t)|^{\alpha_{l+1}}\operatorname{sgn} y'(t)\right)'+R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}\rho_{n-l-1}(t)|y(t)|^{\alpha_{l+1}}\operatorname{sgn} y(t)\le 0, \qquad t\ge t_0,$$

and, by Lemma 2.4, there exists a nonoscillatory solution $z(t)$ (of degree 1) of the corresponding differential equation

$$\left(|z'(t)|^{\alpha_{l+1}}\operatorname{sgn} z'(t)\right)'+R_l^{\alpha_1\cdots\alpha_{l+1}}(t-t_0)^{(r_{l-1}-1)\alpha_1\cdots\alpha_{l+1}}\rho_{n-l-1}(t)|z(t)|^{\alpha_{l+1}}\operatorname{sgn} z(t)=0, \qquad t\ge t_0, \tag{4.4_l}$$

which is the same as (3.6_l).

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References

[1] T. A. Čanturija, On some asymptotic properties of solutions of linear ordinary differential equations (in Russian), *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys.* **25**(1977), No. 8, 757–762. MR0481247; Zbl 0375.34022

[2] T. A. Čanturija, Integral tests for oscillation of solutions of higher-order linear differential equations, II (in Russian) *Differentsial'nye Uravneniya* **16**(1980), No. 4, 635–644. MR0569994; Zbl 0479.34014

[3] O. Došlý, P. Řehák, *Half-linear differential equations*, North-Holland Mathematics Studies, Vol. 202, Elsevier, Amsterdam, 2005. MR2158903; Zbl 1090.34001

[4] O. Došlý, V. Růžička, Nonoscillation of higher order half-linear differential equations, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 19, 1–15. https://doi.org/10.14232/ejqtde.2015.1.19; MR3325922; Zbl 1349.34109

[5] K. E. Foster, R. C. Grimmer, Nonoscillatory solutions of higher order differential equations, *J. Math. Anal. Appl.* **71**(1979), 1–17. https://doi.org/10.1016/0022-247X(79)90214-2; MR0545858; Zbl 0428.34029

[6] J. Jaroš, An integral oscillation criterion for even order half-linear differential equations, *Appl. Math. Lett.* **104**(2020), 106257. https://doi.org/10.1016/j.aml.2020.106257; MR4061795; Zbl 07196209

[7] K. Kamo, H. Usami, Oscillation theorems for fourth-order quasilinear ordinary differential equations, *Studia Sci. Math. Hungar.* **39**(2002), 385–406. https://doi.org/10.1556/sscmath.39.2002.3-4.10; MR1956947; Zbl 1026.34054

[8] K. Kamo, H. Usami, Nonlinear oscillations of fourth order quasilinear ordinary differential equations, *Acta Math. Hungar.* **132**(2011), No. 3, 207–222. https://doi.org/10.1007/s10474-011-0127-x; MR2818904; Zbl 1249.34111

[9] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer, Dordrecht (1993) MR1220223; Zbl 0782.34002

[10] T. Kusano, M. Naito, Oscillation criteria for general linear ordinary differential equations, *Pacific J. Math.* **92**(1981), No. 1, 345–355. MR0618070; Zbl 0475.34019

[11] T. Kusano, M. Naito, F. Wu, On the oscillation of solutions of 4-dimensional Emden–Fowler differential systems, *Adv. Math. Sci. Appl.* **11**(2001), No. 2, 685–719. MR1907463; Zbl 1008.34028

[12] T. Kusano. T. Tanigawa, On the structure of positive solutions of a class of fourth order nonlinear differential equations, *Ann. Mat. Pura Appl.* **185**(2006), 521–536. https://doi.org/10.1007/s10231-005-0165-5; MR2230581.34056; Zbl 1232.34056

[13] T. Kusano, J. Manojlović, T. Tanigawa, Sharp oscillation criteria for a class of fourth order nonlinear differential equations, *Rocky Mountain J. Math.* **41**(2011), No. 1, 249–274. https://doi.org/10.1216/RMJ-2011-41-1-249; MR2845944; Zbl 1232.34053

[14] D. L. Lovelady, Oscillation and even order linear differential equations, *Rocky Mountain J. Math.* **6**(1976), 299–304. https://doi.org/10.1216/RMJ-1976-6-2-299; MR0390370; Zbl 0335.34019

[15] D. L. Lovelady, Asymptotic analyses of two fourth order linear differential equations, *Ann. Polon. Math.* **38**(1980), 109–119. https://doi.org/10.4064/ap-38-2-109-119; MR0599235; Zbl 0453.34044

[16] M. Naito, F. Wu, A note on the existence and asymptotic behavior of nonoscillatory solutions of fourth order quasilinear differential equations, *Acta Math. Hungar.* **102**(2004), No. 3, 177–202. https://doi.org/10.1023/B:AMHU.0000023215.24975.ee; MR2035369; Zbl 1048.34077

[17] M. Naito, F. Wu, On the existence of eventually positive solutions of fourth-order quasilinear differential equations, *Nonlinear Anal.* **57**(2004), No. 2, 253–263. https://doi.org/10.1016/j.na.2004.02.012; MR2056430; Zbl 1058.34066

[18] M. Naito, Existence and asymptotic behavior of positive solutions of higher-order quasilinear ordinary differential equations, *Math. Nachr.* **279**(2006), No. 1–2, 198–216. https://doi.org/10.1002/mana.200510356; MR2193618; Zbl 1100.34040

[19] M. Naito, Existence of positive solutions of higher-order quasilinear ordinary differential equations, *Ann. Mat. Pura Appl.* **186**(2007), No. 1, 59–84. https://doi.org/10.1007/s10231-005-0168-2; MR2263331; Zbl 1232.34054

[20] T. Tanigawa, Oscillation and nonoscillation theorems for a class of fourth order quasilinear functional differential equations, *Hiroshima Math. J.* **33**(2003), 297–316. https://doi.org/10.32917/hmj/1150997976; MR2040899; Zbl 1065.34062

[21] T. Tanigawa, Oscillation criteria for a class of higher order nonliear differential equations, *Mem. Differential Equations Math. Phys.* **37**(2006), 137–152. MR2223229; Zbl 1101.34020

[22] T. Tanigawa, Oscillation theorems for differential equations involving even order nonlinear Sturm–Liouville operator, *Georgian Math. J.* **14**(2007), No. 4, 737–768. MR2389034; Zbl 1139.34031

[23] W. F. Trench, An oscillation condition for differential equations of arbitrary order, *Proc. Amer. Math. Soc.* **82**(1981), No. 4, 548–552. https://doi.org/10.1090/S0002-9939-1981-0614876-4; MR0614876; Zbl 0481.34023

[24] F. Wu, Nonoscillatory solutions of fourth order quasilinear differential equations, *Funkcial. Ekvac.* **45**(2002), No. 1, 71–88. MR1913681; Zbl 1157.34319

[25] F. Wu, Existence of eventually positive solutions of fourth order quasilinear differential equations, *J. Math. Anal. Appl.* **389**(2012), 632–646. https://doi.org/10.1016/j.jmaa.2011.11.061; MR2876527; Zbl 1244.34054